1) Find two linearly independent solutions of \( x^2y'' + x(1-x)y' - (1+3x)y = 0, x > 0 \).

The origin is a regular singular point of the differential equation. (Why?) Let us assume that there is a solution of the form 
\[ \sum_{n=0}^{\infty} a_n x^{n+r}, x > 0. \]

Direct substitution of this into the equation gives 
\[ L(y) = \sum_{n=0}^{\infty} (n+r+1)(n+r-1)a_n x^{n+r} - \sum_{n=1}^{\infty} (n+r+2)a_{n-1} x^{n+r} = 0. \]
Then 
\[ (r+1)(r-1)a_0 x' = 0, a_n = \frac{(n+r+2)a_{n-1}}{(n+r+1)(n+r-1)}, n \geq 1 \]
from which it follows at once that 
\[ a_n = \frac{(n+r+2)a_0}{(r+2)[(r+1)\ldots(r+n-1)]}, n \geq 1. \] 
Then 
\[ y(x, r) = a_0 x^r + \sum_{n=1}^{\infty} \frac{(n+1)x^{n-1}}{(n-2)!} \]
which implies that 
\[ y_1 = y(x, 1) = -3x - \sum_{n=1}^{\infty} \frac{(n+1)x^{n-1}}{(n-2)!} \quad \text{and} \quad y_2 = \left[ \frac{\partial}{\partial r} (r+1)y(x, r) \right] \text{at} (r = -1) \text{ where} \]
\[ \frac{\partial}{\partial r} ((r+1)y(x, r)) = y(x, r) \ln x + x^r + \frac{(r+1)(r+3)x^{1+r}}{(r+2)r} \left\{ \frac{1}{r+1} + \frac{1}{r+3} - \frac{1}{r+2} - \frac{1}{r} \right\} \]
\[ + \frac{(r+4)x^{2+r}}{(r+2)r} \left\{ \frac{1}{r+4} - \frac{1}{r+2} - \frac{1}{r} \right\} + \sum_{n=3}^{\infty} \frac{(r+n+2)x^{n+r}}{(r+2)(r+3)\ldots(n+r-1)} \left\{ \frac{1}{r+n+2} - \frac{1}{r+2} - \frac{1}{r} - \frac{1}{r+3} - \ldots - \frac{1}{n+r-1} \right\}. \]

Hence, 
\[ y_2 = y_1 \ln x + x^{-1} - 2 - x - \sum_{n=1}^{\infty} \frac{(1-(n+1)H_{n-2})x^{n-1}}{(n-2)!} \quad \text{where} \quad H_n = \sum_{k=0}^{n-1} \frac{1}{k}. \]

2) Consider \( x^2y'' + (3x-1)y' + y = 0. \)

a) Show that the origin is the irregular singular point of the differential equation.

b) Obtain solutions of the differential equation for large \( x, \).

Hint: Put \( x = \frac{1}{w}. \)

a) \( \lim_{x \to 0} x^2 \frac{3x-1}{x^2} = -1, \) but \( \lim_{x \to 0} \frac{1}{x^2} \) doesn’t exist means this equation has an irregular singular point at the origin.
b) To investigate the nature of the equation for large \( x \), put \( x = \frac{1}{w} \). Then
\[
y'(x) = -w^2 y'(w), \quad \text{and} \quad y''(x) = w^2 y''(w) + 2w^3 y'(w).
\]
With the aid of these we see that the differential equation becomes
\[
w^2 y''(w) - w(w - 1)y'(w) + y(w) = 0 \ldots \ldots (*),
\]
an equation that we wish to solve about \( w = 0 \).
Since \( w = 0 \) is a regular singular point of equation (*), the point at infinity is a regular singular point of the original equation. From the assumed form
\[
y = \sum a_n w^{n+r}, \text{it follows from eqn.*} \text{by our usual methods that}
\]
\[
L(y) = \sum_{n=0}^{\infty} (n + r - 1) a_n w^{n+r} + \sum (n + r - 1)a_{n-1} w^{n+r} = 0. \text{The indicial equation has roots}
\]
r = 1, 1. From the recurrence relation
\[
a_n = \frac{(-1)^n a_0}{r(r+1)(r+2) \ldots (r+n-1)}, \quad n \geq 1. \quad \text{Hence}
\]
\[
y(w, r) = w^r + \sum_{n=1}^{\infty} \frac{(-1)^n w^{n+r}}{r(r+1)(r+2) \ldots (r+n-1)} \quad \text{and we find}
\]
\[
\frac{\partial y(w, r)}{\partial r} = y(w, r) \ln w - \sum_{n=1}^{\infty} \frac{(-1)^n w^{n+r}}{r(r+1)(r+2) \ldots (r+n-1)}.
\]

Employing the root \( r = 1 \), we obtain the solutions
\[
y_1(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n!}, \quad \text{and} \quad y_2 = y_1 \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n H_n x^{n-1}}{n!}, \quad \text{where}
\]
\[
H_n = \sum_{k=1}^{n} \frac{1}{k}.
\]
Therefore, the original differential equation has two linearly independent solutions
\[
y_1 = \sum_{n=1}^{\infty} \frac{(-1)^n x^{-(n+1)}}{n!} = x^{-1} e^{-x^{-1}}
\]
and
\[
y_2 = y_1 \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n H_n x^{-(n-1)}}{n!}.
\]

3) Solve the problem \( y'' + 2y' + y = t; y(0) = -3, y(1) = -1 \), using the Laplace transform method.

The transformed problem is
\[
s^2 Y(s) + 3s - C + 2sY(s) + 6 + Y(s) = \frac{1}{s^2} \quad \text{where} \ C = y'(0), \text{and} \ L\{y(t)\} = L\{y(t)\},
\]
from which
\[
Y(s) = \frac{-3(s+1) + C - 3}{(s+1)^2} + \frac{1}{s^2(s+1)^2} \quad \text{or} \quad Y(s) = \frac{1}{s^2} - \frac{2}{s} - \frac{1}{s+1} + \frac{C-2}{(s+1)^2},
\]
from which we obtain
\[ y(t) = t - 2 - e^{-t} + (C - 2)te^{-t}. \] After imposing the condition \( y(1) = -1 \) we obtain \( C = 3 \). Thus \( y(t) = t - 2 - e^{-t} + te^{-t} \).

4) Solve \( y'' + 4y = \begin{cases} 4t & 0 \leq t \leq 1 \\ 4 & t > 1 \end{cases} \), \( y(0) = 1, y'(0) = 0 \) using the Laplace transform technique.

Taking the Laplace Transform of the equation we obtain
\[
(s^2 + 4)Y(s) = s + \frac{4}{s^2} - \frac{4e^{-s}}{s^3}\quad \text{and} \quad Y(s) = \frac{1}{(s^2 + 4)} \left( s + \frac{4}{s^2} - \frac{4e^{-s}}{s^3} \right).\]

Hence,
\[
y(t) = \cos 2t + \frac{2t - \sin 2t}{2} - \begin{cases} 0 & t \in [0, 1] \\ 2(t - 1) - \sin 2(t - 1) & t \in (1, \infty) \end{cases}.
\]

5) If \( f(t) \) is to be continuous for \( t \geq 0 \) and \( \mathcal{L}^{-1}\left\{ \frac{e^{-3s}}{(s+1)^3} \right\} \) evaluate \( f(2), f(5), f(7) \).

Remember: \( \mathcal{L}\{e^{-s}\} = (s + 1)^{-1}, \{t^2e^{-t}\} = (-1)^2 \frac{d^2}{ds^2} (s + 1)^{-1} = 2(s + 1)^{-3} \), and \( \mathcal{L}^{-1}\{(s+1)^{-3}\} = \frac{1}{2} t^2 e^{-t} \). So,
\[
f(t) = \begin{cases} 0 & t \in [0, 3] \\ \frac{1}{2} (t-3)^2 e^{-(t-3)} & t \in (3, \infty) \end{cases}.
\]

Then \( f(2) = 0, f(5) = 2e^{-2}, \) and \( f(7) = 8e^{-4} \).