5) Find a Taylor polynomial that will approximate \( F(x) = \int_0^x \sin(t^2) \, dt \) throughout the interval \([0,1]\) with an error of magnitude less than \(10^{-4}\). What polynomial has the smallest degree? 

Show all your work to explain your answer clearly.

We know that 
\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \forall x.
\]

Then 
\[
F(x) = \int_0^x \sin(t^2) \, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+2} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+3}}{(2n+3)}, \quad \forall x.
\]

The Taylor series for \( F(x) \) is an alternating series for every nonzero value of \( x \), and for \( 0 < x \leq 1 \), \((u_{n+1}/u_n) < 1\) and \( u_n \to 0 \) as \( n \to \infty \) where \( u_n = \frac{x^{4n-1}}{(4n-1)(2n-1)!} \). According to the Alternating Series Estimation Theorem, the error in truncating 
\[
F(x) = \frac{x^3}{3.1!} - \frac{x^7}{7.3!} + \cdots + (-1)^{n-1} \frac{x^{4n-1}}{(4n-1)(2n-1)!} + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \cdots
\]

is no greater than 
\[
\left| \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} \right|, \quad \text{ie.}
\]

\[
|\text{error}| \leq \left| \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} \right| \leq \frac{1}{(4n+3)(2n+1)!}, \quad x \in [0,1]
\]

\[
< 10^{-4}
\]

Find \( n \) s.t. 
\[
\frac{n}{(4n+3)(2n+1)!} > 10^{-4}
\]

\[
\begin{align*}
2 & \quad (11)(5) = 1320 + 10^4 \\
3 & \quad (15)(7) = 75600 > 10^4
\end{align*}
\]

\( n \geq 3 \).

\[
F(x) \approx \frac{x^3}{3.1!} - \frac{x^7}{7.3!} + \frac{x^{11}}{11.5!}, \quad \text{with error} \leq 10^{-4} \quad \forall x \in [0,1].
\]

\[
p(x) = \frac{x^3}{3.1!} - \frac{x^7}{7.3!} + \frac{x^{11}}{11.5!} \]

is the polynomial with the smallest degree that the Alternating Series Estimation Thm. yields.