1. Assume $f$ is a differentiable function with an inverse, and also assume that $f(3) = e^3, f'(3) = 7$.

Let $y = f^{-1}(e^{x^2 + x})$. Find $\frac{dy}{dx}$ at $x = 1$.

Let $u = e^{x^2 + x}$. Then $y = f^{-1}(u)$.

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{f'(f^{-1}(u))} \frac{du}{dx}, \quad \frac{du}{dx} = e^{x^2 + x}(2x+1)
\]

At $x = 1$, $\left.\frac{du}{dx}\right|_{x=1} = e^2 \cdot 3$

$x = 1 \Rightarrow u = e^2, f^{-1}(e^2) = 3$. So $\frac{1}{f'(f^{-1}(e^2))} = \frac{1}{f'(3)} = \frac{1}{7}$.

\[
\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{7} \cdot e^2 \cdot 3 = \frac{3e^2}{7}.
\]
2. Let \( f \) be a function so that \( f' \) is continuous on the open interval \((3,7)\). Assume that the linearization of \( f(x) \) at \( x = 4 \) is \( 2x + 5 \) and the linearization of \( f(x) \) at \( x = 5 \) is \( 3x - 5 \). Then

a) Find \( f(4), f(5), f'(4), \) and \( f'(5) \).

b) Show that there exists \( c \in (3,7) \) such that \( f'(c) = \frac{5}{2} \).

c) Show that there exists \( d \in (3,7) \) such that \( f'(d) = -2 \).

\[
\begin{align*}
\text{a)} & \\
L_1(x) &= 2x + 5 \quad \text{linearization of } f \text{ at } x = 4 \\
L_2(x) &= 3x - 5 \quad \text{at } x = 5 \\
L_1(4) &= f(4) = 13, \quad f'(4) = L_1'(4) = 2 \\
f(5) &= L_2(5) = 10, \quad f'(5) = L_1'(5) = 3
\end{align*}
\]

b) Let \( g(x) = f'(x) - \frac{5}{2} \).

\[
\begin{align*}
g(4) &= f'(4) - \frac{5}{2} = 2 - \frac{5}{2} = -\frac{1}{2} < 0 \\
g(5) &= f'(5) - \frac{5}{2} = 3 - \frac{5}{2} = \frac{1}{2} > 0
\end{align*}
\]

\[
\begin{align*}
\text{Since the open interval } (4,5) \text{ is contained in} \\
\text{the open interval } (3,7), \quad c \in (3,7). \\
\text{or} \quad f'(4) = 2 < \frac{5}{2} < f'(5) = 3 \quad \text{and } f' \text{ is cont. on } (3,7) \quad \Rightarrow \exists c \in (4,5) \text{ s.t. } f'(c) = \frac{5}{2}
\end{align*}
\]

c) \( f' \in C(4,5) \) \quad \text{MVT} \quad \exists c_1 \in (4,5) \text{ s.t. } f'(c_1) = \frac{f(5) - f(4)}{5 - 4} = \frac{10 - 13}{1} = -3

\[
\begin{align*}
f'(4) &= 2 \\
-3 &= \frac{-3}{-2} < 2 \quad \text{IVT} \quad \exists d \in (4, c_1) \text{ s.t. } f'(d) = -2.
\end{align*}
\]
3. A particle is moving on the curve \( y = e^{-x} \) in such a way that \( x \)-coordinate is increasing at a rate of 3 cm/sec. Find the rate of change of the area of the triangle formed by the tangent line to the curve at the location of the particle, the \( x \)-axis and the \( y \)-axis when the particle is at the point \( P(2,e^{-2}) \).

Denote the coordinates of the particle by \( P(a, e^{-a}) \).

\[
\frac{dy}{dx} = -e^{-x} \quad \Rightarrow \quad m = -e^{-a}.
\]

\[
\theta_{tg}: \quad y - e^{-a} = -e^{-a}(x-a).
\]

At \( A \), \( y = 0 \). So \( -e^{-a} = -e^{-a}(x-a) \)

\[
\Rightarrow \quad x = a + 1, \quad \text{i.e.,} \quad A(a+1,0)
\]

At \( B \), \( x = 0 \). So \( y - e^{-a}(-a) = 0 \) \( \Rightarrow \) \( y = (a+1)e^{-a} \), i.e., \( B(0,(a+1)e^{-a}) \).

Then \( \text{Area} = \frac{1}{2} (a+1)(a+1) e^{-a} = \frac{1}{2} (a+1)^2 e^{-a} \).

\[
\frac{d}{dt}(\text{Area}) = \frac{1}{2} \left[ 2(a+1)e^{-a} + (a+1)^2 e^{-a} (-1) \right] \frac{da}{dt}
\]

\[
= \frac{1}{2} (a+1) e^{-a} \left[ \frac{2-(a+1)}{1-a} \right] \frac{da}{dt}
\]

\[
= \frac{1}{2} (1-a^2) e^{-a} \frac{da}{dt} \quad \text{where} \quad \frac{da}{dt} = 3 \quad \text{and} \quad a = 2
\]

So \( \frac{d}{dt}(\text{Area}) = \frac{1}{2} (1-4) e^{-2} \cdot 3 = \frac{-9 e^{-2}}{2} \).
4. Find an equation of the
   a) horizontal tangent line(s), and
   b) vertical tangent line(s)
   to the curve \( x^3 + y^3 - 9xy = 0 \).

First note that if a point \((a, b)\)
   is on the curve \( x^3 + y^3 - 9xy \) then \((b, a)\)
   is also on it. Hence if the curve has a
   horizontal tangent at \((a, b)\) then it has a vertical
   tangent at \((b, a)\). Therefore the curve \( x^3 + y^3 - 9xy \)
   does not have a horizontal tangent or vertical tangent
   at \((0, 0)\). Suppose the curve \( x^3 + y^3 - 9xy \) has
   a horizontal tangent at \((a, b)\). Then since we have
\[
\begin{align*}
x^3 + y^3 - 9xy &= 0 \\
3x^2 + 3y^2 - 9y - 9x \frac{dy}{dx} &= 0
\end{align*}
\]
We get
\[
\frac{dy}{dx} = \frac{-x^2 + 3y}{y^2 - 3x}
\]
when \( y^2 - 3x \neq 0 \).

Then \(-a^2 + 3b = 0\) and \(a^3 + b^3 - 9ab = 0\).

So \( a^3 + \frac{a^6}{27} - 9a \frac{a^2}{3} = 0 \). So \( a^6 = 54a^3 \).

Hence \( a = 0 \) or \( a = 3\sqrt{54} \). But \( a \neq 0 \) because
otherwise \( b = 0 \) and we already know that there
is no horizontal tangent line at \((0, 0)\).

So \( a = 3\sqrt{54} \) and \( b = \frac{3\sqrt{54}}{2} \).

So the only horizontal tangent line is \( y = \frac{3\sqrt{54}}{3} \) and
by symmetry the only vertical tangent line is \( x = \frac{\sqrt{54}}{3} \).
5. (1+1+2+2+2+2+2+6 pts.) Consider \( f(x) = \frac{(x+1)^2}{x^2+1} \). Find:

a) The domain \( D \) of \( f \).

b) The \( x \)-intercept:
   - The \( y \)-intercept:

c) Asymptote(s):

d) All critical points:

e) All intervals on which \( f \) is increasing, decreasing:

f) All local extrema:

g) All intervals on which \( f \) is concave up, concave down:

h) All points of inflection:

AND

i) Sketch the graph of \( y = f(x) \).

\[ a) \ D = \mathbb{R} \]

\[ b) \ P_0 (-1, 0) : x \text{-intercept} \]
\[ P_1 (0, 1) : y \text{-intercept} \]

\[ c) \ (V.a) \ None \]

\[ (V.a) \ \lim_{x \to \pm \infty} f(x) = \pm \infty \Rightarrow y = 1 \text{ is a h.a.} \]

\[ (V.a) \ \lim_{x \to \pm \infty} \frac{2(1-x^2)}{(1+x^2)^2} = 0 \Rightarrow x = \pm 1 \text{ (c.pts.)} \]

\[ d) \ f'(x) = \frac{4x(x^2-3)}{(1+x^2)^3} \]

\[ e) \ f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\sqrt{3})</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>( \sqrt{3} )</th>
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<td>( f' )</td>
<td>-</td>
<td>-</td>
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<tr>
<td>( f'' )</td>
<td>-</td>
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<td>+</td>
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</tbody>
</table>

\[ \frac{A = f(-\sqrt{3})}{B = f(\sqrt{3})} = \frac{(1-\sqrt{3})^2}{4} < \frac{(1+\sqrt{3})^2}{4} \]

e) \( f \) is decreasing on the intervals \((-\infty, -1)\) and \((1, \infty)\)

f) \( f \) is increasing on the interval \((-1, 1)\)

\[ f(-1) = 0 \text{ is a local min.} \]
\[ f(1) = 2 \text{ is max.} \]
g) the curve is concave down on the intervals \((-\infty, -\sqrt{3})\) and \((0, \sqrt{3})\), and the curve is concave up on the intervals \((-\sqrt{3}, 0)\) and \((\sqrt{3}, \infty)\).

h) The second derivative changes sign at the points \(x = -\sqrt{3}, x = 0,\) and \(x = \sqrt{3}\). Thus each point is a point of inflection.