MATH-213-01 1st Midterm

DATE: 18 October 2003

TIME: 14:00–16:00

Question 1(a): (5 points) Prove that if $0 < a < 1$ and $b = 1 - \sqrt{1 - a}$, then $0 < b < a$.

Question 1(b): (5 points) Suppose $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and calculate its limit.

Question 1(c): (10 points) For the sequence $(x_n)_{n \in \mathbb{N}}$ as in Question 1(b), prove or disprove that the sequence $y_n = x_{n+1}/x_n$ is convergent.

Question 2: (20 points) Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of real sets, such that for all $n \in \mathbb{N}$ the following hold:
   (a) $A_n$ is non-empty;
   (b) $A_n$ is bounded above;
   (c) $A_n \subseteq A_{n+1}$.
Prove or disprove that
\[
\sup \bigcup_{n \in \mathbb{N}} A_n = \sup \{ \sup A_n \mid n \in \mathbb{N} \}.
\]

Question 3(a): (15 points) Prove or disprove the following statement: A real sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy (fundamental) if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $n \in \mathbb{N}$, we have $|x_{n+1} - x_n| < \varepsilon$.

Question 3(b): (5 points) Prove or disprove the following statement: A real sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy (fundamental) if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $n \in \mathbb{N}$, we have $|x_{n+N} - x_n| < \varepsilon$.

Question 4: (20 points) Prove or disprove the following statement: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and
\[
\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty,
\]
then $f$ has a finite infimum that is attained, more precisely, there exists $x_m \in \mathbb{R}$ such that $f(x_m) = \inf \{ f(x) \mid x \in \mathbb{R} \} > -\infty$.

Question 5: (20 points) Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous and have the property that for all $x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$. 