Problem (3/218)

Show that \( p(x) = \lim sup \xi_n \) where \( x = (\xi_n) \in l^\infty \), \( \xi_n \) real, defines a sublinear functional on \( l^\infty \).

Solution

Let \( x = (\xi_n), y = (\eta_n) \) be in \( l^\infty \) and \( \alpha \) is a positive real number. Then

\[
p(x + y) = \lim sup \xi_n + \eta_n \leq \lim sup \xi_n + \lim sup \eta_n = p(x) + p(y)
\]

\[
p(\alpha x) = \lim sup \alpha \xi_n = \alpha \lim sup \xi_n = \alpha p(x).
\]

Problem (5/218)

(Convex set) If \( p \) is a sublinear functional on a vector space \( X \), show that \( M = \{ x \mid p(x) \leq \gamma, \gamma > 0 \text{ fixed} \} \), is a convex set.

Solution

Let \( x, y \in M \) and \( \alpha \in [0, 1] \). Since \( p(x) \leq \gamma, p(y) \leq \gamma, 1 - \alpha \geq 0, \)

\[
p(\alpha x + (1 - \alpha)y) \leq p(\alpha x) + p((1 - \alpha)y) = \alpha \underbrace{p(x)}_{\leq \gamma} + (1 - \alpha) \underbrace{p(y)}_{\leq \gamma} \leq \gamma.
\]

So \( \alpha x + (1 - \alpha)y \in M \), which means \( M \) is convex.

Problem (6/218)

If a subadditive functional \( p \) on a normed space \( X \) is continuous at 0 and \( p(0) = 0 \), show that \( p \) is continuous for all \( x \in X \).

Solution

Firstly notice that

\[
p(x) = p(x - y + y) \leq p(x - y) + p(y) \implies p(x) - p(y) \leq p(x - y)
\]

\[
p(y) = p(y - x + x) \leq p(y - x) + p(x) \implies -p(y - x) \leq p(x) - p(y).
\]

Hence

\[-p(y - x) \leq p(x) - p(y) \leq p(x - y).
\]
We know that given $\epsilon > 0$ there is $\delta > 0$ such that $$||x - 0|| < \delta \implies |p(x) - p(0)| < \epsilon$$ that is $$||x|| < \delta \implies |p(x)| < \epsilon.$$ This means that $$||x - y|| < \delta \implies -\epsilon < p(x) - p(y) < \epsilon \implies |p(x) - p(y)| < \epsilon.$$ Hence $p$ is continuous.

**Problem (7/224)**

Give another proof of Theorem 4.3-3 in the case of a Hilbert space.

**Solution**

We will show that if $H$ is a Hilbert space and $x_0 \in H$ is non-zero then there exists $f \in H'$ such that $||f|| = 1$ and $f(x_0) = ||x_0||$.

Define $$f : H \rightarrow k(\mathbb{C} \text{ or } \mathbb{R}) \text{ by } f(x) = \langle x, \frac{x_0}{||x_0||} \rangle.$$ Clearly $f$ is linear s.t. $||f|| = ||\frac{x_0}{||x_0||}|| = 1$ (see Theorem 3.8-1) and $f(x_0) = ||x_0||$.

**Problem (14/224)**

(Hyperplane) Show that for any sphere $S(0; r)$ in a normed space $X$ and any point $x_0 \in S(0; r)$ there is a hyperplane $H_0$ such that $x_0 \in H_0$ and ball $\tilde{B}(0; r)$ lies entirely in one of the two half spaces determined by $H_0$.

**Solution**

(See pg.111 ques. 15). By Theorem 4.3-3 there exists $f \in X'$ such that $||f|| = 1$ and $f(x_0) = ||x_0||$. Consider hyperplane $H_0$ defined by $$H_0 = \{x \mid f(x) = r\}.$$ Since $f(x_0) = ||x_0|| = r$, $x_0 \in H_0$. Also if $x \in \tilde{B}(0; r)$ then $||x|| \leq r$ so $$|f(x)| \leq ||f|| ||x|| \leq r \implies f(x) \leq r.$$ Hence $\tilde{B}(0; r)$ lies entirely in the half space $\{x \mid f(x) \leq r\}$. 

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