Problem (2/175)

Show that if the orthogonal dimension of a Hilbert space $H$ is finite, it equals the dimension of $H$ regarded as a vector space; conversely, if the latter is finite, show that so is the former.

Solution

Let orthogonal dimension of $H$ finite, say $n$. So there is a subset $\{e_1, e_2, \ldots, e_n\}$ of $H$ which is total orthonormal in $H$. But this set is linearly independent and spans $H$ so it is a basis for $H$, which means that $\dim H = n$.

Let's now suppose $\dim H = n$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for $H$. Apply Gram-Schmidt process to $x_1, x_2, \ldots, x_n$ and let $e_1, e_2, \ldots, e_n$ be the derived vectors. Then $\{e_1, e_2, \ldots, e_n\}$ is total orthonormal in $H$. Because it is orthonormal and $\text{span}\{e_1, e_2, \ldots, e_n\} = \text{span}\{x_1, x_2, \ldots, x_n\} = H$. So orthogonal dimension of $H$ is $n$.

Remark 1 $\{e_1, e_2, \ldots, e_n\}$ orthonormal $\Rightarrow$ it is linearly independent.

Let $\alpha_1 e_1 + \ldots + \alpha_n e_n = 0$. Then

$$\langle \alpha_1 e_1 + \ldots + \alpha_n e_n, e_j \rangle = \alpha_j \langle e_j, e_j \rangle = \alpha_j = 0 \text{ for all } j = 1, 2, \ldots, n.$$ 

Remark 2 For any vectors $y_1, y_2, \ldots, y_n$,

$$\text{span}\{y_1, y_2, \ldots, y_n\} = \text{span}\{y_1, y_2, \ldots, y_n\}$$

since it is a finite dimensional vector space and any finite dimensional vector space is closed in a normed space, in particular in an inner product space.

Problem (7/175)

Show that if a Hilbert space $H$ is separable, the existence of a total orthonormal set in $H$ can be proved without the use of Zorn’s lemma.

Solution Let $M$ be a countable dense subset of $H$. Choose a linearly independent subset $C$ in $M$ such that $\text{span} C \supseteq M$ (just remove $n^{th}$ term if it is in the span of first $n - 1$ element for $n \geq 2$ and wlog we may assume first element of $M$ is not 0). Surely $C$ is countable. Apply Gram-Schmidt process to $C$ and let $F$ be the derived set. Then $F$ is total orthonormal in $H$. Indeed $F$ is orthonormal and $\text{span} F = \overline{\text{span} C} \supseteq M = H$. 


Problem (8/175)

Show that for any orthonormal sequence $F$ in a separable Hilbert space $H$ there is a total orthonormal sequence $\tilde{F}$ which contains $F$.

Solution

Let $Y = \text{span} F$. Since $Y$ is closed $H = Y \oplus Y^\perp$ (see Theorem 3.3-4). Clearly $Y^\perp$ is also separable and indeed it is easy to show it is closed subspace of $H$. This means that we can view $Y^\perp$ as a Hilbert space and so it has a countable total orthonormal set, say $E$ (see previous question and Theorem 3.6-4). Set $\tilde{F} = F \cup E$. Then $\tilde{F}$ is total orthonormal sequence in $H$ that contains $F$.

$\tilde{F}$ is orthonormal since $F$ and $E$ are orthonormal and $F \perp E$.

$\tilde{F}$ is total, i.e. $\text{span} \tilde{F} = H$. Given $x \in H$, let $x = y + z$ where $y \in Y$ and $z \in Y^\perp$. Since $\text{span} F \supseteq \text{span} \tilde{F} = Y$ and $\text{span} F \supseteq \text{span} E = Y^\perp$, we obtain $y, z \in \text{span} \tilde{F}$. But $\text{span} \tilde{F}$ is a vector space so $x = y + z \in \text{span} \tilde{F}$.

Problem (9/175)

Let $M$ be a total set in an inner product space $X$. If $\langle v, x \rangle = \langle w, x \rangle$ for all $x \in M$, show that $v = w$.

Solution

$\langle v, x \rangle = \langle w, x \rangle$ for all $x \in M \Rightarrow \langle v - w, x \rangle = 0$ for all $x \in M \Rightarrow v - w \perp M \Rightarrow v - w = 0 \Rightarrow v = w$ (see Theorem 3.6-2(a)).

Problem (10/175)

Let $M$ be a subset of a Hilbert space $H$, and let $v, w \in H$. Suppose that $\langle v, x \rangle = \langle w, x \rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$, show that $M$ is total in $H$.

Solution

$v \perp M \Rightarrow \langle v, x \rangle = 0$ for all $x \in M \Rightarrow \langle v, x \rangle = \langle 0, x \rangle$ for all $x \in M \Rightarrow v = 0$.

So totality of $M$ follows from Theorem 3.6-2(b).

[Prove or disprove: The second part of Theorem 3.6-2 is valid for non-complete inner product spaces.]