**Problem (6/159)**

(Minimum property of Fourier coefficients) Let \(\{e_1, ..., e_n\}\) be an orthonormal set in an inner product space \(X\), where \(n\) is fixed. Let \(x \in X\) be fixed and \(y = \beta_1 e_1 + \cdots + \beta_n e_n\). Then \(||x - y||\) depends on \(\beta_1, ..., \beta_n\). Show by direct calculation that \(||x - y||\) is minimum if and only if \(\beta_j = \langle x, e_j \rangle\), where \(j = 1, ..., n\).

**Solution**

\[
||x - y||^2 = \langle x - y, x - y \rangle
= \langle x - \sum_{k=1}^{n} \beta_k e_k, x - \sum_{k=1}^{n} \beta_k e_k \rangle
= \langle x, x \rangle - \sum_{k=1}^{n} \beta_k \langle e_k, x \rangle - \sum_{k=1}^{n} \beta_k \langle e_k, e_k \rangle
= ||x||^2 - \sum_{k=1}^{n} \beta_k \langle x, e_k \rangle - \sum_{k=1}^{n} \beta_k \langle e_k, x \rangle + \sum_{k=1}^{n} \beta_k \overline{\beta_k}
= ||x||^2 + \sum_{k=1}^{n} [-\overline{\beta_k} \langle x, e_k \rangle - \beta_k \overline{\langle x, e_k \rangle}] + \sum_{k=1}^{n} \beta_k \overline{\beta_k}.
\]

So let’s add and remove \(\sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle}\). Then

\[
||x - y||^2 =
= ||x||^2 + \sum_{k=1}^{n} [-\overline{\beta_k} \langle x, e_k \rangle - \beta_k \overline{\langle x, e_k \rangle}] + \overline{\beta_k} \beta_k + \langle x, e_k \rangle \overline{\langle x, e_k \rangle} - \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle}
= ||x||^2 + \sum_{k=1}^{n} \overline{\beta_k - \langle x, e_k \rangle} (\beta_k - \langle x, e_k \rangle) - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2
= ||x||^2 + \sum_{k=1}^{n} |\beta_k - \langle x, e_k \rangle|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2.
\]

Since \(x, e_1, ..., e_n\) are fixed, clearly, \(||x - y||\) is minimum if and only if \(\beta_j = \langle x, e_j \rangle\), where \(j = 1, ..., n\).
Problem (7/159)

Let \((e_k)\) be any orthonormal sequence in an inner product space \(X\). Show that for any \(x, y \in X\),
\[
\sum_{k=1}^{\infty} |\langle x, e_k \rangle\langle y, e_k \rangle| \leq \|x\| \|y\|.
\]

Solution
\[
\sum_{k=1}^{\infty} |\langle x, e_k \rangle\langle y, e_k \rangle| \leq \left( \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} |\langle y, e_k \rangle|^2 \right)^{1/2} \leq \|x\| \|y\|,
\]
where first inequality is Cauchy-Schwartz and second inequality is Bessel.

Problem (9/160)

Orthonormalize the first three terms of the sequence \((x_0, x_1, x_3, \ldots)\), where \(x_j(t) = t^j\), on the interval \([-1, 1]\) wrt inner product
\[
\langle x, y \rangle = \int_{-1}^{1} x(t) y(t) \, dt.
\]

Solution
\[
e_0(t) = \frac{1}{\sqrt{2}} \quad e_0 = \frac{x_0}{\|x_0\|},
\]
\[
e_1(t) = \frac{\sqrt{3}}{\sqrt{2}} t \quad e_1 = \frac{v_1}{\|v_1\|} \quad \text{where } v_1 = x_1 - \langle x_1, e_0 \rangle e_0.
\]
\[
e_2(t) = \frac{\sqrt{5}}{2\sqrt{2}} (3t^2 - 1) \quad e_2 = \frac{v_2}{\|v_2\|} \quad \text{where } v_2 = x_2 - \langle x_2, e_0 \rangle e_0 - \langle x_2, e_1 \rangle e_1.
\]

Problem (5/167)

Show that in a Hilbert space \(H\), convergence of \(\sum \|x_j\|\) implies convergence of \(\sum x_j\).

Solution
Let \(s_n = x_1 + \cdots + x_n\). It is enough to show \(\{s_n\}_{n=1}^{\infty}\), the sequence of partial sums, is Cauchy. Let \(n > m\) then
\[
\|s_n - s_m\| = \| \sum_{k=m+1}^{n} x_k \| \leq \sum_{k=m+1}^{n} \|x_k\| \to 0 \quad \text{as } n, m \to \infty.
\]
Problem (6/167)

Let \((e_j)\) be an orthonormal sequence in a Hilbert space \(H\). Show that if

\[
x = \sum_{j=1}^{\infty} \alpha_j e_j, \quad y = \sum_{j=1}^{\infty} \beta_j e_j,
\]

then

\[
\langle x, y \rangle = \sum_{j=1}^{\infty} \alpha_j \bar{\beta}_j.
\]

Solution

Let \(x_n = \sum_{j=1}^{n} \alpha_j e_j\) and \(y_n = \sum_{j=1}^{n} \beta_j e_j\). Then \(x_n \to x\) and \(y_n \to y\).

Since \((e_j)\) is an orthonormal sequence we get

\[
\langle x_n, y_n \rangle = \left\langle \sum_{j=1}^{n} \alpha_j e_j, \sum_{j=1}^{n} \beta_j e_j \right\rangle = \sum_{j=1}^{n} \alpha_j \bar{\beta}_j.
\]

But we know that inner product is continuous, that is \(x_n \to x\) and \(y_n \to y\) implies \(\langle x_n, y_n \rangle \to \langle x, y \rangle\). Hence

\[
\langle x_n, y_n \rangle \to \sum_{j=1}^{\infty} \alpha_j \bar{\beta}_j = \langle x, y \rangle.
\]

To show the absolute convergence, notice that \(\alpha_j = \langle x, e_j \rangle\) and \(\beta_j = \langle y, e_j \rangle\). So it follows from the solution of 7/159.