Problem (5/140)

Show that for a sequence \((x_n)\) in an inner product space the conditions 
\[||x_n|| \to ||x||\] and 
\[\langle x_n, x \rangle \to \langle x, x \rangle\] imply convergence 
\[x_n \to x.\]

Solution

\[||x_n - x||^2 = \langle x_n - x, x_n - x \rangle = \langle x_n, x_n \rangle - \langle x, x_n \rangle - \langle x_n, x \rangle + \langle x, x \rangle = ||x_n||^2 + ||x||^2 - \langle x, x \rangle - \langle x_n, x \rangle = 0\] as \(n \to \infty.\)

Problem (8/141)

Show that in an inner product space, 
\[x \perp y\] if and only if 
\[||x + \alpha y|| \geq ||x||\] for all scalars \(\alpha\).

Solution

\((\Rightarrow)\) Let \(x \perp y\). Pick arbitrary scalar \(\alpha\). Then

\[||x - \alpha y||^2 = \langle x - \alpha y, x - \alpha y \rangle = \langle x, x \rangle - \alpha \langle y, x \rangle - \alpha \langle x, y \rangle + |\alpha|^2 \langle y, y \rangle = ||x||^2 + |\alpha|^2 ||y||^2 \geq ||x||^2.\]

\((\Leftarrow)\) Now let \(\langle x, y \rangle \neq 0\). We will find \(\alpha\) such that 
\[||x + \alpha y|| < ||x||.\] Indeed choosing

\[\alpha = -\frac{\langle x, y \rangle}{||y||^2}\] (notice that \(y \neq 0\)) implies that

\[||x - \alpha y||^2 = \cdots = ||x||^2 - \frac{||\langle x, y \rangle||^2}{||y||^2} < ||x||^2.\]

Problem (9/141)

Let \(V\) be a vector space of all continuous complex-valued functions on 
\(J = [a, b]\). Let 
\[X_1 = (V, ||\cdot||_\infty),\] where

\[||x||_\infty = \max_{t \in J} |x(t)|;\]

and let 
\[X_2 = (V, ||\cdot||_2)\] where

\[||x||_2 = \langle x, x \rangle^{1/2}, \quad \langle x, y \rangle = \int_a^b x(t)\overline{y(t)} dt.\]
Show that the identity mapping \( x \mapsto -x \) of \( X_1 \) onto \( X_2 \) is continuous. (It is not a homeomorphism. \( X_2 \) is not complete.)

**Solution**

Since the operator is linear, boundedness and continuity are equivalent. We will show it is bounded.

\[
||Ix||^2 = ||x||^2 = \int_a^b |x(t)|^2 \, dt \leq \max_{t \in J} |x(t)|^2 \int_a^b dt = ||x||_\infty^2 (b - a).
\]

(Here it can be shown that \( ||I||_op = \sqrt{b - a} \) and this may not be 1, of course this is because underlying norms are different. Indeed the inverse of this operator, which is identity again, is not continuous.)

**Problem (1/150)**

Let \( H \) be a Hilbert space, \( M \subset H \) a convex subset, and \((x_n)\) a sequence in \( M \) such that

\[
||x_n|| \rightarrow d, \text{ where } d = \inf_{x \in M} ||x||.
\]

Show that \((x_n)\) converges in \( H \). Give an illustrative example in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

**Solution**

Firstly observe that if \( x \) and \( y \) in \( M \) then \( \alpha x + (1 - \alpha)y \) is in \( M \) for \( \alpha \in [0, 1] \). Putting \( \alpha = 1/2 \) we have

\[
||x + y|| = 2||\frac{x + y}{2}|| \geq 2d
\]

We also have parallelogram equality

\[
||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)
\]

since the norm is induced by an inner product. So we have

\[
||x_m - x_n||^2 = 2(||x_m||^2 + ||x_n||^2) - ||x_m + x_n||^2
\]

\[
\leq 2(||x_m||^2 + ||x_n||^2) - 4d^2 \rightarrow 2(d^2 + d^2) - 4d^2 = 0
\]

as \( n, m \to \infty \).

So \((x_n)\) is Cauchy. Since \( H \) is complete \((x_n)\) converges in \( H \).
Problem (2/150)

Show that the subset \( M = \{ y = (\eta_j) \mid \sum \eta_j = 1 \} \) of complex space \( \mathbb{C}^n \) is complete and convex. Find the vector of minimum norm in \( M \).

Solution

Completeness of \( M \): We know that \( \| x \|_1 = |\xi_1| + \cdots + |\xi_n| \) where \( x = (\xi_1, \cdots, \xi_n) \) is a norm on \( \mathbb{C}^n \). Since every norm on a finite dimensional vector space produces the same topology, it is enough to show that \( M \) is complete with respect to this norm. So let \((x_m)\) be Cauchy sequence in \( M \) where \( x_m = (\xi_{m1}, \xi_{m2}, \cdots, \xi_{mn}) \).

Since \( \mathbb{C}^n \) is complete, there exists a vector \( x = (\xi_1, \xi_2, \cdots, \xi_n) \) such that \( x_n \) converges to \( x \). We know that \( \sum \eta_j = 1 \) for all \( m \). So

\[
\left| 1 - \sum_j \xi_j \right| \leq \left| 1 - \sum_j \xi_{mj} \right| + \left| \sum_j \xi_{mj} - \sum_j \xi_j \right| \leq \sum_j |\xi_{mj} - \xi_j| = \|x_m - x\| \rightarrow 0.
\]

So \( x \in M \). This shows that \( M \) is closed so it is complete since \( \mathbb{C}^n \) is complete.

Convexity of \( M \): Let \( x = (\xi_j) \) and \( y = (\eta_j) \) be in \( M \) and \( \alpha \in [0, 1] \). Since

\[
\sum [\alpha \xi_j + (1-\alpha)\eta_j] = \alpha \sum \xi_j + (1-\alpha) \sum \eta_j = \alpha + (1-\alpha) = 1
\]

we obtain \( \alpha x + (1-\alpha)y \) is in \( M \). So \( M \) is a convex set.

Vector in \( M \) of minimum norm (wrt \( \| \cdot \|_2 \) ): Let \( x = (\xi_j) \in M \). Then by Hölder inequality

\[
\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum \xi_j \leq \frac{1}{\sqrt{n}} \sum |\xi_j| = \sum \frac{|\xi_j|}{\sqrt{n}}
\]

\[
\leq \left( \sum_{j=1}^n \frac{1}{\sqrt{n}} \right)^{1/2} \left( \sum |\xi_j|^2 \right)^{1/2}
\]

\[
= \left( \sum |\xi_j|^2 \right)^{1/2} = \|x\|_2
\]

This means that the vectors of \( M \) cannot have a norm less then \( 1/\sqrt{n} \). On the other hand \( y = (\frac{1}{n}, \cdots, \frac{1}{n}) \in M \) has norm \( \|y\|_2 = 1/\sqrt{n} \). So the vector of \( M \) with minimum norm with respect to \( \| \cdot \|_2 \) is \( y = (\frac{1}{n}, \cdots, \frac{1}{n}) \).