(Continuity of vector space operations) Show that in a normed space $X$, vector addition and multiplication by scalars are continuous operations with respect to norm; that is, the mapping defined by $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$ are continuous.

**Solution**

To show continuity of vector addition we have to show that if $x_n \mapsto x$ and $y_n \mapsto y$ then $x_n + y_n \mapsto x + y$. But this follows from $||x_n + y_n - x - y|| \leq ||x_n - x|| + ||y_n - y|| \to 0 + 0$ as $n \to \infty$.

Similarly to show continuity of scalar multiplication we must show $\alpha_n \mapsto \alpha$ and $x_n \mapsto x$ implies $\alpha_n x_n \mapsto \alpha x$. This follows from

$$||\alpha_n x_n - \alpha x|| = ||\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x||$$

$$\leq ||\alpha_n x_n - \alpha_n x|| + ||\alpha_n x - \alpha x||$$

$$= \limsup \sqrt[\beta_{n, d}] {||x_n - x|| + |\alpha_n - \alpha||x||} \to 0 + 0.$$

Remark: If $(X, d)$ and $(Y, \rho)$ are metric spaces then $X \times Y$, with product topology on it, is also a metric space with metric $D$ given by

$$D((x, y), (x', y')) = d(x, x') + \rho(y, y').$$

So this means that $(x_n, y_n) \mapsto (x, y)$ iff $x_n \mapsto x$ and $y_n \mapsto y$.

**Problem (7/71)**

(Absolute convergence) Show that convergence of $||y_1|| + ||y_2|| + ||y_3|| + \cdots$ may not imply convergence of $y_1 + y_2 + y_3 \cdots$.

**Solution** Let $F = \{(\xi_n)_{n=1}^\infty \in l^\infty |$ only finitely many terms of $(\xi_n)_{n=1}^\infty$ is non-zero}. Clearly $F$ is a linear subspace of of $l^\infty$, and so $F$ is a normed space with sup norm. Consider $y_n \in F$ defined by $y_n = (0, 0, \ldots, \frac{1}{n}, 0, \ldots)$ where $\frac{1}{n}$ is $n^{th}$ component of $y_n$. Since $||y_n|| = \frac{1}{n}$, $||y_1|| + ||y_2|| + ||y_3|| + \cdots$ converges. However $y_1 + y_2 + y_3 \cdots$ converges to $(\frac{1}{n^2})_{n=1}^\infty$ in $l^\infty$ which is not in $F$. By uniqueness of limit we obtain $y_1 + y_2 + y_3 \cdots$ doesn’t converge in $F$.

Remark: This is because $F$ is not complete. In fact in a Banach space you
can’t find such an example but in an incomplete normed space it is always possible to find a sequence as above, so this is a characterization of normed spaces. Try to prove this fact.

**Problem (10/71)**

**(Schauder basis)** Show that if a normed space has a Schauder basis, it is separable.

**Solution** Let normed space $X$ has a Schauder basis $\{e_n\}_{n=1}^{\infty}$. lets define $M \subset X$ as follows

$$M = \{ \sum_{n=1}^{N} \beta_n e_n : N \in \mathbb{N} \text{ and } \beta_1, \beta_2, ... \beta_N \in \mathbb{Q} + i\mathbb{Q} \}.$$ 

It is not hard to show $M$ is countable, in fact it is countable union of countable sets. $M$ is dense in $X$. To see this lets fix arbitrary $x \in X$. By definition $\exists! (\alpha_n)_{n=1}^{\infty}$ in $K$ such that $\sum_{n=1}^{\infty} \alpha_n e_n = x$. Lets fix arbitrary $\epsilon > 0$. By definition of convergence $\exists N \in \mathbb{N}$ such that

$$\| \sum_{n=1}^{N} \alpha_n e_n - x \| < \frac{\epsilon}{2}.$$ 

Lets for $n = 1, 2, ..., N$ choose $\beta_n \in \mathbb{Q} + i\mathbb{Q}$ such that

$$|\beta_n - \alpha_n| < \frac{\epsilon}{\|e_n\|N(N+1)}.$$ 

Clearly $\sum_{n=1}^{N} \beta_n e_n$ is in $M$ and we have

$$\| \sum_{n=1}^{N} \beta_n e_n - x \| \leq \| \sum_{n=1}^{N} \beta_n e_n - \sum_{n=1}^{N} \alpha_n e_n \| + \| \sum_{n=1}^{N} \alpha_n e_n - x \|$$

$$< \sum_{n=1}^{N} \| \beta_n - \alpha_n \| |e_n\| + \frac{\epsilon}{2}$$

$$< \sum_{n=1}^{N} \frac{\epsilon}{\|e_n\|N(N+1)} |e_n\| + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

So we can approximate $x$ by the elements of $M$. Hence $M$ is dense in $X$.

Remark: The converse of this question is not true, that is, there exists separable Banach space with no Schauder basis on it.
Problem (14/71)

(Quotient space) Let \( Y \) be a closed subspace of a normed space \((X, ||.||)\). Show that \( ||.||_o \) on \( X/Y \) is defined by
\[
||\hat{x}||_o = \inf_{x \in \hat{x}} ||x||
\]
where \( \hat{x} \in X/Y \), that is, \( \hat{x} \) is any coset of \( Y \).

Solution

(N1) Clearly \( ||\hat{x}||_o \) is a non-negative real number for any coset \( \hat{x} \).

(N2) \( \hat{x} = \hat{0} \) implies \( 0 \in \hat{x} \), so \( ||\hat{x}||_o = 0 \). Conversely let \( ||\hat{x}||_o = 0 \). Then there is a sequence \( \{x_n\} \) in \( \hat{x} \) such that \( ||x_n|| \to 0 \). But \( Y \) is closed so any coset of \( Y \) must be closed. Hence \( 0 \in \hat{x} \). This means that \( \hat{x} = \hat{0} \).

(N3) Notice that
\[
||\hat{x}||_o = ||\hat{0}||_o = 0 = 0 ||\hat{x}||_o.
\]
And for any non-zero \( \alpha \) in \( K \),
\[
||\alpha \hat{x}||_o = ||\alpha \hat{0}||_o = ||\hat{0}||_o = 0 = ||\alpha ||_o ||\hat{x}||_o.
\]

(N4)
\[
||\hat{x} + \hat{y}||_o = ||\hat{x} + \hat{y}||_o = \inf_{z \in \hat{x} + \hat{y}} ||z|| = \inf_{z \in \hat{x} + \hat{y}} ||z|| = \inf_{u \in \hat{x}, v \in \hat{y}} ||u + v|| \leq \inf_{u \in \hat{x}, v \in \hat{y}} ||u|| + ||v|| = \inf_{u \in \hat{x}} ||u|| + \inf_{v \in \hat{y}} ||v|| = ||\hat{x}||_o + ||\hat{y}||_o
\]

(Product of normed spaces) If \((X_1, ||.||_1)\) and \((X_2, ||.||_2)\) are normed spaces, show that the product vector space \( X = X_1 \times X_2 \) becomes a normed space if we define
\[
||x|| = \max(||x_1||_1, ||x_2||_2) \quad [x = (x_1, x_2)].
\]

Solution In fact all axioms are easy to verify. Lets, for example, show (N4).
\[
||(x_1, x_2) + (y_1, y_2)|| = ||(x_1 + y_1, x_2 + y_2)|| = \max(||x_1 + y_1||_1, ||x_2 + y_2||_2) \leq \max(||x_1||_1 + ||y_1||_1, ||x_2||_2 + ||y_2||_2) \leq \max(||x_1||_1, ||x_2||_2) + \max(||y_1||_1, ||y_2||_2) = ||(x_1, x_2)|| + ||(y_1, y_2)||.
\]