Problem (12/9)

(Triangle inequality) By using triangle inequality show that

\[ |d(x, y) - d(z, w)| \leq d(x, z) + d(y, w). \]

Solution

We have

\[
\begin{align*}
  d(x, y) &\leq d(x, z) + d(z, w) + d(w, y) \\
  d(z, w) &\leq d(z, x) + d(x, y) + d(y, w)
\end{align*}
\]

\[ \Rightarrow \]

\[
\begin{align*}
  d(x, y) - d(z, w) &\leq d(x, z) + d(y, w) \\
  d(z, w) - d(x, y) &\leq d(z, x) + d(y, w)
\end{align*}
\]

Since \( d(x, z) = d(z, x) \) and \( d(y, w) = d(w, y) \), we get

\[ |d(x, y) - d(z, w)| \leq d(x, z) + d(y, w). \]

Problem (4/16)

(Space \( l^p \)) Find a sequence which converges to 0, but is not in any space \( l^p \), where \( 1 \leq p < +\infty \).

Solution Consider

\[
x = (x_n) = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, \ldots).
\]

This sequence certainly converges to 0. But it is not in \( l^p \) for any \( 1 \leq p < +\infty \).

\[
\sum_{n=1}^{\infty} |x_n|^p = 1 + \frac{2^2}{2^p} + \frac{3^3}{3^p} + \ldots + \frac{n^n}{n^p} \cdots = \infty
\]

because given \( 1 \leq p < +\infty \), \( n^n/n^p > 1 \) for all \( n > 1 \).

Also \( x = (x_n) \) where \( x_1 = 0 \) and \( x_n = \frac{1}{\log n} \) for \( n \geq 2 \) works. (You can use integral test to show \( x \) is not in any \( l^p \), where \( 1 \leq p < +\infty \).)
Problem (12/25)

(Space $B[a,b]$) Show that $B[a,b]$, $a < b$, is not separable.

Solution For each $c \in [a, b]$ define $x_c$ as follows

$$x_c(t) = \begin{cases} 1 & t = c \\ 0 & \text{otherwise} \end{cases}.$$

Then certainly $x_c$ is in $B[a,b]$ for all $c \in [a, b]$. Let $A$ be the set containing all such elements. Observe that if $x_c$ and $x_d$ two distinct elements of $A$, then

$$d(x_c, x_d) = \sup_{t \in [a, b]} |x_c(t) - x_d(t)| = 1.$$

Now let $M$ is a dense subset of $B[a,b]$. It is easy to see that

$$B[a,b] = \bigcup_{m \in M} B(m, \frac{1}{3}).$$

But observe that every ball of radius $1/3$ can contain at most one element of $A$. Since $A$ has uncountably many elements we see that $M$ cannot be a countable set. So we conclude $B[a,b]$ is not separable.

Problem (6/40)

Show that the set of all real numbers constitutes an incomplete metric space if we choose

$$d(x,y) = |\arctan x - \arctan y|.$$ 

Solution

Consider the sequence $\{x_n\}_{n=1}^\infty$ in real line where $x_n = n$. Then $\{x_n\}_{n=1}^\infty$ is Cauchy but not convergent.

We know that given $\epsilon > 0$, $\exists N$ such that for $n > N$,

$$|\arctan n - \frac{\pi}{2}| < \frac{\epsilon}{2}.$$

So for $n, m > N$,

$$d(x_m, x_n) = |\arctan m - \arctan n| = |\arctan m - \frac{\pi}{2}| + |\frac{\pi}{2} - \arctan n| < \epsilon.$$ 

On the other hand $\{x_n\}_{n=1}^\infty$ is not a convergent sequence. If not $\exists x \in \mathbb{R}$ such that $d(x_n, x) \to 0$ as $n \to \infty$. But

$$d(x_n, x) = |\arctan n - \arctan x| \geq |\arctan n - \frac{\pi}{2}| - |\frac{\pi}{2} - \arctan x|.$$

So $\arctan x = \frac{\pi}{2}$. But this a contradiction because for any $x \in \mathbb{R}$, $\arctan x < \frac{\pi}{2}$. 

2
Problem (6/46)

Show that $C[0, 1]$ and $C[a, b], a < b$, isometric.

Solution

Clearly $\varphi : [a, b] \rightarrow [0, 1]; \varphi(t) = \frac{t-a}{b-a}$ is a homeomorphism.
Define $T : C[0, 1] \rightarrow C[a, b]$ by $Tx = x \circ \varphi$. Then $T$ preserves the distance since

$$d(Tx, Ty) = \sup_{t \in (a, b)} \{Tx(t) - Ty(t)\} = \sup_{t \in (a, b)} \{x(\frac{t-a}{b-a}) - y(\frac{t-a}{b-a})\}$$

$$= \sup_{t \in (0, 1)} \{x(t) - y(t)\} = d(x, y).$$

Surjectivity of $T$ follows from the fact that $h \in C[a, b]$ implies $h \circ \varphi^{-1} \in C[0, 1]$ such that $T(h \circ \varphi^{-1}) = h$.

$T$ is injective for $Tx = Ty$ implies $d(Tx, Ty) = d(x, y) = 0$, so $x = y$.

Hence $C[0, 1]$ and $C[a, b]$ are isometric.