Problem (8/255)

Show that the completeness of $X$ is essential in Theorem 4.3-7 and cannot be omitted.

Solution

Consider $X = F_{00} := \{(\xi_n) \in l_\infty : \text{only finitely many terms of } (\xi_n) \text{ is non-zero} \}$ with sup norm. And choose $Y$ as scalar field of $F_{00} (\mathbb{R} \text{ or } \mathbb{C})$. Define $T_n : X \to Y$ by

$$T_n(x) = n\xi_n \ \text{ where } \ x = (\xi_1, \xi_2, \xi_3, ...).$$

Clearly $T_n$ is linear and bounded for all $n$. Now fix $x \in X$. We will show that $\{|T_n x|\}_{n=1}^\infty$ is a bounded sequence. We know that $x$ is of the form

$$x = (\xi_1, \xi_2, \xi_3, ..., \xi_J, 0, 0, 0, ...).$$

This means that

$$T_1(x) = 1\xi_1 \ T_2(x) = 2\xi_2 \ \cdots \ T_J(x) = J\xi_J$$

but,

$$T_{J+1}(x) = 0 \ T_{J+2}(x) = 0 \ \cdots \ \text{ since } \xi_n = 0 \text{ for } n > J.$$

So we obtain

$$|T_n x| \leq J||x|| \ \text{ for all } n.$$

Clearly $J||x||$ depends on only $x$. So $\{|T_n x|\}_{n=1}^\infty$ is a bounded sequence. Since $x$ is arbitrary we obtain the condition. But it is easy to show $||T_n|| = n$. That is $\{|T_n||\}_{n=1}^\infty = \{n\}_{n=1}^\infty$ is not bounded.

Problem (10/255)

(Space $c_0$) Let $y = (\eta_j), \ \eta_j \in \mathbb{C},$ be such that $\sum \xi_j\eta_j$ converges for every $x = (\xi_j) \in c_0$. Show that $\sum |\eta_j|$ converges.

Solution

Lets define $f_n : c_0 \to \mathbb{C}$ by

$$f_n(x) = \xi_1\eta_1 + \xi_2\eta_2 + ... + \xi_n\eta_n \ \text{ where } \ x = (\xi_1, \xi_2, \xi_3, ...).$$

Clearly $f_n$ is linear and bounded for all $n.$
We will show \( \{f_n\} \) is pointwise bounded, that is if we fix \( x = (\xi_1, \xi_2, \xi_3, \ldots) \) in \( c_0 \) then \( \{|f_n(x)|\} \) is bounded. Indeed

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \xi_k \eta_k = \sum_{k=1}^{\infty} \xi_k \eta_k.
\]

Since \( \{f_n(x)\} \) convergent it must be bounded so \( \{|f_n(x)|\} \) is also bounded.

We know that \( c_0 \) is complete. So by UBP \( \{|f_n|\} \) must be bounded.

Let's show \( |f_n| = |\eta_1| + |\eta_2| + \cdots |\eta_n| \).

\[
|f_n(x)| \leq \sum_{k=1}^{n} |\xi_k \eta_k| \leq \max_{k=1}^{n} |\xi_k| \sum_{k=1}^{n} |\eta_k| \leq \|x\| \sum_{k=1}^{n} |\eta_k| \Rightarrow \|f_n\| \leq \sum_{k=1}^{n} |\eta_k|.
\]

On the other hand \( x = (|\eta_1|, |\eta_2|, \ldots, |\eta_n|, 0, 0, \ldots) \) is in \( c_0 \) such that \( \|x\| = 1 \). So

\[
|f_n(x)| \leq \|f_n\| \|x\| \implies \sum_{k=1}^{n} |\eta_k| \leq \|f_n\|.
\]

Since there exists constant \( c \) such that \( \|f_n\| \leq c \) for all \( n \), we obtain

\[
\sum_{k=1}^{n} |\eta_k| \leq c \text{ for all } n \implies \sum_{k=1}^{\infty} |\eta_k| \leq c \quad \text{so it converges}.
\]

Problem (1/262)

(Pointwise convergence) If \( x_n \in C[a, b] \) and \( x_n \xrightarrow{w} x \in C[a, b] \), show that \( (x_n) \) is pointwise convergent on \( C[a, b] \), that is, \( (x_n(t)) \) converges for every \( t \in [a, b] \).

Solution

Fix a point \( t \in [a, b] \). It is easy to show that \( f : C[a, b] \to \mathbb{C} \) defined by \( f(x) = x(t) \) is a bounded linear functional. So \( f(x_n) \to f(x) \), that is, \( x_n(t) \to x(t) \). Since we chose \( t \) arbitrary \( (x_n(t)) \) converges for every \( t \in [a, b] \).

Problem (2/262)

Let \( X \) and \( Y \) be normed spaces, \( T \in B(X, Y) \) and \( (x_n) \) a sequence in \( X \). If \( x_n \xrightarrow{w} x_0 \), show that \( T x_n \xrightarrow{w} T x_0 \).
Solution

Observe that if \( g \in Y' \) then \( g^T \) is in \( X' \) where \( gT(x) = g(Tx) \). Because if \( x, z \in X \) and \( \alpha, \beta \) scalars then

\[
gT(\alpha x + \beta y) = g(\alpha T(x) + \beta T(y)) = \alpha gT(x) + \beta gT(y)
\]

and \( |gT(x)| \leq ||g|| ||T|| ||x|| \).

So \( gT(x_n) \to gT(x) \) equivalently \( g(Tx_n) \to g(Tx) \). Since \( g \) was arbitrary element of \( Y' \), \( Tx_n \overset{w}{\longrightarrow} Tx_0 \).

Problem (4/262)

Show that \( x_n \overset{w}{\longrightarrow} x_0 \) implies \( \liminf ||x_n|| \geq ||x_0|| \). (Use theorem 4.3-3.)

Solution

The case \( x_0 = 0 \) is trivial. So assume \( x_0 \neq 0 \). Then theorem 4.3-3 says that there is an \( f \in X' \) such that

\[
||f|| = 1 \quad \text{and} \quad f(x_0) = ||x_0||.
\]

This means that \( |f(x_n)| \leq ||x_n|| \) for all \( n \). We also know \( f(x_n) \to f(x_0) \) which implies that \( |f(x_n)| \to |f(x_0)| = ||x_0|| \). Since

\[
\liminf ||x_n|| \geq \liminf |f(x_n)| = \lim |f(x_n)| = ||x_0||,
\]

we get the desired result.

Q Find an example in a Banach space such that the inequality is strict.