Problem (3/245)

If a normed space $X$ is reflexive, show that $X'$ is reflexive.

Solution

Since $X$ is reflexive, we know that all elements $g$ in $X''$ can be written $g = g_x$ where $g_x(f) = f(x)$. Observe that if $f \in X'$ then $G_f$ is a linear functional on $X''$ where $G_f(g_x) = g_x(f)$. Our aim is to show that the canonical mapping

$$\tilde{C} : X' \rightarrow X'''$$

defined by $f \mapsto G_f$

is surjective. So let $G \in X'''$, then define $h(x) = G(g_x)$. Then $h$ is a linear and bounded functional on $X$. Continuity is easy to show and let's see linearity.

$$h(\alpha x + \beta y) = G(\alpha g_x + \beta g_y) = \alpha G(g_x) + \beta G(g_y) = \alpha h(x) + \beta h(y).$$

We claim that $G = G_h$. $G(g_x) = h(x)$ and $G_h(g_x) = g_x(h) = h(x)$. Hence $\tilde{C}$ is surjective equivalently $X'$ is reflexive.

Problem (7/246)

Let $Y$ be a closed subspace of a normed space $X$ such that every $f \in X'$ which is zero everywhere on $Y$ is zero everywhere on the whole space $X$. Show that then $Y = X$.

Solution

Suppose that $Y \neq X$. So let $x_0 \in X - Y$. Since $Y$ is closed there exists $f \in X'$ such that $f(Y) = 0$ but $f(x_0) \neq 0$ (see lemma 4.6-7). Contradiction.

Problem (8/246)

Let $M$ be any subset of a normed space $X$. Show that an $x_0 \in X$ is an element of $A = \text{span} M$ if and only if $f(x_0) = 0$ for every $f \in X'$ such that $f|_M = 0$.

Solution

We will show that

$$x_0 \in \text{span} M \iff f(x_0) = 0 \ \forall f \in X' \text{ satisfying } f|_M = 0.$$
⇒ Let \( x_0 \in \text{span}M \) and take arbitrary \( f \in X' \) which satisfies \( f|_M = 0 \). But linearity and continuity of \( f \) implies that
\[
f|_M = 0 \implies f|_{\text{span}M} = 0 \implies f|_{\text{span}X'} = 0.
\]
So \( f(x_0) = 0 \).

⇐ If \( x_0 \notin \text{span}M \), then we know that there is an \( f \in X' \) such that \( f|_{\text{span}M} = 0 \) but \( f(x_0) \neq 0 \) (see lemma 4.6-7). So this means that there exists \( f \in X' \) such that \( f|_M = 0 \) but \( f(x_0) \neq 0 \).

**Problem (9/246)**

(Total set) Show that a subset \( M \) of a normed space \( X \) is total in \( X \) if and only if every \( f \in X' \) which is zero everywhere on \( M \) is zero everywhere on \( X \).

**Solution**

This is a direct result of previous question. Or it can be solved in the same way.

**Problem (10/246)**

Show that if a normed space \( X \) has a linearly independent subset of \( n \) elements, so does the dual space \( X' \).

**Solution**

Let the set is \( \{x_1, x_2, \ldots, x_n\} \). Then consider the subspaces
\[
Y_1 = \text{span}\{x_2, x_3, \ldots, x_n\} \\
Y_2 = \text{span}\{x_1, x_3, \ldots, x_n\} \\
\vdots \\
Y_n = \text{span}\{x_1, x_2, \ldots, x_{n-1}\}.
\]
Then since \( Y_j \) is finite dimensional it is closed and also \( x_j \notin Y_j \). So we know that there exists \( f_j \in X' \) such that \( f_j \) is zero on \( Y_j \) with \( f_j(x_j) \neq 0 \) for \( j = 1, 2, \ldots, n \). We claim that \( \{f_1, f_2, \ldots, f_n\} \) is linearly independent in \( X' \). Let
\[
\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0
\]
then
\[
(\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n)(x_j) = 0 = \alpha_j f_j(x_j) \implies \alpha_j = 0 \text{ for } j = 1, 2, \ldots, n.
\]
So \( \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0 \). That is \( \{f_1, f_2, \cdots, f_n\} \) is a linearly independent set.