**Subgraphs, Complements**

**Definition.** Let $G = (V, E)$ be a graph (directed or undirected). Then $G_1 = (V_1, E_1)$ is called a **subgraph** of $G$ if $V_1$ is a nonempty subset of $V$, $E_1$ is a nonempty subset of $E$, and each edge in $E_1$ is incident with vertices in $V_1$.

**Examples**

\[ G = (V, E) \]
\[ V = \{a, b, c, d, e\} \]

\[ G_1 = (V_1, E_1) \text{ is a subgraph of } G \]
\[ V_1 = \{a, b, c, e\} \]

\[ G_2 = (V_2, E_2) \]
\[ V_2 = \{a, b, c, d\} \]

**Definition.** Let $V$ be a set of $n$ vertices. The **complete graph** on $V$, denoted $K_n$, is a loop-free undirected graph where for all $a, b \in V$, $a \neq b$, there is an edge $\{a, b\}$.

**Examples**

\[ K_2 \]

\[ K_3 \]

\[ K_4 \]

**Remark:** Let $K_n = (V, E_n)$. Then $|E_n| = \frac{n(n-1)}{2}$.

**Definition.** Let $G$ be a loop-free undirected graph on $n$ vertices. The **complement of $G$**, denoted $\overline{G}$, is the subgraph of $K_n$ consisting of the $n$ vertices.
in \( G \) and all edges that are not in \( G \).

**Remark.** Let \( G = (V, E) \), \( |V| = n \). Then, if \( K_n = (V, E_n) \), \( \overline{G} = (V, E_n \setminus E) \).

**Example 1.** Let \( G \) be an undirected graph with \( n \) vertices. If the number of edges in \( G \) is equal to the number of edges in \( \overline{G} \), how many edges \( G \) has?

**Solution.**
Let \( G = (V, E) \), \( |V| = n \), \( |E| = e \). Then \( \overline{G} = (V, E_n \setminus E) \). Therefore, \( |E_n \setminus E| = \frac{n(n-1)}{2} - e \). We have,
\[
e = \frac{n(n-1)}{2} - e, \quad \text{i.e.} \quad 2e = \frac{n(n-1)}{2}, \quad \text{i.e.} \quad e = \frac{n(n-1)}{4}.
\]

**Example 2.** Give an example of a graph on \( 4 \) vertices s.t. the number of edges in \( G \) is the same as the number of edges in \( \overline{G} \).

**Solution.**

\[
\begin{align*}
G &= (V, E) \\
\overline{G} &= (V, E_n \setminus E)
\end{align*}
\]

**Example 3.** How many subgraphs \( H = (V, E) \) of \( K_6 \) satisfy \( |V| = 3 \)?

**Solution.** We can choose 3 vertices out of 6 in \( C(6,3) = \frac{6!}{3!3!} = 20 \) different ways. Let us choose \( x, y, z \) vertices.

- 0 edges: 1 possibility
- 1 edge: 3 possibilities
- 2 edges: 3 possibilities
- 3 edges: 1 possibility
Totally there are $20 \times (1+3+3+1) = 160$ different subgraphs of $K_6$ with 3 vertices.

Multigraphs.

**Definition.** Let $V$ be a finite nonempty set. We say that the pair $(V, E)$ determines a multigraph $G$ with vertex set $V$ and edge set $E$ if, for some $x, y \in V$, there are two or more edges in $E$ of the form
- (a) $(x, y)$ (for a directed multigraph), or
- (b) $\{x, y\}$ (for an undirected multigraph).

**Example.**

![Diagram of multigraph](image)

**Definition.** Let $G$ be an undirected graph or multigraph. For each vertex $v$ of $G$, the degree of $v$, $\deg(v)$, is the number of edges in $G$ that are incident with $v$. A loop at a vertex $v$ is considered as two incident edges for $v$.

**Example.**

![Diagram of graph with vertex degrees](image)

- $\deg(e) = 5$
- $\deg(d) = 3$
- $\deg(a) = 2$
- $\deg(f) = 0$

**Remark.** If $G = (V, E)$ is an undirected graph or multigraph, then $\sum_{v \in V} \deg(v) = 2|E|$. 

(3)
Remark 2. For any undirected graph or multigraph, the number of vertices of odd degree must be even.

Example. Determine $|V|$ for the following graphs or multigraphs:
(a) $G$ has 9 edges and all vertices have degree 3.
(b) $G$ has 10 edges with two vertices of degree 4 and all others of degree 3.

Solution: By Remark 1,
(a) $3 |V| = 2 \cdot |E| = 2 \cdot 9 = 18 \Rightarrow |V| = 6.$
(b) $4 \cdot 2 + 3 (|V| - 2) = 2 \cdot |E| = 2 \cdot 10 = 20 \Rightarrow |V| = 6.$

Example. Is it possible to have a graph with 15 edges and such that all vertices have degree 4.

Solution. If such graph $G = (V,E)$ exists then $4 |V| = 2 \cdot |E| = 2 \cdot 15 = 30$. Since 30 is not divisible by 4 then such graph does not exist.

Weighted graphs.

Definition. Let $G = (V,E)$ be a loop-free connected directed graph. To each edge $e = (a,b)$ we assign a positive real number called the weight of $e$, denoted by $\text{wt}(e)$, or $\text{wt}(a,b)$. If $x,y \in V$ but $(x,y) \notin E$ we define $\text{wt}(x,y) = \infty$.

Graph $G = (V,E)$, where for each edge the number is assigned (as the weight of this edge) is called a weighted graph.

\[
\begin{align*}
\text{wt}(a,b) &= 11, & \text{wt}(b,a) &= \infty, \\
\text{wt}(c,b) &= 3, & \text{wt}(a,c) &= \infty. \\
\text{wt}(d,a) &= 5, & \text{wt}(a,d) &= 7.
\end{align*}
\]
Let $G = (V, E)$ be a directed weighted graph. For each $e = (x, y) \in E$, $w(e)$ may represent the length of a road from $x$ to $y$, the time it takes to travel on this road from $x$ to $y$, the cost of traveling from $x$ to $y$ on this road.

**Definition.** For $a, b \in V$, suppose that $v_1, v_2, \ldots, v_n \in V$ and that the edges $(a, v_1), (v_1, v_2), \ldots, (v_n, b)$ provide a directed path in $G = (V, E)$ from $a$ to $b$. The **length of this path** is defined as

$$w_t(a, v_1) + w_t(v_1, v_2) + \ldots + w_t(v_n, b).$$

The length of a shortest directed path in $G$ from $a$ to $b$ is called the *(shortest) distance from $a$ to $b* and denoted by $d(a, b)$.

**Agreement:**
1. $\forall a \in V \quad d(a, a) = 0$
2. if there is no path in $G$ from $a$ to $b$ then we define $d(a, b) = \infty$.

**Properties of $d(a, b)$:** Let $v_0 \in V, S \subseteq V$

Define the distance from $v_0$ to $S$ by

$$d(v_0, S) = \min_{v \in S} \{ d(v_0, v) \}$$

If $d(v_0, S) < \infty$ then $\exists V_{m+1} \subseteq S$ s.t. $d(v_0, S) = d(v_0, V_{m+1})$.

Here $P$: $(V_0, V_1), (V_1, V_2), \ldots, (V_m, V_{m+1})$ is a shortest directed path in $G$ from $v_0$ to $\overline{S}$, let us show that

1. $V_0, V_1, V_2, \ldots, V_m \subseteq \overline{S}$ and

2. $P': (V_0, V_1), (V_1, V_2), \ldots, (V_{k-1}, V_k)$ is a shortest directed path from $v_0$ to $V_k$, for each $1 \leq k \leq m$.

**Proof of 1:** Assume that $\exists V_i, 1 \leq i \leq m$ s.t. $V_i \subseteq \overline{S}$. Then $P'': (V_0, V_1), (V_1, V_2), \ldots, (V_{i-1}, V_i)$ is a path from $v_0$ to an element in $\overline{S}$.
Therefore, \( d(\bar{v}_0, \overline{S}) \leq wt(\bar{v}_0, u_i) + \ldots + wt(u_{i-1}, v_i) < \)
\(< wt(\bar{v}_0, u_i) + \ldots + wt(u_{i-1}, v_i) + wt(u_i, v_{i+1}) + \ldots + wt(u_m, v_{m+1}) \)

It contradicts the definition of \( d(\bar{v}_0, \overline{S}) \). So, our assumption was wrong. Hence, among \( \bar{v}_0, \bar{v}_1, \ldots, v_m \) we do not have an element from \( \overline{S} \).

Proof of 2): Assume that \( d(\bar{v}_0, v_k) < wt(\bar{v}_0, v_i) + \ldots + wt(v_k, v_i) \).

Then \( d(\bar{v}_0, \overline{S}) < wt(\bar{v}_0, v_k) + wt(v_k, v_{k+1}) + \ldots + wt(v_m, v_{m+1}) \).

It contradicts the definition of \( \overline{S} \). Hence, our assumption was wrong. Therefore, \( P' \) is a shortest directed path from \( \bar{v}_0 \) to \( v_k \).

We have, from (1) and (2),
\[ d(\bar{v}_0, \overline{S}) = \min \{ d(\bar{v}_0, u) + wt(u, w) \} \]
where minimum is evaluated over all \( u \in S \), \( v \in \overline{S} \).

If a minimum occurs for \( u = x \) and \( w = y \) then
\( d(\bar{v}_0, y) = d(\bar{v}_0, x) + wt(x, y) \)
is the shortest distance from \( \bar{v}_0 \) to \( y \).

**Problem.** Let \( G = (V, E) \) be a weighted graph with \( |V| = n \).

Let \( \bar{v}_0 \) be a fixed vertex. Find the shortest distance from \( \bar{v}_0 \) to all other vertices in \( G \).

To solve this problem, follow the following procedure (discovered by Dijkstra).

**Step 1.** Assign to \( \bar{v}_0 \) the label \((-\infty, 0)\).

**Step 2.** 
(a) For each labeled vertex \( u(x, d) \) and for each unlabeled vertex \( v \) adjacent to \( u \) (there is an edge \( (u, v) \)) compute \( d + wt(u, v) \).

(b) For each labeled vertex \( u(x, d) \) and unlabeled adjacent vertex \( v \) giving minimum \( d' = d + wt(u, v) \), assign to \( v \) the label \( (x, d') \). If a vertex can be labeled \( (x, d') \) for various vertices \( x \), make any choice.
Let us find distances from A to all other vertices for the following weighted graph.

First, give A the label (−, 0). There are three edges incident with A with weights 7, 5, 8. Since \( d = 0 \), vertex H gives the smallest value \( d + \text{wt}(AH) \), so H acquires the label (A, 5).

Now we repeat Step 2 for the two vertices labeled so far. There are two unlabeled vertices adjacent to the vertex A. The numbers \( d + \text{wt}(e_3) \) are 0 + 7 = 7 and 0 + 8 = 8. There are also two unlabeled vertices adjacent to the other labeled vertex H, and for these \( d + \text{wt}(e_3) \) are 5 + 4 = 9 and 5 + 5 = 10. The smallest \( d + \text{wt}(e_3) \) is 7 corresponding to the labeled vertex A and the unlabeled \( v = B \). Thus, B is labeled (A, 7). Again we repeat step 2.

Now there are three labeled vertices:

- A → one adjacent vertex G: \( d + \text{wt}(e_3) = 0 + 8 = 8 \)
- B → adjacent unlabeled vertex C: \( d + \text{wt}(e_3) = 7 + 8 = 15 \)
- H → one adjacent unlabeled vertex I: \( d + \text{wt}(e_3) = 7 + 3 = 10 \)

The smallest \( d + \text{wt}(e_3) \) is 8, corresponding to edge G. So G acquires the label (A, 8).

We repeat step 2. There are 4 labeled vertices A, B, H, G. All vertices adjacent to A, H are already labeled. We looked only for B and G:

- B → two adj. unl. vertices: C: \( d + \text{wt}(e_3) = 15 \)
- H → two adj unl. vertices: I: \( d + \text{wt}(e_3) = 10 \)

The minimum \( d + \text{wt}(e_3) \) occurs only with I and either edge \( I, B, I^2 \) or \( I, G, I^3 \). We can therefore assign to I either the label (B, 10) or (G, 10). We opt for (B, 10).

\[ \text{Week 9} \]
Repeating Step 2, we have to look only for vertices B, G, I:

B → adj. unl.  C → \( d + w(e_3) = 7 + 8 = 15 \)

G → adj. unl.  F → \( d + w(e_3) = 7 + 8 = 15 \)

I → adj. unl.  C → \( d + w(e_3) = 10 + 5 = 15 \)

\( Y \) → \( d + w(e_3) = 10 + 7 = 17 \)

We have to label two vertices C and F. We assign labels (B, 15) for C and (G, 15) for F.

We have to look now at vertices C, I, F:

C → adj. unl.  D → \( d + w(e_3) = 15 + 8 = 23 \)

I → adj. unl.  J → \( d + w(e_3) = 10 + 7 = 17 \)

F → adj. unl.  J → \( d + w(e_3) = 15 + 10 = 25 \)

\( E \) → \( d + w(e_3) = 15 + 6 = 21 \)

We label \( Y \) with (I, 17).

Next, we look at C, F, J:

C → adj. unl.  D → \( d + w(e_3) = 23 \)

F → adj. unl.  E → \( d + w(e_3) = 24 \)

\( Y \) → adj. unl.  D → \( d + w(e_3) = 17 + 3 = 20 \)

We label D with (Y, 20).

Finally, consider D, F:

D → adj. unl.  E → \( d + w(e_3) = 20 + 4 = 24 \)

F → adj. unl.  E → \( d + w(e_3) = 15 + 6 = 21 \)

We label E with (F, 21).

Since E was labeled last, the algorithm has actually found the length of a shortest route from A to any vertex.

For example,

A → B → I → Y is a shortest path to Y of length 17.

A → G → F → E is a shortest path to E, of length 21.
Problem 1: Apply The Traveling Salesman's Procedure to find the length of the shortest path from A to every other vertex. Show the final labels on all vertices. Also find the shortest path from A to H.

1. \( A \rightarrow \text{adj. unl.} \quad B \rightarrow 5 \quad \text{d+w+l+e+3} \)
   \( \text{d+w+l+e+3} \)
   \( C \rightarrow 2 \)
   \( E \rightarrow 10 \)
   \( \text{label C with (A,2)} \)

2. \( A \rightarrow \text{adj. unl.} \quad B \rightarrow 5 \quad \text{d+w+l+e+3} \)
   \( \text{d+w+l+e+3} \)
   \( D \rightarrow 4 \)
   \( E \rightarrow 10 \)
   \( \text{C} \rightarrow \text{adj. unl.} \quad E \rightarrow 2+7=9 \)
   \( \text{label D with (A,4)} \)

3. \( A \rightarrow \text{adj. unl.} \quad B \rightarrow 5 \quad \text{d+w+l+e+3} \)
   \( \text{d+w+l+e+3} \)
   \( D \rightarrow \text{adj. unl.} \quad G \rightarrow 6 \quad \text{E \rightarrow 10} \)
   \( B \rightarrow 12 \)
   \( C \rightarrow \text{adj. unl.} \quad E \rightarrow 9 \)
   \( F \rightarrow 7 \)
   \( \text{label B with (A,5)} \)

4. We have to look at \( A, B, C, D \).
   \( B \) does not have unl. adj. vertices.
   We are left with \( A, C, D \); i.e., see 3 without edge \((A,B)\).
   Then we have to label \( G \) with \((D,6)\).

5. \( A \rightarrow \text{adj. unl.} \quad E \rightarrow 10 \)
   \( \text{D} \rightarrow \text{adj. unl.} \quad E \rightarrow 10 \)
   \( C \rightarrow \text{adj. unl.} \quad E \rightarrow 9 \)
   \( \text{F} \rightarrow 7 \)
   \( \text{G} \rightarrow \text{adj. unl.} \quad E \rightarrow 8 \)
   \( \text{label F with (C,7)} \)

6. \( A \rightarrow E \rightarrow 10 \)
   \( D \rightarrow E \rightarrow 10 \)
   \( C \rightarrow E \rightarrow 9 \)
   \( G \rightarrow E \rightarrow 9 \)
   \( H \rightarrow 9 \)
   \( F \rightarrow E \rightarrow 10 \)
   \( H \rightarrow 9 \)
   \( I \rightarrow 11 \)

   \( \text{label E with (G,8)} \)

7. \( G \rightarrow H \rightarrow 9 \)
   \( F \rightarrow H \rightarrow 9 \)
   \( I \rightarrow 11 \)
   \( E \rightarrow H \rightarrow 11 \)

   \( \text{label H with (G,9)} \)
   \( \text{or} \quad (F,9) \)

8. \( F \rightarrow I \rightarrow 11 \)
   \( H \rightarrow I \rightarrow 14 \)
   \( \text{label I with (F,11)} \)

9. Answer: The shortest path from A to H is \( A \rightarrow D \rightarrow G \rightarrow H \), of length 9.