Problem 1. (15 points) Among the many rooms in an old house, there is a ghost in each room that has an even number of doors. If the house has only one entrance, prove that a person entering from outside can always reach a room in which there is no ghost.
Problem 2. (a) (15 points) For

\[ A = \{ (-4, -20), (-3, -9), (-2, -4), (-1, -11), (-1, -3), (1, 2), (1, 5), (2, 10), (2, 14), (3, 6), (4, 8), (4, 12) \} \]

define the relation \( R \) on \( A \) as follows:

\[ R = \{ ((a, b), (c, d)) \mid ad = bc \} \]

Is \( R \) an equivalence relation on \( A \)? Explain. If yes, find the equivalence classes \([ (2, 14)] \)
and \([ (-3, -9)] \).

(b) (10 points) If \(| B | = 30\) and the equivalence relation \( R \) on \( B \) partitions \( B \) into three
disjoint equivalence classes \( B_1, B_2, \) and \( B_3 \), where \(| B_1 | = | B_2 | = | B_3 |\), what is \(| R | \)?
**Problem 3.** (20 points) Let $S$ be a set of five positive integers the maximum of which is at most 9. Prove that the sum of the elements in all the nonempty subsets of $S$ cannot be distinct.
Problem 4. (a) (10 points) Let $f$ and $g$ be homomorphisms from a group $(G, \circ)$ to a group $(H, \ast)$. Is

$$K = \{ x \in G \mid f(x) = g(x) \}$$

a subgroup of $G$? Explain.

(b) (10 points) Let $(G, \circ)$ be a group that consists of 8 elements, i.e. $|G| = 8$. Show that there is an element $a \in G$ such that $a \circ a = e$, where $e$ is the identity element in $G$ and $a \neq e$. 


Problem 5. (a) (10 points) Let $H$ and $K$ be subgroups of a group $G$. Show that if $|H| = 10$ and $|K| = 21$, then $H \cap K = \{e\}$, where $e$ is the identity element of $G$.

(b) (10 points) Let $(G, \circ)$ be a cyclic group that consists of 7 elements, i.e. $|G| = 7$. How many distinct generators does $G$ have? Explain.