BİLKENT UNIVERSITY
Department of Mathematics

Date: 9 December 2010 | NAME: ....

Time: 18:00-20:00 | STUDENT NO: ....

Fall 2010-11, U. Muğan & Y. Kurtulmaz | **SECTION:** 01 02 03 04

Math 225.01-04, Linear Algebra & Differential Eq. Midterm Exam # 2

1	2	3	4	5	TOTAL
20	20	20	20	20	100
20	20	20	20	20	100

(Do not write anything on the above table)

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1) If $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for the finite dimensional vector space V, then show that $T = \{\vec{u} - 2\vec{v} + 3\vec{w}, 2\vec{u} + \vec{v} - \vec{w}, \vec{u} - \vec{v} + \vec{w}\}$ is also a basis for V.

(20 points)

$\dim V = 3$ because S contains 3 vectors. Thus it's enough to prove that T is linearly independent since $\dim V = 3$ because we know that: If $\dim V = n$ then n-linearly independent vectors span V.

$c_1(u-2v+3w) + c_2(2u+v-w) + c_3(u-v+w) = 0$; $c_1, c_2, c_3 \in \mathbb{R}$

$(c_1+2c_2+c_3)u + (-2c_1+c_2-c_3)v + (3c_1-c_2+c_3)w = 0$

and using $\{u,v,w\}$ is lin independent we obtain the following homog. eqn.

$$(*)\begin{cases} c_1+2c_2+c_3=0 \\ -2c_1+c_2-c_3=0 \\ 3c_1-c_2+c_3=0 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \xrightarrow[-3R_1+R_3]{2R_1+R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & -7 & -2 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/5 \\ 0 & -7 & -2 \end{bmatrix} \xrightarrow[7R_2+R_3]{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & -3/5 \end{bmatrix}$$

$$\xrightarrow[\substack{3R_3+R_2 \\ -\frac{5}{3}R_3}]{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$ gives the unique solution $c_1 = c_2 = c_3 = 0$.

Thus the set S lin. indep & spans V. So S is a basis for V.

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2) For the given matrix A,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \end{bmatrix}_{3\times 5},$$

find,
a) A basis for Row(A),
b) A basis for Col(A),
c) A basis for Null(A).
d) rank(A).

(6 + 6 + 6 + 2 = 20 points)

Find the reduced echelon (or echelon) form of A.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \end{bmatrix} \xrightarrow[-4R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 1 & 2 & 0 & -2 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

a) $\mathcal{B}_R = \{[0\ 1\ 2\ 0\ -2], [0\ 0\ 0\ 1\ 2]\}$ is a basis for the Row(A).

b) $\mathcal{B}_C = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$ is a basis for the Col(A).

c) Using R:

$x_1 = r$; $r, s, t \in \mathbb{R}$.

$x_2 = -2s + 2t$

$x_3 = s$

$x_4 = -2t$

$x_5 = t$

$S = \{(r, -2s+2t, s, -2t, t) \mid r, s, t \in \mathbb{R}\}$

$= \{r\underbrace{(1,0,0,0,0)}_{\vec{u}_1} + s\underbrace{(0,-2,1,0,0)}_{\vec{u}_2} + t\underbrace{(0,2,0,-2,1)}_{\vec{u}_3} \mid r,s,t \in \mathbb{R}\}$

$= \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ & $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is lin. independent

Thus $\mathcal{B}_N = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

d) rank(A) = 2.

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3) Let $\mathbf{E}_1$ be the elementary matrix obtained by the row operation $Swap\,(R_i,\ R_j)$, $\mathbf{E}_2$ be the elementary matrix obtained by the row operation cR_i, c =scalar, $\mathbf{E}_3$ be the elementary matrix obtained by the row operation $cR_i + R_j$, c =scalar, and let $\mathbf{A}$ be $n \times n$ matrix and $\det \mathbf{A} = -5$. Evaluate the followings for $c = 4$:

a) $\det(\mathbf{E}_1\mathbf{A})$,
b) $\det(\mathbf{E}_2\mathbf{E}_1\mathbf{A})$,
c) $\det(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\mathbf{A})$,
d) $\det(\mathbf{E}_1\mathbf{E}_2^{-1}\mathbf{A})$.

(5 + 5 + 5 + 5 = 20 points)

Remember that elementary matrix is obtained from the nxn identity matrix by using a single elementary row operation.

Note that:

$\det I = 1$, $\det E_1 = -1$, $\det E_2 = c = 4$, $\det E_3 = 1 = \det I$ and

$\det(AB) = \det A \cdot \det B$. Then

a) $\det(E_1 A) = \det E_1 \det A = (-1)(-5) = 5$.

b) $\det(E_2 E_1 A) = \det E_2 \cdot \det E_1 \cdot \det A = 4 \cdot (-1) \cdot (-5) = 20$

c) $\det(E_1 E_2 E_3 A) = \det E_1 \cdot \det E_2 \cdot \det E_3 \cdot \det A = (-1)(4)\cdot(1)\cdot(-5) = 20$

d) $\det(E_1 E_2^{-1} A) = \det E_1 \cdot (\det E_2)^{-1} \cdot \det A = \left(-\frac{1}{4}\right)(-5) = \frac{5}{4}$.

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4) Let $W = Span\{p_1(x),\ p_2(x),\ p_3(x),\ p_4(x)\}$ be a subspace of the vector space $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ spanned by the polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ where $p_1(x) = 1 - x + 2x^2,\ p_2(x) = 3 + x,\ p_3(x) = 5 - x + 4x^2,\ p_4(x) = -2 - 2x + 2x^2$.

a) Find a basis for W.

b) What is the dimension of W?

c) Is $W = \mathcal{P}_2$? Why? Explain your answer.

(16 + 2 + 2 = 20 points)

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a) W is already spanned by P_1, P_2, P_3 & P_4. Thus we should choose lin. independent members among P_1, P_2, P_3 & P_4.

$$c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) + c_4 p_4(x) = 0$$

$$c_1(1 - x + 2x^2) + c_2(3 + x) + c_3(5 - x + 4x^2) + c_4(-2 - 2x + 2x^2) = 0$$

$$\begin{cases} c_1 + 3c_2 + 5c_3 - 2c_4 = 0 \\ -c_1 + c_2 - c_3 - 2c_4 = 0 \\ 2c_1 + 4c_3 + 2c_4 = 0 \end{cases}$$

$$A = \begin{bmatrix} 1 & 3 & 5 & -2 \\ -1 & 1 & -1 & -2 \\ 2 & 0 & 4 & 2 \end{bmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 4 & 4 & -4 \\ 0 & -6 & -6 & 6 \end{bmatrix} \xrightarrow[\substack{-\frac{1}{6}R_3 \\ R_2+R_3}]{\frac{1}{4}R_2} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

lin. independent

leading 1's correspond to lin. independent columns in A.

Thus $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ are lin. independent column vectors in A.

<u>Note:</u> $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \longleftrightarrow a_0 + a_1 x + a_2 x^2 = p(x) \in P_2$

Then $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \longleftrightarrow 1 - x + 2x^2 = p_1(x)$ & $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \longleftrightarrow 3 + x = p_2(x)$.

Hence $\mathcal{B}_W = \{p_1(x), p_2(x)\}$ & $p_3(x) = 5p_1(x) + p_2(x)$, $p_4(x) = -2p_1(x) - p_2(x)$ are in $span\{p_1(x), p_2(x)\}$.

b) $\dim W = 2$

c) No!. Because $\dim W = 2 \neq \dim P_2 = 3$.

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5) Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ with usual definitions of addition and scalar multiplication:

$$(f+g)(x) = f(x) + g(x), \qquad (cf)(x) = cf(x), \qquad c = \text{scalar} \in \mathbb{R}$$

Prove that,
a) If W_1 is the set of all even functions in V, i.e. $W_1 = \{f(x) \in V | f(-x) = f(x)\}$,
b) If W_2 be the set of all odd functions in V, i.e. $W_2 = \{f(x) \in V | f(-x) = -f(x)\}$
both W_1 and W_2 are subspaces of V.

(10+10 points)

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a) $W_1 \neq \emptyset$ because $f(x) = \cos x \in W_1$ (There are many functions in W_1)

1) Let f & g be in W_1. Then $f(-x) = f(x)$ & $g(-x) = g(x)$.

$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x) \Rightarrow f+g \in W_1$.

2) Let $f \in W_1$ & $c \in \mathbb{R}$. Then $(cf)(-x) = c(f(-x)) = cf(x)$

gives $cf \in W_1$.

Hence W_1 is a subspace of V.

b) $W_2 \neq \emptyset$ because $g(x) = \sin x \in W_2$. (There are many functions in W_2)

1) Let f & g be in W_2. Then $f(-x) = -f(x)$ & $g(-x) = -g(x)$.

$(f+g)(-x) = f(-x) + g(-x) = -(f+g)(x) \Rightarrow f+g \in W_2$.

2) Let $f \in W_2$ & $c \in \mathbb{R}$. Then $(cf)(-x) = c(f(-x)) = -(cf(x))$

implies that $cf \in W_2$.

Thus W_2 is a subspace of V.

<u>Note</u>! It's a proof question and you should prove it.
Giving one or two examples is <u>not</u> a proof & you will <u>not</u> get any points.