1) Prove that a square matrix $A$ is singular if and only if $0$ is an eigenvalue of $A$.

2) What must be $c$ if the matrix

$$
\begin{bmatrix}
3 & -1 & 1 \\
0 & 2 & c \\
0 & 1 & -1
\end{bmatrix}
$$

has three distinct real eigenvalues?

3) Given the matrix $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 1 & -2 \\ 3 & 2 & 0 \end{bmatrix}$.

   a) Find its characteristic polynomial.

   b) Use the characteristic polynomial to evaluate the determinant of $A$.

   c) Use the Cayley-Hamilton Theorem to find the inverse of $A$.

   d) Compute eigenvalues and corresponding eigenvectors of $A$.

   e) If possible find an invertible matrix $P$ such that $P^{-1}AP$ is diagonal and write this diagonal matrix.

4) Find the general solutions of the following systems of linear differential equations:

   a) \[ \frac{dx_1}{dt} = 2x_1 + 4x_2 \]

   \[ \frac{dx_2}{dt} = x_1 - x_2 \]

   b) \[ \frac{dx_1}{dt} = -x_3 \]

   \[ \frac{dx_2}{dt} = x_1 \]

   \[ \frac{dx_3}{dt} = -14x_1 - 8x_2 + 7x_3 \]
\begin{align*}
\frac{dx_1}{dt} &= 4x_1 - 12x_2 - x_3 \\
\frac{dx_2}{dt} &= x_1 - 3x_2 - x_3 \\
\frac{dx_3}{dt} &= x_1 - 4x_2
\end{align*}

c) The \( nxn \) matrix \( A \) is said to be idempotent if \( A^2 = A \). If \( \lambda \) is an eigenvalue of such a matrix, show that \( \lambda \) is either 0 or 1. What can be said about a non-singular idempotent matrix?

6) If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of the \( nxn \) matrix \( A \) and \( c \) is a scalar, show that the matrix \( A + cI \) has eigenvalues \( \lambda_1 + c, \lambda_2 + c, \ldots, \lambda_n + c \).

7) A certain \( 4x4 \) real matrix is known to have these properties:
   1. Two of the eigenvalues of \( A \) are \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \).
   2. The number 3 is an eigen value of the matrix \( A + 2I \).
   3. \( \det A = 12 \).

   Use this information to answer the following questions about \( A \).
   (i) What are the other two eigenvalues of \( A \)?
   (ii) What is the characteristic polynomial of \( A \)?
   (iii) What is the characteristic polynomial of \( A^{-1} \)?

8) The characteristic polynomial of a certain \( 3x3 \) matrix is \( p(\lambda) = \lambda^3 - 7\lambda^2 + 5\lambda - 9 \). Use this fact to express \( \text{adj} A \) as a linear combination of \( A^2, A \) and \( I \).