

1) a) (8 pts.) Let A be an $n \times n$ matrix. Show that A can be written as the sum of a symmetric and a skew symmetric matrix. Show all your work.

$$A = \left(\frac{A+A^T}{2} \right) + \left(\frac{A-A^T}{2} \right)$$

$$(A+A^T)^T = A^T+A \text{ thus } \frac{A+A^T}{2} \text{ is symmetric}$$

$$(A-A^T)^T = A^T-A = -(A-A^T) \text{ thus } \frac{A-A^T}{2} \text{ is skew symmetric}$$

$$\text{Thus } A = S + K \text{ where } S = \frac{A+A^T}{2} \text{ is symmetric}$$

$$K = \frac{A-A^T}{2} \text{ is skew symmetric.}$$

b) (12 pts.) Suppose $\{u_1, u_2, u_3, u_4, u_5\}$ be a basis for \mathbb{R}^5 . If c_1, c_2, c_3, c_4, c_5 are scalars with $c_3 \neq 0$, show that $\{u_1, u_2, c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5, u_4, u_5\}$ is also a basis for \mathbb{R}^5 .

It's enough to show that they're lin. independent because 5 vectors $\Rightarrow 5 = \dim \mathbb{R}^5$.

$$\text{Suppose that } d_1u_1 + d_2u_2 + d_3(c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5) + d_4u_4 + d_5u_5 = 0$$

$$(d_1 + d_3c_1)u_1 + (d_2 + d_3c_2)u_2 + d_3c_3u_3 + (d_3c_4 + d_4)u_4 + (d_3c_5 + d_5)u_5 = 0$$

Since $\{u_1, u_2, u_3, u_4, u_5\}$ is a basis for \mathbb{R}^5 , these vectors are linearly independent. Thus

$$d_1 + d_3c_1 = 0 \Rightarrow d_1 = 0$$

$$d_2 + d_3c_2 = 0 \Rightarrow d_2 = 0$$

$$d_3c_3 = 0 \implies d_3 = 0 \text{ since } c_3 \neq 0. \text{ Using this result we obtain}$$

$$d_3c_4 + d_4 = 0 \Rightarrow d_4 = 0$$

$$d_1 = d_2 = d_3 = d_4 = d_5 = 0$$

$$d_3c_5 + d_5 = 0 \Rightarrow d_5 = 0$$

Thus the given set is also a basis for \mathbb{R}^5 .

2) Let $C[0,1]$ be the vector space of all continuous real valued functions

with domain $[0,1]$. Let $\langle \cdot, \cdot \rangle$ be the inner product defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ in

$C[0,1]$. Let $f(x) = x$ and $g(x) = \cos x$

a) Find $\langle f, g \rangle$ and $|g|$ where $|\cdot|$ denotes the length induced by this inner product. Show all your work.

$$\langle f, g \rangle = \int_0^1 x \cos x dx = x \sin x \Big|_0^1 - \int_0^1 \sin x dx = (x \sin x + \cos x) \Big|_0^1 = [\sin 1 + \cos 1 - 1]$$

$u = x \quad du = dx$
 $dv = \cos x dx \quad v = \sin x$

$$|g| = \sqrt{\langle g, g \rangle}$$

$$\langle g, g \rangle = \int_0^1 \cos^2 x dx = \int_0^1 \frac{1 + \cos 2x}{2} dx = \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) \Big|_0^1 = \frac{1}{2} + \frac{\sin 2}{4}$$

$$|g| = \sqrt{\frac{1}{2} + \frac{\sin 2}{4}}$$

b) Determine the scalar c so that $f - cg$ is orthogonal to f . Show all your work.

$f - cg$ is orthogonal to $f \Leftrightarrow \langle f, f - cg \rangle = 0$

$$\langle f, f - cg \rangle = \langle f, f \rangle - c \langle f, g \rangle \quad (\text{Using the properties of inner product})$$

$$\langle f, f \rangle = \int_0^1 x^2 dx = \frac{1}{3} \quad \langle f, g \rangle = \sin 1 + \cos 1 - 1 \quad (\text{from part a})$$

$$\langle f, f - cg \rangle = \frac{1}{3} - c(\sin 1 + \cos 1 - 1) = 0$$

$$\boxed{c = \frac{1}{3(\sin 1 + \cos 1 - 1)}}$$

3) Let $A = \begin{bmatrix} a & a & -3a \\ 0 & 2 & 9 \\ 0 & 1 & 2 \end{bmatrix}$ where a is a real number.

a) Find the characteristic polynomial of A . (Of course, it may depend on a). Show all your work.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} a-\lambda & a & -3a \\ 0 & 2-\lambda & 9 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (a-\lambda) \begin{vmatrix} 2-\lambda & 9 \\ 1 & 2-\lambda \end{vmatrix} = (a-\lambda)[(2-\lambda)^2 - 9] \\ &= (a-\lambda)(\lambda^2 - 4\lambda - 5) = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 5, \lambda_3 = a. \end{aligned}$$

$(\lambda+1)(\lambda-5)$

b) What must be a if A is not diagonalizable? Explain your answer.

If A has 3 distinct eigenvalues then it is diagonalizable. Thus A is diagonalizable if $a \neq -1$ & $a \neq 5$; we must check the cases when $a = -1$ & $a = 5$.

For $a = -1$: $\lambda_1 = -1$ is an eigenvalue with multiplicity 2.

$$A + I = \begin{bmatrix} 0 & -1 & 3 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 0, t \in \mathbb{R} \\ x_2 = 0 \\ x_3 = t \end{array}$$

Thus $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the only eigenvector corresponding to $\lambda_1 = -1$. Thus A is not diagonalizable when $a = -1$.

For $a = 5$: $\lambda_2 = 5$ is an eigenvalue with multiplicity 1:

$$A - 5I = \begin{bmatrix} 0 & 5 & -15 \\ 0 & -3 & 9 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 3t ; s, t \in \mathbb{R} \\ x_2 = s \\ x_3 = t \end{array}$$

Thus $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are the eigenvectors corresponding to $\lambda_2 = 5$.

$$A + I = \begin{bmatrix} 6 & 5 & -15 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 0, t \in \mathbb{R} \\ x_2 = -3t \\ x_3 = t \end{array} \quad \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \text{ is an eigenvector for } \lambda_1 = -1.$$

Thus A is diagonalizable when $a = 5$.

Consequently, A is not diagonalizable $\Leftrightarrow a = -1$.

4) Let $(D-1)^2(D^2+4)y = x^5e^x + 7\sin 2x$ be the given 4-th order nonhomogeneous differential equation with constant coefficients.

a) Find the general solution of the corresponding homogeneous differential equation.

$(D-1)^2(D^2+4)y = 0$ gives the characteristic equation

$$(r-1)^2(r^2+4) = 0 \Rightarrow 1, 1, \pm 2i$$

$$y = c_1 e^x + c_2 x e^x + c_3 \cos 2x + c_4 \sin 2x$$

b) Write down the form of the particular solution y_p , but DO NOT calculate the coefficients.

Suppose that $R(x) = x^5e^x + 7\sin 2x$ is a solution of some homog. d.e.

$x^5e^x \rightarrow 1$ is a root with multiplicity 6.

$\sin 2x \rightarrow \pm 2i$ are the complex roots.

$s^6(s^2+4) = 0$ is the characteristic eqn.

$$D^6(D^2+4)R(x) = 0$$

$$(D-1)^2(D^2+4)y = R(x)$$

$$(D-1)^2(D^2+4)D^6(D^2+4)y = D^6(D^2+4)R(x) = 0$$

1, 1, $\pm 2i$, 1, 1, 1, 1, 1, 1, $\pm 2i$ are the roots of the charact. eqn

$$y = e^x(c_1 + x c_2 + x^2 c_3 + x^3 c_4 + x^5 c_5 + x^6 c_6 + x^7 c_7) + (c_8 \cos 2x + c_9 \sin 2x) + x(c_{10} \cos 2x + c_{11} \sin 2x)$$

Thus

$$y_p = e^x(x^2 c_3 + x^3 c_4 + x^5 c_5 + x^6 c_6 + x^7 c_7) + x(c_{10} \cos 2x + c_{11} \sin 2x)$$

is the particular solution.

5) Use the method of variation of parameters to find a particular solution of the differential equation $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$. Show all your work.

From $y'' - 3y' + 2y = 0$ we get $r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0$

$$r_1 = 2 \text{ & } r_2 = 1.$$

Thus $y = c_1 e^{2x} + c_2 e^x$ is the solution of the corresponding homog. d.e. with $y_1 = e^{2x}$ & $y_2 = e^x$.

Suppose that $y_p = u_1(x)e^{2x} + u_2(x)e^x$.

$$y'_p = u_1' e^{2x} + u_2' e^x + 2u_1 e^{2x} + u_2 e^x \text{ & suppose that } \boxed{u_1' e^{2x} + u_2' e^x = 0} \quad (1)$$

$$\text{thus } y_p' = 2u_1 e^{2x} + u_2 e^x \Rightarrow y_p'' = 2u_1' e^{2x} + 4u_1 e^{2x} + u_2' e^x + u_2 e^x$$

Put y_p, y_p', y_p'' in the given equation:

$$2u_1' e^{2x} + 4u_1 e^{2x} + u_2' e^x + u_2 e^x - 6u_1 e^{2x} - 3u_2 e^x + 2u_1 e^{2x} + 2u_2 e^x = \frac{1}{1+e^{-x}}$$

$$\boxed{2u_1' e^{2x} + u_2' e^x = \frac{1}{1+e^{-x}}} \quad (2) \quad \text{Using (1) & (2):}$$

$$\begin{vmatrix} e^{2x} & e^x \\ 2e^{2x} & e^x \end{vmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{1+e^{-x}} \end{pmatrix} \quad \& \quad \begin{vmatrix} e^{2x} & e^x \\ 2e^{2x} & e^x \end{vmatrix} = -e^{3x} \neq 0. \text{ Using Cramer's rule:}$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^x \\ \frac{1}{1+e^{-x}} & e^x \end{vmatrix}}{-e^{3x}} = \frac{-\frac{e^x}{1+e^{-x}}}{-e^{3x}} = \frac{e^{-2x}}{1+e^{-x}}$$

$$u_1 = \int \frac{e^{-2x}}{1+e^{-x}} dx = \int \left(e^{-x} - \frac{e^{-x}}{1+e^{-x}} \right) dx = -e^{-x} + \ln(1+e^{-x}) + C_1$$

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{1}{1+e^{-x}} \end{vmatrix}}{-e^{3x}} = \frac{\frac{e^{2x}}{1+e^{-x}}}{-e^{3x}} = \frac{e^{-x}}{1+e^{-x}}$$

$$u_2 = \int \frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x}) + C_2. \text{ Choosing } C_1 = C_2 = 0 \text{ we get}$$

$$\boxed{y_p = (-e^{-x} + \ln(1+e^{-x})) \cdot e^{2x} + \ln(1+e^{-x}) \cdot e^x}$$