

1) Let W be a subspace of \mathbb{R}^4 spanned by the vectors $u = (3, -2, -1, 0)$, $v = (0, 2, -2, 0)$, $w = (3, 0, 0, -3)$ and $z = (0, -4, 0, 4)$.

Determine

- a basis for W ,
- $\dim W$,
- whether or not $W = \mathbb{R}^4$.

Show all your steps explicitly.

a) It is enough to choose linearly independent vectors because W is already spanned by the given vectors.

$$c_1 u + c_2 v + c_3 w + c_4 z = 0$$

$$c_1(3, -2, -1, 0) + c_2(0, 2, -2, 0) + c_3(3, 0, 0, -3) + c_4(0, -4, 0, 4) = (0, 0, 0, 0)$$

$$\begin{cases} 3c_1 + 3c_3 = 0 \\ -2c_1 + 2c_2 - 4c_4 = 0 \\ -c_1 - 2c_2 = 0 \\ -3c_3 + 4c_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc} 3 & 0 & 3 & 0 \\ -2 & 2 & 0 & -4 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -3 & 4 \end{array} \right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ -2 & 2 & 0 & -4 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -3 & 4 \end{array} \right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -3 & 4 \end{array} \right] \xrightarrow{R_1+R_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 4 \end{array} \right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -3 & 4 \end{array} \right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -4/3 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_3+R_1} \left[\begin{array}{cccc} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & -4/3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The vectors u, v, w are lin. indep & they form a basis for W .

b) $\dim W = 3$

c) $W \neq \mathbb{R}^4$ because $\dim W \neq \dim \mathbb{R}^4$.

2) Solve the system of differential equations

$$\frac{dx_1}{dt} = x_1 + x_2 + x_3$$

$$\frac{dx_2}{dt} = x_1 + x_2 - x_3$$

$$\frac{dx_3}{dt} = x_1 - x_2 + x_3.$$

Show all your work.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = -(1-\lambda)^2(\lambda+1) = 0$$

$\lambda_1 = 2$ is an eigenvalue with multiplicity 2.

$\lambda_2 = -1$ is the 2nd eigenvalue.

For $\lambda_1 = 2$: Solve $(A - 2I)v = 0$

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= s+t \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

Thus $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are the corresponding eigenvectors.

For $\lambda_2 = -1$: Solve $(A + I)v = 0$

$$A + I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= -t, t \in \mathbb{R} \\ x_2 &= t \\ x_3 &= t \end{aligned} \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ is a corresponding eigenvector.}$$

∴ A is diagonalizable & $P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ diagonalizes A.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P \cdot \begin{bmatrix} K_1 e^{2t} \\ K_2 e^{-t} \\ K_3 e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 e^{2t} \\ K_2 e^{-t} \\ K_3 e^{-t} \end{bmatrix} \quad \text{gives}$$

$$x_1 = K_1 e^{2t} + K_2 e^{-t} - K_3 e^{-t}$$

$$x_2 = K_1 e^{2t} + K_3 e^{-t}$$

$$x_3 = K_2 e^{-t} - K_3 e^{-t}$$

3) Use the method of reduction of order to determine the general solution to the given differential equation $x^2y'' - 3xy' + 4y = 8x^4$, $x > 0$ and $y_1(x) = x^2$ is one of the solutions to the associated homogeneous differential equation
Show all your steps explicitly.

Let $y = ux^2$. Then $y' = u'x^2 + 2xu$ & $y'' = u''x^2 + 4xu' + 2u$

$$x^2y'' - 3xy' + 4y = 8x^4 \text{ becomes}$$

$$x^2(u''x^2 + 4xu' + 2u) - 3x(u'x^2 + 2xu) + 4ux^2 = 8x^4$$

$$u''x^4 + 4x^3u' + 2x^2u - 3x^3u' - 6x^2u + 4u = 8x^4$$

$$u''x^4 + x^3u' = 8x^4 \Rightarrow u'' + \frac{1}{x}u' = 8 \text{ is 2nd order reducible d.e.} \\ x > 0 \quad (*)$$

Say $u' = w$. Then $(*)$ becomes

$$(**) \quad w' + \frac{1}{x}w = 8 \text{ linear 1st order d.e. with } g(x) = e^{\int \frac{1}{x} dx} = x$$

$$xw' + w = 8x$$

$$\frac{d(w \cdot x)}{dx} = 8x \Rightarrow w \cdot x = 4x^2 + C_1 \text{ gives } w = 4x + \frac{C_1}{x}$$

$$\text{and } u' = 4x + \frac{C_1}{x} \Rightarrow \boxed{u = 2x^2 + C_1 \ln x + C_2}$$

Thus $y = ux^2 = \underbrace{2x^4}_{y_p} + \underbrace{C_1x^2 \ln x + C_2x^2}_{y_c}$ is the general solution.

4) Let $y''' + y'' + 9y' + 9y = 4xe^{-x} + 5e^{2x} \cos 3x$ be a non-homogeneous 3rd order differential equation with constant coefficients.

a) Find the solution of the corresponding homogeneous differential equation. Show all your steps explicitly.

$$y''' + y'' + 9y' + 9y = 0$$

$L(r) = r^3 + r^2 + 9r + 9 = (r+1)(r^2 + 9) = 0$ & $-1, \pm 3i$ are the roots.

Thus $\boxed{y_c = c_1 e^{-x} + c_2 \cos 3x + c_3 \sin 3x}$

b) Write down the form of the particular solution y_p , but do not calculate the coefficients. Show all your steps explicitly.

Let $R(x) = 4xe^{-x} + 5e^{2x} \cos 3x$ be a particular solution of some homog. d.e. with constant coefficients.

$x e^{-x} \rightarrow -1$ is a real root with multiplicity 2

$e^{2x} \cos 3x \rightarrow 2 + 3i$ are complex roots.

Thus $g(s) = (s+1)^2(s^2 - 4s + 13)$ is the corr. charac. eqn. &

$$\boxed{(D+1)^2(D^2 - 4D + 13)R(x) = 0} \quad (1)$$

Using the given d.e. $(D^3 + D^2 + 9D + 9)y = R(x)$ & eqn (1) we obtain the homog. eqn.

$$(D+1)^3(D^2 + 9)(D^2 - 4D + 13)y = 0 \quad . \text{ Its charact. eqn. has the roots}$$

$$-1, -1, -1, \pm 3i, 2 \mp 3i$$

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + c_4 \cos 3x + c_5 \sin 3x + e^{2x} (c_6 \cos 3x + c_7 \sin 3x)$$

Using $y_c = c_1 e^{-x} + c_2 \cos 3x + c_3 \sin 3x$ we obtain

$$y_p = c_2 x e^{-x} + c_3 x^2 e^{-x} + e^{2x} (c_6 \cos 3x + c_7 \sin 3x).$$

5) a) Is there a 3×3 matrix A such that the matrices A^3, A^2, A and I are linearly independent? Explain your answer.

By Cayley Hamilton Thm. A satisfies its characteristic polynomial

$$p(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \text{ where } a, b, c, d \in \mathbb{R}. \text{ Thus}$$

$$aA^3 + bA^2 + cA + d \cdot I = 0 \text{ for some real numbers } a, b, c, d \in \mathbb{R}.$$

Thus A^3, A^2, A & I cannot be linearly independent.

b) Let A be a 3×3 matrix such that $\text{rank}(A) = 1$ and $\text{rank}(A + I) = 2$. Is A diagonalizable? Explain your answer.

$\text{rank } A = 1 \Rightarrow \text{nullity}(A) = 2$ i.e. the homog system $Ax = 0$ has non-trivial solutions & 0 is an eigenvalue of A having two linearly independent eigenvectors.

$\text{rank}(A + I) = 2 \Rightarrow \text{nullity}(A + I) = 1$ i.e. the homog system $(A + I)x = 0$ has non-trivial solutions & -1 is an eigenvalue of A having one eigenvector.

Thus A is a 3×3 matrix having 3 distinct eigenvectors & A is diag'ble.

c) Write 5 statements each of which equals to the invertibility of a square matrix A ?

- 1) A is row-equivalent to I .
- 2) $AX=0$ has only the trivial solution $x=0$.
- 3) $AX=b$ " " " " " $x=A^{-1}b$
- 4) $\det A \neq 0$
- 5) $\text{rank}(A)=n$

These are some possibilities, you can find different sentences.

d) Let V be a vector space which can be spanned by 7 linearly dependent vectors.

If V contains some 4 linearly independent vectors whose span is not equal to V , what are the possibilities for $\dim V$?

Since V contains 4 linearly independent vectors whose span not equal to V , $\dim V > 4$ i.e. $\dim V = 5, 6, 7$.
On the other hand V can be spanned by 7 linearly dependent vectors. Thus $\dim V \leq 7$ i.e. $\dim V = 5$ or 6.