

3) (20 pts.) Show all your work.

Solve the following initial value problem.

$$\begin{cases} \frac{dx_1}{dt} = 4x_1 + 2x_2 \\ \frac{dx_2}{dt} = 3x_1 - x_2 \end{cases}, x_1(0) = 1 \text{ and } x_2(0) = 2.$$

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \quad \& \quad \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6 \\ = \lambda^2 - 3\lambda - 10 = 0 \\ (\lambda+2)(\lambda-5) = 0$$

$\lambda_1 = -2$ & $\lambda_2 = 5$ are the eigenvalues.

For $\lambda_1 = -2$:

$$A + 2I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = -\frac{1}{3}t \\ x_2 = t \end{array}$$

$v_1 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda_1 = -2$.

For $\lambda_2 = 5$:

$$A - 5I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 2s \\ x_2 = s \end{array}$$

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda_2 = 5$.

Thus $P = \begin{bmatrix} -1/3 & 2 \\ 1 & 1 \end{bmatrix}$.

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = P \cdot \begin{bmatrix} K_1 e^{-2t} \\ K_2 e^{5t} \end{bmatrix} = \begin{bmatrix} -1/3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 e^{-2t} \\ K_2 e^{5t} \end{bmatrix}$$

$$x_1(t) = -\frac{K_1}{3} e^{-2t} + 2K_2 e^{5t}$$

$$x_2(t) = K_1 e^{-2t} + K_2 e^{5t}$$

Using the initial conditions $x_1(0) = 1$ & $x_2(0) = 2$,

$$\left. \begin{array}{l} -\frac{K_1}{3} + 2K_2 = 1 \\ K_1 + K_2 = 2 \end{array} \right\} \quad K_1 = \frac{9}{7} \quad K_2 = \frac{5}{7}$$

Thus

$$x_1(t) = -\frac{3}{7} e^{-2t} + \frac{10}{7} e^{5t}$$

$$x_2(t) = \frac{9}{7} e^{-2t} + \frac{5}{7} e^{5t}$$