MATH 225
Spring 2007-2008
SOLUTIONS OF MIDTERM 1

1) (10+10 pts.)

Consider the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{6y(y-2)} \\ y(2) = 1 \end{cases}$$

a) Using the Existence and Uniqueness Theorem show that the given initial value problem has a unique solution.

Solution:

The functions $f(x,y) = \dfrac{x}{6y(y-2)}$ and $f_y(x,y) = -\dfrac{12x(y-1)}{[6y(y-2)]^2}$ are continuous everywhere except the points $y = 0$ and $y = 2$. Thus we can find a rectangle R such as R=$\left\{(x,y) \mid |x-2| < 1, |y-1| < \dfrac{1}{2}\right\}$ around the point (2,1) such that $f(x,y)$ and $f_y(x,y)$ are continuous in R. Thus by the Existence and Uniqueness Theorem there exists a unique solution around the point (2,1).

b) Solve the initial value problem and find this unique solution.

Solution:

The equation $\dfrac{dy}{dx} = \dfrac{x}{6y(y-2)}$ is separable.

We obtain the equation $(6y^2 - 12y)dy = xdx$ and it gives the result $2y^3 - 6y^2 = \dfrac{x^2}{2} + C$. Using the initial point $y(2) = 1$, C=-6.

Thus $2y^3 - 6y^2 - \dfrac{x^2}{2} + 6 = 0$ is the unique solution of this problem.

2)(20 pts.) Solve the following initial value problem

$$x\frac{dy}{dx}+y=(xy)^{\frac{3}{2}}, \quad y(1)=4.$$

Solution:

This equation is Bernoulli type.

(1) $\frac{dy}{dx}+\frac{1}{x}y=x^{1/2}y^{3/2}$ with $n=\frac{3}{2}$.

Multiplying the equation (1) by $y^{-3/2}$,

$$y^{-3/2}\frac{dy}{dx}+\frac{1}{x}y^{-1/2}=x^{1/2}.$$

Let $v=y^{-1/2}$. Then $\frac{dv}{dx}=-\frac{1}{2}y^{-3/2}\frac{dy}{dx}\Rightarrow y^{-3/2}\frac{dy}{dx}=-2\frac{dv}{dx}$.

Thus

(2) $\quad -2\frac{dv}{dx}+\frac{1}{x}v=x^{1/2}\Rightarrow\frac{dv}{dx}-\frac{1}{2x}v=-\frac{1}{2}x^{1/2}$

is a linear equation with an integrating factor $\rho(x)=e^{\int -\frac{1}{2x}dx}=x^{-1/2}$.

Multiply the equation (2) by $\rho(x)$ and obtain

$$\frac{d}{dx}\left(x^{-1/2}v\right)=-\frac{1}{2}\Rightarrow x^{-1/2}v=-\frac{1}{2}x+C\Rightarrow v=-\frac{1}{2}x^{3/2}+Cx^{1/2}.$$

Put $v=y^{-1/2}$:

$y^{-1/2}=-\frac{1}{2}x^{3/2}+Cx^{1/2}$. To find C use the initial condition $y(1)=4\Rightarrow C=1$.

Thus $y=\dfrac{1}{\left(-\frac{1}{2}x^{3/2}+x^{1/2}\right)^2}$.

OR

Use the substitution $v=xy\Rightarrow y=\frac{v}{x}\Rightarrow\frac{dy}{dx}=\frac{-1}{x^2}v+\frac{1}{x}\frac{dv}{dx}$ and equation (1) becomes

$\frac{dv}{dx}=v^{\frac{3}{2}}$ and the rest is the standard application of the separable differential equations.

3)(20 pts.) Consider the initial value problem

$$\left(4x+3y^2\right)dx+2xydy=0\,,\ \ y(1)=1.$$

Show that the given differential equation is not exact. Find an integrating factor and solve the initial value problem.

Solution:

Since $\dfrac{\partial M}{\partial y}=6y\neq\dfrac{\partial N}{\partial x}=2y$ the given equation is not exact.

But $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right]=\dfrac{2}{x}$ depends on x only. Therefore an integrating factor is

$e^{\int\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]dx}=x^2$. Multiplying the given equation by x^2

$$\left(4x^3+3x^2y^2\right)dx+2x^3ydy=0$$

we get an exact equation (check it!).

Thus $F(x,y)=\int\left(4x^3+3x^2y^2\right)dx+g(y)\Rightarrow F(x,y)=x^4+x^3y^2+g(y)$.

On the other hand

$$\frac{\partial F}{\partial y}=2x^3y+g'(y)=N(x,y)=2x^3y\Rightarrow g'(y)=0\Rightarrow g(y)=C.$$

It gives $x^4+x^3y^2+C=0.$

Using the initial condition $y(1)=1\Rightarrow C=-2$ and we conclude the result

$$x^4+x^3y^2-2=0\,.$$

4)(20 pts.) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}$.Calculate a, b, c such that $A^3 = aA^2 + bA + cI$.

Solution: The solution of this question is not given in details but you are expected to solve it in details.

$$A^2 = \begin{bmatrix} -2 & 0 & 5 \\ 5 & -3 & -2 \\ 0 & 9 & 6 \end{bmatrix},\ A^3 = \begin{bmatrix} -7 & 12 & 13 \\ 1 & -12 & 2 \\ 12 & 21 & 9 \end{bmatrix}.$$

Using $A^3 = aA^2 + bA + cI$ and identifying the first row we get

$$\begin{cases} -2a + b + c = -7 \\ \quad -b = 12 \\ 5a + b = 13 \end{cases}$$

with solution $a = 5, b = -12, c = 15$.

Finally the relation has to be verified for all the other entries.

5)(20 pts.) The elementary row operations

$E_1 : 4R_3$,

$E_2 : SWAP(R_2, R_3)$,

$E_3 : -2R_2 + R_1$

are applied in the given order to a $3X3$ real matrix A and the identity matrix I is obtained.

a)(6 pts.) Express A^{-1} as a product of elementary matrices.

Solution:

$A^{-1} = E_3 E_2 E_1$

b)(7 pts.) Find A^{-1}.

Solution:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{4R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = E_1.$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{SWAP(R_2,R_3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_2.$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{3.}$$

$$A^{-1} = E_3 E_2 E_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

c) (7 pts.) If $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the unique solution of the system $AX = B$.

Solution: The solution will be $X = A^{-1}B = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -23 \\ 12 \\ 2 \end{bmatrix}$.

Thus $x = -23, y = 12, z = 2$ is the unique solution.