1) **(20 pts.)** Solve $\begin{cases} \dfrac{dx}{dt} = 4x+z \\ \dfrac{dy}{dt} = 2x+3y+2z \\ \dfrac{dz}{dt} = x+4z \end{cases}$. **Show your work.**

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}, \quad |A-\lambda I| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)^2(3-\lambda) - (3-\lambda) = -(\lambda-3)^2(\lambda-5)$$

$\lambda_1 = 3$ is an eigenvalue with multiplicity 2,
$\lambda_2 = 5$ " .

For $\lambda_1 = 3$: Solve $(A-3I)v = 0$

$$A-3I = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x &= -t \\ y &= s \\ z &= t \end{aligned} \; ; \; s,t \in \mathbb{R}$$

$S_1 = \text{span}\{\underbrace{(-1,0,1)}_{v_1}, \underbrace{(0,1,0)}_{v_2}\}$ is an eigenspace for $\lambda_1 = 3$ & v_1, v_2 are the eigenvectors.

For $\lambda_2 = 5$: Solve $(A-5I)v = 0$

$$A-5I = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x &= t \\ y &= 0 \\ z &= t \end{aligned} \; , \; t \in \mathbb{R}$$

$S_2 = \text{span}\{\underbrace{(1,0,1)}_{v_3}\}$ is an eigenvector associated with $\lambda_2 = 5$ & v_3 is an eigenvector.

$\therefore$ A is diag'ble with $P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ & $D = \text{diag}(3,3,5)$.

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = P \cdot \begin{bmatrix} K_1 e^{3t} \\ K_2 e^{3t} \\ K_3 e^{5t} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1 e^{3t} \\ K_2 e^{3t} \\ K_3 e^{5t} \end{bmatrix}$$

$$\begin{cases} x(t) = -K_1 e^{3t} + K_3 e^{5t} \\ y(t) = K_2 e^{3t} \\ z(t) = K_1 e^{3t} + K_3 e^{5t} \end{cases}$$

2) a) (10 pts.) Calculate A^{-1} by using **Cayley-Hamilton Theorem** where

$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ is the coefficient matrix in **Question 1). Show your work.**

From Question 1;

$p(\lambda) = -(\lambda-3)^2(\lambda-5) = -(\lambda^3 - 5\lambda^2 - 6\lambda^2 + 30\lambda + 9\lambda - 45)$

$p(\lambda) = -\lambda^3 + 11\lambda^2 - 39\lambda + 45$ & by CAYLEY - HAMILTON THM

$p(A) = 0 \Rightarrow -A^3 + 11A^2 - 39A + 45I = 0$

$A(A^2 - 11A + 39I) = 45I$

$A^{-1} = \frac{1}{45}[A^2 - 11A + 39I]$

$$A^{-1} = \frac{1}{45}\left\{ \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} - 11\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 39 & 0 & 0 \\ 0 & 39 & 0 \\ 0 & 0 & 39 \end{bmatrix} \right\}$$

$$A^{-1} = \begin{bmatrix} 4/15 & 0 & -1/15 \\ -2/15 & 1/3 & -2/15 \\ -1/15 & 0 & 4/15 \end{bmatrix}$$

b)(2.5 pts each) Suppose v_1, v_2, v_3, v_4 are vectors in R^3. **Complete the following sentences** using this information:

a) These four vectors are linearly **dependent** because $\dim \mathbb{R}^3 = 3$ & # vectors $= 4 > 3$.

b) The two vectors v_1 and v_2 are linearly **dependent** if $v_1 = k v_2$, $k \in \mathbb{R}$.

c) The vectors v_1 and (0,0,0) are linearly **dependent** because $c_1 v_1 + c_2 0_v = 0_v$ has non-zero solutions. e.g. Take $c_1 = 0$, $c_2 = -7$.

d) v_1, v_2, v_3 form a basis for R^3 if they're lin. independent (or they span $\mathbb{R}^3$).

3)(20 pts.) Let $W = span\{\overbrace{(-1,1,-1,3,0)}^{v_1}, \overbrace{(0,1,-3,4,2)}^{v_2}, \overbrace{(-3,1,3,1,-4)}^{v_3}\}$ be a subspace of R^5 spanned by the given vectors. Find a **basis** for the orthogonal complement $W^\perp$. **Show your work.**

Let $A = \begin{bmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{bmatrix}$.

We know that; if $W = Row(A)$, then $W^\perp = Null(A)$.
Thus if we find a basis for $Null(A)$, then we find a basis for $W^\perp$.

$$A = \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -3 & 4 & 2 \\ -3 & 1 & 3 & 1 & -4 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -3 & 4 & 2 \\ 0 & -2 & 6 & -8 & -4 \end{bmatrix}$$

$$\xrightarrow[2R_2+R_3]{-R_2+R_1} \begin{bmatrix} -1 & 0 & 2 & -1 & -2 \\ 0 & 1 & -3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 & -2 & 1 & 2 \\ 0 & 1 & -3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading var.

$x_1 = 2a - b - 2c$; $a, b, c \in \mathbb{R}$.
$x_2 = 3a - 4b - 2c$
$x_3 = a$
$x_4 = b$
$x_5 = c$

$\mathcal{B}_N = span\{\underbrace{(2,3,1,0,0)}_{w_1}, \underbrace{(-1,-4,0,1,0)}_{w_2}, \underbrace{(-2,2,0,0,1)}_{w_3}\}$

and $c_1 w_1 + c_2 w_2 + c_3 w_3 = 0_V$ has only the trivial solution $c_1 = c_2 = c_3 = 0$ & $\{w_1, w_2, w_3\}$ is lin. independent.
$\therefore$ $\{w_1, w_2, w_3\}$ is a basis for $W^\perp$.

4) **(20 pts.)** **Using reduction of order**, solve $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = x$, $y(1) = 3$, $y'(1) = 1$

where $y_1(x) = x^2$ is the given solution. **Show your work.**

$y(x) = ux^2$, $y'(x) = u'x^2 + 2xu$, $y''(x) = u''x^2 + 4xu' + 2u$

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \underbrace{u''x^2 + 4xu' + 2u + \frac{1}{x}(u'x^2 + 2xu) - \frac{4}{x^2}(ux^2) = x}_{u''x^2 + 5xu' = x}$$

$u''x^2 + 5xu' = x$, $x = 1 > 0$ given, say $u' = w$

(1) $w' + \frac{5}{x}w = \frac{1}{x}$ linear in w

$$\rho(x) = e^{\int \frac{5}{x}dx} = x^5$$

Then (1) becomes:

$$\frac{d}{dx}(x^5 \cdot w) = x^4 \Rightarrow x^5 w = \frac{x^5}{5} + A$$

$$u' = w = \frac{1}{5} + Ax^{-5}$$

$$u = \frac{x}{5}\underbrace{\left(-\frac{A}{4}\right)}_{C_1}x^{-4} + \underbrace{(B)}_{C_2} \Rightarrow \boxed{u = \frac{x}{5} + \frac{C_1}{x^4} + C_2}$$

$$y = ux^2 = \underbrace{\frac{C_1}{x^2} + C_2x^2}_{y_c} + \underbrace{\frac{x^3}{5}}_{y_p} \Rightarrow y'(x) = -\frac{2C_1}{x^3} + 2C_2x + \frac{3}{5}x^2$$

$y(1) = 3 \Rightarrow \cancel{C_1} + C_2 = \frac{14}{5}$

$y'(1) = 1 \Rightarrow -\cancel{C_1} + C_2 = \frac{1}{5}$

$2C_2 = 3 \Rightarrow \boxed{C_2 = \frac{3}{2}}$ & $\boxed{C_1 = \frac{13}{10}}$

$$\therefore \boxed{y(x) = \frac{13}{10}\cdot\frac{1}{x^2} + \frac{3}{2}x^2 + \frac{x^3}{5}}$$

5)(20 pts.) Find the general solution of

$$y^{(4)} - 3y^{(3)} - 3y'' + 11y' - 6y = 3e^x + 6xe^{2x} - 5\cos x.$$

Determine the complementary solution y_c and particular solution y_p. **DO NOT** evaluate the coefficients of y_p. **SHOW YOUR WORK.**

$(D^4 - 3D^3 - 3D^2 + 11D - 6)y = R(x)$ where $R(x) = 3e^x + 6xe^{2x} - 5\cos x$

$L(r) = r^4 - 3r^3 - 3r^2 + 11r - 6 = 0$

$\mp 1 \quad \mp 2 \quad \mp 3 \quad \mp 6$

$L(1) = 0 \Rightarrow 1$ is a root.

$$\begin{array}{l|l} r^4 - 3r^3 - 3r^2 + 11r - 6 & r-1 \\ \hline -r^4 + r^3 & r^3 - 2r^2 - 5r + 6 \\ \hline -2r^3 - 3r^2 + 11r - 6 & \\ -2r^3 + 2r^2 & \\ \hline -5r^2 + 11r - 6 & \\ -5r^2 + 5r & \\ \hline 6r - 6 & \\ 0 & \end{array}
\qquad
\begin{array}{l|l} r^3 - 2r^2 - 5r + 6 & r-1 \\ \hline -r^3 + r^2 & r^2 - r - 6 = (r-3)(r+2) \\ \hline -r^2 - 5r + 6 & \\ -r^2 + r & \\ \hline -6r + 6 & \\ 0 & \end{array}$$

$\therefore L(r) = (r-1)^2(r+2)(r-3) = 0 \Rightarrow y_c = (c_1 + c_2x)e^x + c_3e^{-2x} + c_4e^{3x}$

$1, 1, -2, 3$

$R(x) = 3e^x + 6xe^{2x} - 5\cos x$ is a particular solution of some homog. d.e.

$e^x \to 1$

$xe^{2x} \to 2, 2$

$\cos x \to \mp i$

$g(s) = (s-1)(s-2)^2(s^2+1) = 0$

$(D-1)(D-2)^2(D^2+1)y = 0$ is the corresponding d.e.

and $R(x)$ is a sol. of this d.e. gives

$(D-1)(D-2)^2(D^2+1)R(x) = 0.$

$\therefore$ We obtain a homog. eqn.

$(D-1)^2(D+2)(D-3)(D-1)(D-2)^2(D^2+1)y = 0$ has the roots

$3, -2, 1, 1, 1, 2, 2, \mp i$

$$y(x) = \underbrace{A_1e^{3x} + A_2e^{-2x} + A_3e^x + A_4xe^x}_{y_c} + \underbrace{A_5x^2e^x + A_6e^{2x} + A_7xe^{2x} + A_8\cos x + A_9\sin x}_{y_p}$$