BİLKENT UNIVERSITY
Department of Mathematics

Date: 22 October 2009 NAME: ....

Time: 18:00-20:00 STUDENT NO: ....

Fall 2009-10, Y. Kurtulmaz & U. Muğan SECTION: 01 02 03 04

Math 225.01-04, Linear Algebra & Differential Eq. Midterm Exam # 1

1	2	3	4	5	TOTAL
15	20	20	20	25	100

(Do not write anything on the above table)

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1) Solve the following initial value problem

$$\underbrace{(4e^{2x} + 2xy - y^2)}_{M(x,y)}dx + \underbrace{(x-y)^2}_{N(x,y)}dy = 0, \qquad y(0) = 0.$$

(15 points)

$\frac{\partial M}{\partial y} = 2x - 2y = \frac{\partial N}{\partial y}$. It's an exact d.e.

$F(x,y) = \int (4e^{2x} + 2xy - y^2)\,dx + g(y) = 2e^{2x} + x^2y - xy^2 + g(y)$

$\frac{\partial F}{\partial y} = \cancel{x^2} - \cancel{2xy} + g'(y) = N(x,y) = \cancel{x^2} - \cancel{2xy} + y^2$

$g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + C_1$

$\therefore F(x,y) = 2e^{2x} + x^2y - xy^2 + \frac{y^3}{3} + C_1 = 0$

$y(0) = 0$ gives $C_1 = -2$ & we get

$$2e^{2x} + x^2y - xy^2 + \frac{y^3}{3} = 2$$

Ans. :

NAME:..

2) Solve the following differential equations

$$\frac{dy}{dx} = \frac{x+2y-1}{2x-y+3} \quad .$$

(20 points)

The given d.e. is <u>not</u> homog. type.

Use $x = u + h$ & $y = v + k$ transformation to make it homog.

$$\left.\begin{array}{l} h + 2k = 1 \\ 2h - k = -3 \end{array}\right\} \begin{array}{l} h = -1 \\ k = 1 \end{array} \quad \& \quad \begin{array}{l} x = u - 1 \\ y = v + 1 \end{array}$$

$$\frac{dy}{dx} = \frac{x+2y-1}{2x-y+3} \quad \text{becomes} \quad \frac{dv}{du} = \frac{u+2v}{2u-v} = \frac{1+2\frac{v}{u}}{2 - \frac{v}{u}} \quad \text{homog. } (1)$$

$$\text{Let } z = \frac{v}{u} \quad \& \quad v = uz \quad \text{gives} \quad \frac{dv}{du} = z + u \cdot \frac{dz}{du}.$$

$$(1) \text{ becomes} \quad z + u\frac{dz}{du} = \frac{1+2z}{2-z} \Rightarrow u\frac{dz}{du} = \frac{1+2z}{2-z} - z = \frac{1+z^2}{2-z}$$

is sep.

$$\frac{2-z}{1+z^2}\,dz = \frac{du}{u} \Rightarrow 2\arctan z - \frac{1}{2}\ln(1+z^2) = \ln|u| + C.$$

$$2\arctan\left(\frac{v}{u}\right) - \frac{1}{2}\ln\left(1 + \frac{v^2}{u^2}\right) = \ln|u| + C$$

Put $v = y - 1$ & $u = x + 1$:

$$2\arctan\left(\frac{y-1}{x+1}\right) - \ln\sqrt{\frac{(x+1)^2 + (y-1)^2}{(x+1)^2}} = \ln|x+1| + C$$

$$2\arctan\left(\frac{y-1}{x+1}\right) = \ln\sqrt{(x+1)^2 + (y-1)^2} + C.$$

Ans. :

NAME:...

3) Find the solution of the following second order differential equation

$$y'' = -\frac{2}{1-y}\,(y')^2.$$

(20 points)

"x" is missing.

$y' = u(y)$, $y'' = u'(y)\cdot u(y) = u'\cdot u$

$y'' = -\frac{2}{1-y}(y')^2$ becomes

$u'\cdot \cancel{u} = -\frac{2}{1-y}u^{\cancel{2}} \Rightarrow \frac{du}{dy} = -\frac{2}{1-y}u \Rightarrow \frac{du}{u} = -\frac{2}{1-y}dy$ sep.

$\ln|u| = 2\ln|1-y| + \ln C_1$, assuming $u>0, C_1>0, 1-y>0$

$\ln(u) = \ln\left[(1-y)^2\cdot C_1\right]$

$u = (1-y)^2\cdot C_1 \Rightarrow \frac{dy}{dx} = (1-y)^2\cdot C_1$

$$\frac{dy}{(1-y)^2} = C_1\cdot dx$$

$\frac{1}{1-y} = C_1 x + C_2 \overset{\text{OR}}{\Longrightarrow} \left[\frac{1}{1-y} - C_2\right]\cdot\frac{1}{C_1} = x$

$$\boxed{x = \frac{C_2 y + (1-C_2)}{C_2(1-y)}}$$

$1-y = \frac{1}{C_1 x + C_2}$

$1 - \frac{1}{C_1 x + C_2} = y$

$$\boxed{\frac{C_1 x + (C_2 - C_1)}{C_1 x + C_2} = y}$$

NAME:..

4) Determine all values of the constant k for which the following system has, a) no solution, b) an infinite number of solutions, c) a unique solution.

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &= 3\\
2x_1 + 5x_2 + x_3 &= 7\\
x_1 + x_2 - k^2 x_3 &= -k.
\end{aligned}
$$

Find the solution set when it exits.

(20 points)

$$
\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & -k \end{bmatrix}
\xrightarrow[-R_1+R_3]{-2R_1+R_2}
\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & 1-k^2 & -k-3 \end{bmatrix}
$$

$$
\xrightarrow{R_2+R_3}
\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4-k^2 & -k-2 \end{bmatrix} = E
$$

a) If $4-k^2=0$ but $-k-2\neq 0$ then there is no solution.

It's possible when $k=2$.

b) If $k=-2$ then we've!

$$
E = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\xrightarrow{-2R_2+R_1}
\begin{bmatrix} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\qquad
\begin{aligned} x_1 &= 7t+1 \\ x_2 &= -3t+1 \\ x_3 &= t \quad , t\in\mathbb{R}. \end{aligned}
$$

Leading var.

In this case there are infinitely many solutions.

c) If $k\neq\pm 2$ then we've!

$$
E = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4-k^2 & -k-2 \end{bmatrix}
\xrightarrow[\substack{-2R_2+R_1 \\ -3R_3+R_2 \\ 7R_3+R_1}]{\frac{1}{4-k^2}R_3}
\begin{bmatrix} 1 & 0 & 0 & \frac{k+5}{k-2} \\ 0 & 1 & 0 & \frac{k-5}{k-2} \\ 0 & 0 & 1 & \frac{1}{k-2} \end{bmatrix}
$$

Leading

$x_1 = \frac{k+5}{k-2}$, $x_2 = \frac{k-5}{k-2}$ & $x_3 = \frac{1}{k-2}$. There is a unique solution if $k\neq\mp 2$.

NAME:..

5) Classify the the following differential equations.
e.g. Linear in y, Homogenous, Separable, Exact, and so on.
DO NOT find the solutions and write <u>ONLY ONE</u> answer in each box.

a) $(x+y)dx + (x\ln x)dy = 0.$

Ans. : Linear in y

b) $y' - ay + f(x)y^3 = 0\,;\quad a > 0.$

Ans. : Bernoulli

c) $xdx - ydy = (xy)^{1/2}dx.$

Ans. : Homog.

d) $(x + e^y)dy - dx = 0.$

Ans. : Linear in x

e) $y' + 5x^2y = 2 + x^3 + 3xy^2.$

Ans. : Riccati

($5 \times 5 = 25$ points)

You can use the space below for your own calculations