MATH 225
SOLUTIONS OF MIDTERM II

Question 1. **(a)** Let $B = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 2 & 0 \\ 8 & 7 & 1 \end{bmatrix}$. Compute the following determinants:

$$\det\big(\frac{1}{2}B^{12}\big), \quad \det(3C^{-1}), \quad \det(B^{19}C^3), \quad \det\Big(\big((2B)C\big)^T\Big).$$

(b) Let A be a 3×3 matrix with $\det(A) = -1$. Express the matrix $\operatorname{adj}(2A)$ in terms of A^{-1}.

Solution: **(a):** As B and C are triangular matrices we easily calculate their determinants as the products of entries on the main diagonal of them. Indeed, $\det(B) = -1$ and $\det(C) = 2$. Then, by the properties of determinant function we have:

$$\det\big(\frac{1}{2}B^{12}\big) = (\frac{1}{2})^3\big(\det(B)\big)^{12} = \frac{1}{8}$$

$$\det(3C^{-1}) = 3^3\frac{1}{\det(C)} = \frac{27}{2}$$

$$\det(B^{19}C^3) = \big(\det(B)\big)^{19}\big(\det(C)\big)^3 = (-1)^{19}(2)^3 = (-1)(8) = -8.$$

$$\det\Big(\big((2B)C\big)^T\Big) = \det\Big((2B)C\Big) = 2^3\det(B)\det(C) = 2^3(-1)(2) = -16$$

(b): Recall that for any square matrix B we have the formula

$$\operatorname{adj}(B)B = \det(B)I.$$

Putting $B = 2A$ we get

$$\operatorname{adj}(2A)(2A) = \det(2A)I_3 = 2^3\det(A)I_3 = -8I_3.$$

This gives;

$$\operatorname{adj}(2A)A = -4I_3 \implies \operatorname{adj}(2A)AA^{-1} = -4I_3A^{-1}.$$

Consequently,

$$\operatorname{adj}(2A) = -4A^{-1}.$$

Question 2. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices whose entries are real numbers, and let W be the subset $\{A \in M_{2\times 2} : A^T = -A\}$ of $M_{2\times 2}$.

(i) Show that W is a subspace of $M_{2\times 2}$.

(ii) Find a basis for W. What is the dimension of W?

Solution: (i): As $0_{2\times 2}^T = -0_{2\times 2}$, the set W is a nonempty subset of $M_{2\times 2}$.

Take any two elements A and B from W. Then, $A^T = -A$ and $B^T = -B$, so

$$(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B),$$

implying that $A + B \in W$. Thus W is closed under addition.

Take any element A from W and any real number c. Then, $A^T = -A$ and so

$$(cA)^T = cA^T = c(-A) = -(cA),$$

implying that $cA \in W$. Thus W is closed under scalar multiplication.

(b): An element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $M_{2\times 2}$ is in W if and only if $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$, equivalently

$$a = -a,\ c = -b,\ b = -c,\ d = -d, \qquad \text{or} \qquad a = d = 0,\ c = -b.$$

Hence, any element of W is of the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This means that the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ spans W. Recall that a single vector is linearly independent if and only if it is nonzero. The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, being nonzero, must be linearly independent.

Consequently, the set $\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$ is a basis for W and the dimension of W is 1.

Question 3. Let P_3 is the vector space of all polynomials of degree ≤ 3 with real coefficients, that is $P_3 = \{at^3 + bt^2 + ct + d : a, b, c, d \in \mathbb{R}\}$, and let W be the subspace of P_3 spanned by the following the polynomials

$$t^3 + t^2 - 2t + 1, \quad t^2 + 1, \quad t^3 - 2t, \quad 2t^3 + 3t^2 - 4t + 3.$$

(i) Find a basis for W.

(ii) What is dimension of W?

(iii) Is W equal to P_3?

Solution:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ -2 & 0 & -2 & -4 \\ 1 & 1 & 0 & 3 \end{bmatrix} \begin{matrix} -R_1 + R_2 \\ 2R_1 + R_3 \\ -R_1 + R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \; -R_2 + R_4 \; \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

As the columns of R containing leading entries are the first and the second columns, the first and the second polynomial form a basis for W. That is, the set

$$\{t^3 + t^2 - 2t + 1, \; t^2 + 1\}$$

is a basis for W. Thus the dimension of W is 2. Moreover, $W \neq P_3$ because their dimensions are not equal.

Question 4. Given the 3×6 matrix $A = \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 \\ 3 & 9 & -5 & -13 & 6 & 3 \\ -2 & -6 & 8 & 18 & -4 & -1 \end{bmatrix}$,

(i) Find the <u>reduced</u> row echelon form R of A.

(ii) Find a basis for the row space of A.

(iii) Find a basis for the column space of A.

(iv) Find a basis for the nullspace of A.

(v) Find rank(A) and nullity(A).

Solution: (i): We reduce A as below:

$$\begin{matrix} -3R_1 + R_2 \\ 2R_1 + R_3 \end{matrix} \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 & 1 \end{bmatrix} \begin{matrix} -4R_2 + R_3 \\ 2R_2 + R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-R_3 + R_1 \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R.$$

(ii): The nonzero rows of R form a basis for the row space of A. Hence, the set

$$\{ \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \}$$

is a basis for the row space of A. So rank$(A) = 3$.

(iii): Since the columns of R containing leading entries are its 1^{st}, 3^{rd}, and 6^{th} columns, it follows that the 1^{st}, 3^{rd}, and 6^{th} columns of A form a basis for the column space of A. Hence, the set

$$\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \}$$

is a basis for the column space of A. Note that this basis contains 3 elements as we expected because the dimensions of the row and column space are equal, and this common number is said to be the rank of the matrix.

(iv): We must find a basis for the solution space of $A\vec{x} = \vec{0}$. The reduced row echelon form of the augmented matrix of the system $A\vec{x} = \vec{0}$ is

$$[R|\vec{0}] = \left[\begin{array}{cccccc|c} 1 & 3 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ \underbrace{0}_{x_1} & \underbrace{0}_{x_2} & \underbrace{0}_{x_3} & \underbrace{0}_{x_4} & \underbrace{0}_{x_5} & \underbrace{1}_{x_6} & 0 \end{array}\right]$$

The unknowns x_2, x_4 and x_5 are free. So the solution set of $A\vec{x} = \vec{0}$ is given by

$$x_1 = -3x_2 + x_4 - 2x_5, \quad x_2 = \text{free} \quad x_3 = -2x_4, \quad x_4 = \text{free}, \quad x_5 = \text{free}, \quad x_6 = 0.$$

Hence, the set

$$\left\{ \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{x_2=1; x_4=x_5=0}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{x_4=1; x_2=x_5=0}, \underbrace{\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{x_5=1; x_2=x_4=0} \right\}$$

is a basis for the nullspace of A. Thus nullity$(A) = 3$.

(v): We have already observed that rank$(A) = 3$ and nullity$(A) = 3$.

Question 5. Complete each of the following sentences by using a correct "*number*", or by using a correct word which is one of the words "*row, column, null, consistent, inconsistent, dependent, independent*".

(1) If P_n is the vector space of all polynomials of degree $\leq n$ with real coefficients, then the dimension of P_5 is **6**.

(2) The column space of a 3×5 matrix is a subspace of $\mathbb{R}^{\wedge}$ **3**.

(3) The column vectors of a 5×7 matrix are linearly **dependent**.

(4) The largest possible value for the nullity of a nonzero 5×3 matrix is **2**.

(5) The **row** vectors of a 8×6 matrix can not be linearly independent.

(6) Let A be a matrix and $\vec{b}$ be a column vector satisfying $A\vec{b} = \vec{0}$. Then, $\vec{b}$ is in the **null** space of A.

(7) Let A be a square matrix whose determinant is nonzero. Then, the row vectors of A are linearly **independent**.

(8) Elementary row operations may change the **column** space of a matrix.

(9) Let A be a nonzero 3×3 matrix satisfying nullity$(A) \geq$ rank(A). Then, the rank of A is **1**.

(10) Let $\vec{u}$ and $\vec{v}$ be two linearly independent vectors of a vector space V. If a vector $\vec{w}$ of V is not in span$\{\vec{u}, \vec{v}\}$ then the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly **independent** .

Solution: See above.