Question 1. (a) Let \( B = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \) and \( C = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 2 & 0 \\ 8 & 7 & 1 \end{bmatrix} \). Compute the following determinants:

\[
\begin{align*}
\det\left(\frac{1}{2}B^{12}\right), & \quad \det(3C^{-1}), & \quad \det(B^{19}C^3), & \quad \det\left((2B)C^7\right).
\end{align*}
\]

(b) Let \( A \) be a \( 3 \times 3 \) matrix with \( \det(A) = -1 \). Express the matrix \( \text{adj}(2A) \) in terms of \( A^{-1} \).

Solution: (a): As \( B \) and \( C \) are triangular matrices we easily calculate their determinants as the products of entries on the main diagonal of them. Indeed, \( \det(B) = -1 \) and \( \det(C) = 2 \). Then, by the properties of determinant function we have:

\[
\det\left(\frac{1}{2}B^{12}\right) = \left(\frac{1}{2}\right)^{12}\det(B)^{12} = \frac{1}{8}
\]

\[
\det(3C^{-1}) = 3^3\frac{1}{\det(C)} = \frac{27}{2}
\]

\[
\det(B^{19}C^3) = (\det(B))^{19}(\det(C))^3 = (-1)^{19}(2)^3 = (-1)(8) = -8.
\]

\[
\det\left((2B)C^7\right) = \det\left((2B)C^7\right) = 2^3\det(B)\det(C) = 2^3(-1)(2) = -16
\]

(b): Recall that for any square matrix \( B \) we have the formula

\[
\text{adj}(B)B = \det(B)I.
\]

Putting \( B = 2A \) we get

\[
\text{adj}(2A)(2A) = \det(2A)I_3 = 2^3\det(A)I_3 = -8I_3.
\]

This gives:

\[
\text{adj}(2A)A = -4I_3 \implies \text{adj}(2A)A A^{-1} = -4I_3 A^{-1}.
\]

Consequently,

\[
\text{adj}(2A) = -4A^{-1}.
\]
Question 2. Let $M_{2 \times 2}$ be the vector space of all $2 \times 2$ matrices whose entries are real numbers, and let $W$ be the subset $\{A \in M_{2 \times 2} : A^T = -A\}$ of $M_{2 \times 2}$.

(i) Show that $W$ is a subspace of $M_{2 \times 2}$.

(ii) Find a basis for $W$. What is the dimension of $W$?

Solution: (i): As $0_{2 \times 2}^T = -0_{2 \times 2}$, the set $W$ is a nonempty subset of $M_{2 \times 2}$.

Take any two elements $A$ and $B$ from $W$. Then, $A^T = -A$ and $B^T = -B$, so

$$(A + B)^T = A^T + B^T = (-A) + (-B) = -(A + B),$$

implying that $A + B \in W$. Thus $W$ is closed under addition.

Take any element $A$ from $W$ and any real number $c$. Then, $A^T = -A$ and so

$$(cA)^T = cA^T = c(-A) = -(cA),$$

implying that $cA \in W$. Thus $W$ is closed under scalar multiplication.

(b): An element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $M_{2 \times 2}$ is in $W$ if and only if $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$, equivalently

$a = -a, \ c = -b, \ b = -c, \ d = -d, \ \text{ or } \ a = d = 0, \ c = -b.$

Hence, any element of $W$ is of the form

$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$

This means that the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ spans $W$. Recall that a single vector is linearly independent if and only if it is nonzero. The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, being nonzero, must be linearly independent.

Consequently, the set $\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \}$ is a basis for $W$ and the dimension of $W$ is 1.
Question 3. Let $P_3$ is the vector space of all polynomials of degree $\leq 3$ with real coefficients, that is $P_3 = \{at^3 + bt^2 + ct + d : a, b, c, d \in \mathbb{R}\}$, and let $W$ be the subspace of $P_3$ spanned by the following the polynomials

\[ t^3 + t^2 - 2t + 1, \quad t^2 + 1, \quad t^3 - 2t, \quad 2t^3 + 3t^2 - 4t + 3. \]

(i) Find a basis for $W$.

(ii) What is dimension of $W$?

(iii) Is $W$ equal to $P_3$?

Solution:

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
1 & 1 & 0 & 3 \\
-2 & 0 & -2 & -4 \\
1 & 1 & 1 & 3
\end{bmatrix}
- R_1 + R_2
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1
\end{bmatrix}
- R_2 + R_4
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
= R
\]

As the columns of $R$ containing leading entries are the first and the second columns, the first and the second polynomial form a basis for $W$. That is, the set

\[ \{t^3 + t^2 - 2t + 1, \ t^2 + 1\} \]

is a basis for $W$. Thus the dimension of $W$ is 2. Moreover, $W \neq P_3$ because their dimensions are not equal.
Question 4. Given the $3 \times 6$ matrix $A = \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 \\ 3 & 9 & -5 & -13 & 6 & 3 \\ -2 & -6 & 8 & 18 & -4 & -1 \end{bmatrix}$.

(i) Find the reduced row echelon form $R$ of $A$.

(ii) Find a basis for the row space of $A$.

(iii) Find a basis for the column space of $A$.

(iv) Find a basis for the nullspace of $A$.

(v) Find $\text{rank}(A)$ and $\text{nullity}(A)$.

Solution: (i): We reduce $A$ as below:

$\begin{align*}
-3R_1 + R_2 & \quad \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 8 & 0 & 1 \end{bmatrix} \\
2R_1 + R_3 & \quad \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 \end{bmatrix} \\
-2R_2 + R_3 & \quad \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
-4R_2 + R_3 & \quad \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
2R_2 + R_1 & \quad \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
\end{align*}$

Thus \[R \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.\]

(ii): The nonzero rows of $R$ form a basis for the row space of $A$. Hence, the set

\[\{ \begin{bmatrix} 1 & 3 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \} \]

is a basis for the row space of $A$. So $\text{rank}(A) = 3$.

(iii): Since the columns of $R$ containing leading entries are its $1$st, $3$rd, and $6$th columns, it follows that the $1$st, $3$rd, and $6$th columns of $A$ form a basis for the column space of $A$. Hence, the set

\[\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \} \]

is a basis for the column space of $A$. Note that this basis contains 3 elements as we expected because the dimensions of the row and column space are equal, and this common number is said to be the rank of the matrix.

(iv): We must find a basis for the solution space of $A\vec{x} = \vec{0}$. The reduced row echelon form of the augmented matrix of the system $A\vec{x} = \vec{0}$ is

\[\begin{bmatrix} R | \vec{0} \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & -5 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 \end{bmatrix} \]

The unknowns $x_2$, $x_4$, and $x_5$ are free. So the solution set of $A\vec{x} = \vec{0}$ is given by

\[x_1 = -3x_2 + x_4 - 2x_5, \quad x_2 = \text{free}, \quad x_3 = -2x_4, \quad x_4 = \text{free}, \quad x_5 = \text{free}, \quad x_6 = 0.\]
Hence, the set
\[
\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]
\[x_2=1; x_4=x_5=0 \quad x_4=1; x_2=x_5=0 \quad x_5=1; x_2=x_4=0\]
is a basis for the nullspace of \( A \). Thus \( \text{nullity}(A) = 3 \).

(v): We have already observed that \( \text{rank}(A) = 3 \) and \( \text{nullity}(A) = 3 \).
**Question 5.** Complete each of the following sentences by using a correct “number”, or by using a correct word which is one of the words “row, column, null, consistent, inconsistent, dependent, independent”.

(1) If $P_n$ is the vector space of all polynomials of degree $\leq n$ with real coefficients, then the dimension of $P_5$ is $6$.

(2) The column space of a $3 \times 5$ matrix is a subspace of $\mathbb{R}^3$.

(3) The column vectors of a $5 \times 7$ matrix are linearly dependent.

(4) The largest possible value for the nullity of a nonzero $5 \times 3$ matrix is $2$.

(5) The **row** vectors of a $8 \times 6$ matrix cannot be linearly independent.

(6) Let $A$ be a matrix and $\vec{b}$ be a column vector satisfying $A\vec{b} = \vec{0}$. Then, $\vec{b}$ is in the **null** space of $A$.

(7) Let $A$ be a square matrix whose determinant is nonzero. Then, the row vectors of $A$ are linearly independent.

(8) Elementary row operations may change the **column** space of a matrix.

(9) Let $A$ be a nonzero $3 \times 3$ matrix satisfying $\text{nullity}(A) \geq \text{rank}(A)$. Then, the rank of $A$ is $1$.

(10) Let $\vec{u}$ and $\vec{v}$ be two linearly independent vectors of a vector space $V$. If a vector $\vec{w}$ of $V$ is not in $\text{span}\{\vec{u}, \vec{v}\}$ then the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

**Solution:** See above.