1) (20 pts.) Show all your work

Let $A = \begin{bmatrix} 1 & 6 & -3 & 4 \\ 1 & 7 & -1 & 7 \\ 1 & 8 & 1 & 10 \\ 1 & 0 & -15 & 0 \end{bmatrix}$.

a) Find the reduced row echelon form of *A*.

Reduced row echelon form of A is $R = \begin{bmatrix} 1 & 0 & -15 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) Find a basis for the row space of *A*.

$B_R = \{(1,0,-15,0),(0,1,2,0),(0,0,0,1)\}$ is a basis for the row space.

c) Find a basis for the column space of *A*.

$B_C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 10 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space of *A*.

d) Find a basis for the null space of *A*.

A is row-equivalent to *R* then *Ax=0* and *Rx=0* have the same solution space.
Using *R*,

$x_1 - 15x_3 = 0$

$x_2 + 2x_3 = 0$

$x_4 = 0.$

x_3 is a free variable,

$x_1 = 15t$

$x_2 = -2t$

$x_3 = t$

$x_4 = 0$

where $t \in R$. Thus

$v = \begin{bmatrix} 15 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ spans the null space of *A* and $B_N = \{v\}$ is a basis for the null space of *A*.

e) Find the rank and the dimension of the null space of *A*.

Rank(A) = 3, *Nullity(A)= 1.*

2) **(20 pts.)** Let *W* be a subspace of R^3 spanned by the vector $(1,-1,1)$.

a) Find a basis for the orthogonal complement $W^\perp$ of *W*. **Show all your work.**

$u = (x, y, z) \in W^\perp \Leftrightarrow u \cdot (1,-1,1) = 0 \Leftrightarrow x - y + z = 0.$

Set $y=t$, $z=s$ then $x=s-t$ where $s,t \in R$. Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s-t \\ t \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $W^\perp$.

b) Find a basis for R^3 **consisting of vectors in *W* and** $W^\perp$**, only. Explain your solution.**

The vectors $(1,-1,1), (1,0,1), (-1,1,0)$ form a basis for R^3. $(1,0,1), (-1,1,0)$ are linearly independent by **a).** Mutually orthogonal vectors are linearly independent hence $(1,-1,1), (1,0,1), (-1,1,0)$ are linearly independent and 3 linearly independent vectors in R^3 form a basis.

3) (20 pts.) Show all your work.

Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors in R^n such that for every pair of scalars *x* and *y*, the vectors $x\vec{u}+y\vec{v}$ and $4y\vec{u}-9x\vec{v}$ are orthogonal, and $\left|\vec{u}\right| = 6$.

Compute

a) $\vec{u}\bullet\vec{v}$ **=?**

$(x\vec{u}+y\vec{v})\bullet(4y\vec{u}-9x\vec{v}) = 0$ for all *x* and *y* in *R*.

$4xy\left|\vec{u}\right|^2 + \left(4y^2-9x^2\right)\vec{u}\bullet\vec{v} - 9xy\left|\vec{v}\right|^2 = 0$ for all *x* and *y* in *R*.

Choose *x=0, y=1*. Then $\vec{u}\bullet\vec{v} = 0$.

b) $\left|\vec{v}\right|$ **=?**

Taking *x=y=1*,

$4\left|\vec{u}\right|^2 - 9\left|\vec{v}\right|^2 = 0.$ Put $\left|\vec{u}\right| = 6.$ Then $\left|\vec{v}\right|^2 = 16 \Rightarrow \left|\vec{v}\right| = 4.$

c) $\left|2\vec{u}+3\vec{v}\right|$ **=?**

$\left|2\vec{u}+3\vec{v}\right|^2 = 4\left|\vec{u}\right|^2 + 9\left|\vec{v}\right|^2 + 12\vec{u}\bullet\vec{v} = 4\cdot 36 + 9\cdot 16 = 288$

and $\left|2\vec{u}+3\vec{v}\right| = \sqrt{288} = 12\sqrt{2}.$

4) (20 pts.) Show all your work.

Let A be an nxn matrix such that $A^5 = 0$ and $A^4 \neq 0$.

a) Show that there is a nonzero $\vec{v} \in R^n$ such that $A^4 \vec{v} \neq 0$.

$A^4 \neq 0 \Rightarrow Null(A^4) \neq R^n$.Hence there exists $\vec{v} \in R^n$ such that $\vec{v} \notin Null(A^4)$

i.e. $A^4 \vec{v} \neq 0$.

Or

$$A^4 \vec{v} \neq 0 \Rightarrow 0 \neq A^4 = A^4 I = \begin{bmatrix} A^4 e_1 & A^4 e_2 & A^4 e_3 & \cdots & A^4 e_n \end{bmatrix}$$

i.e. $A^4 e_1 \neq 0$ or $A^4 e_2 \neq 0$ or $A^4 e_3 \neq 0$ or …. $A^4 e_n \neq 0$.

b) Let v be as in part **a)** .Show that the vectors $\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}, A^4\vec{v}$ are linearly independent.

Let $c_0, c_1, c_2, c_3, c_4 \in R$ be such that

(1) $c_0 \vec{v} + c_1 A \vec{v} + c_2 A^2 \vec{v} + c_3 A^3 \vec{v} + c_4 A^4 \vec{v} = 0.$

Since $A^5 = 0 \Rightarrow A^k = 0$ for all $k \geq 5$.

Multiply the equation (1) by A^4 from left we obtain

$$c_0 A^4 \vec{v} = 0 \Rightarrow c_0 = 0. \text{Then}$$

(2) $c_1 A \vec{v} + c_2 A^2 \vec{v} + c_3 A^3 \vec{v} + c_4 A^4 \vec{v} = 0.$

Multiply the equation (2) by A^3 from left we obtain

$$c_1 A^4 \vec{v} = 0 \Rightarrow c_1 = 0.$$

Similarly we get $c_2 = c_3 = c_4 = 0$ and these 5 vectors are linearly independent.

5)(20 pts.) Show all your work.

a) Show that two similar square matrices *A* and *B* have the same eigenvalues with the same multiplicities.

A and *B* are similar if there exists an invertible matrix *P* such that $P^{-1}AP = B$.

$$|B-\lambda I| = |P^{-1}AP - \lambda P^{-1}P| = |P^{-1}AP - P^{-1}\lambda P| = |P^{-1}(A-\lambda I)P| = |P^{-1}||A-\lambda I||P|$$

$$= |P|^{-1}|P||A-\lambda I| = |A-\lambda I|.$$

Thus *A* and *B* have the same characteristic polynomials. Therefore *A* and *B* have the same eigenvalues with the same multiplicities.

b) Given the matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ 5 & -1 & -5 \\ -1 & 0 & 2 \end{bmatrix}$, calculate real numbers *a,c,d* such that

$P^{-1}AP = \begin{bmatrix} a & 0 & 0 \\ 0 & c & d \\ 0 & -d & c \end{bmatrix}$ for some invertible matrix *P*.

Let $B = \begin{bmatrix} a & 0 & 0 \\ 0 & c & d \\ 0 & -d & c \end{bmatrix}$. Then by **a)** , A and B have the same characteristic polynomials.

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & -1 \\ 5 & -1-\lambda & -5 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (-1)\begin{vmatrix} -1 & -1 \\ -1-5\lambda & -5 \end{vmatrix} + (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 5 & -1-\lambda \end{vmatrix}$$

$$= -(\lambda-1)(\lambda^2-2\lambda+2) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 1+i,\ \lambda_3 = 1-i.$$

$$|B-\lambda I| = \begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & c-\lambda & d \\ 0 & -d & c-\lambda \end{vmatrix} = (a-\lambda)\left((c-\lambda)^2+d^2\right) = 0 \Rightarrow \lambda_1 = a, \lambda_{2,3} = c \pm id.$$

Thus *a=1,c=1* and *d=1*.

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THIS IS AN EXTRA PAGE FOR CALCULATIONS.DO NOT SEPARATE.