

1) (20 pts.) Solve the following initial value problem

$$\frac{dy}{dx} = \begin{cases} \frac{y+x}{x} & \text{if } x \geq 2 \\ \frac{1}{2}(y+x) & \text{if } x < 2 \end{cases} \quad \text{and} \quad y(0) = 0.$$

Show all your work.

It was a homework question. Look at the solutions.

2) (20 pts.) Solve the differential equation  $\frac{dy}{dx} = \frac{x+3y-5}{x-y-1}$ . Show all your work.

Use  $x=u+h$  transformation  
 $y=v+k$

$$\begin{cases} h+3k-5=0 \\ h-k-1=0 \end{cases} \Rightarrow \begin{cases} h=1 \\ k=2 \end{cases} \Rightarrow \begin{cases} x=u+2 \\ y=v+1 \end{cases}$$

$$\frac{dy}{dx} = \frac{x+3y-5}{x-y-1} \text{ becomes } \frac{dv}{du} = \frac{u+3v}{u-v} \text{ homog.}$$

$$\frac{dv}{du} = \frac{1+3\frac{v}{u}}{1-\frac{v}{u}} \quad (\text{I})$$

$$\text{Say } z = \frac{v}{u} \Rightarrow v = uz \text{ & } \frac{dv}{du} = z + u \frac{dz}{du}.$$

Eqn (I) becomes:

$$z+u \frac{dz}{du} = \frac{1+3z}{1-z} \text{ gives } \frac{1-z}{(1+z)^2} dz = \frac{du}{u} \quad (\text{II}).$$

$$\frac{1-z}{(1+z)^2} = \frac{-1}{1+z} + \frac{2}{(1+z)^2} \text{ using partial fractions.}$$

Thus using (II)

$$\int \frac{1-z}{(1+z)^2} dz = \int \frac{du}{u}$$

$$\int -\frac{1}{1+z} dz + \int \frac{2}{(1+z)^2} dz = \int \frac{du}{u}$$

$$-\ln|1+z| - \frac{2}{1+z} = \ln|u| + C, \text{ put } z = \frac{v}{u} = \frac{y-1}{x-2}$$

$$\ln|x+y-3| + \frac{2(x-2)}{x+y-3} = C$$

(I)

- 3) (20 pts.) Solve  $y'' + \frac{1}{y}(y')^2 = ye^{-y}(y')^3$  and find the general solution. Show all your work.

The independent variable "x" is missing.

Put  $y' = u(y)$  &  $y'' = u'(y) \cdot y' = u' \cdot u$

(I) becomes

$$u' \cdot u + \frac{1}{y} u^2 = ye^{-y} \cdot u^3$$

$$u' + \frac{1}{y} u = ye^{-y} u^2 \quad \text{is Bernoulli with } n=2.$$

Divide both sides by  $u^2$ :

$$(II) u^{-2} u' + \frac{1}{y} u^{-1} = ye^{-y} \quad \text{& say } v = u^{-1}, \frac{dv}{dy} = -\frac{1}{u^2} \cdot \frac{du}{dy}$$

(II) becomes:

$$u^2 \cdot \left( -u^2 \frac{dv}{dy} \right) + \frac{1}{y} v = ye^{-y} \quad \frac{du}{dy} = -u^2 \frac{dv}{dy}$$

$$(III) \frac{dv}{dy} - \frac{1}{y} v = -ye^{-y} \quad \text{linear}$$

$f(y) = e^{\int -\frac{1}{y} dy} = y^{-1}$  is an integrating factor.

(III) becomes

$$\frac{d}{dy} \left( \frac{1}{y} v \right) = -e^{-y} \quad \text{gives} \quad \frac{1}{y} v = e^{-y} + C$$

$$v = y e^{-y} + Cy$$

$$\frac{1}{u} = y e^{-y} + Cy$$

$$\frac{dx}{dy} = y e^{-y} + Cy$$

$$dx = (y e^{-y} + Cy) dy$$

$$x = x(y) = -e^{-y}(y+1) + C \frac{y^2}{2} + D$$

4) (20 pts.) Use Gauss-Jordan elimination to solve the given system of equations

$$\begin{aligned}x_1 - x_2 - x_3 + 2x_4 &= 1 \\-3x_1 + 2x_2 + x_3 - 7x_4 &= -5 \\5x_1 + x_2 + 7x_3 + 17x_4 &= 18\end{aligned}$$

**Warning:** There is no partial for this question, so make sure that your operations are right and your final answer is correct.

$$[A \ b] = \left[ \begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ -3 & 2 & 1 & -7 & -5 \\ 5 & 1 & 7 & 17 & 18 \end{array} \right] \xrightarrow{3R_1+R_2} \left[ \begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & -1 & -2 & -1 & -2 \\ 5 & 1 & 7 & 17 & 18 \end{array} \right] \xrightarrow{-5R_1+R_3} \left[ \begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & -1 & -2 & -1 & -2 \\ 0 & 6 & 12 & 7 & 13 \end{array} \right]$$

$$\xrightarrow[-R_2+R_1]{, R_2+R_3} \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 3 & 3 \\ 0 & -1 & -2 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow[R_3+R_2]{-3R_3+R_1} \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow[-R_2]{} \left[ \begin{array}{ccccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{aligned}x_1 &= -t & ; \quad t \in \mathbb{R} \\x_2 &= 1-2t \\x_3 &= t \\x_4 &= 1\end{aligned}$$

$$S = \{(x_1, x_2, x_3, x_4) = (-t, 1-2t, t, 1), t \in \mathbb{R}\}$$

5) a) (10 pts.)

Are there a  $225 \times 112$  matrix  $A$  and a  $112 \times 225$  matrix  $B$  such that  $AB = I_{225 \times 225}$ ?

Prove your answer.

Suppose that we can find a  $225 \times 112$  matrix  $A$  and  $112 \times 225$  matrix  $B$  s.t.  $AB = I_{225 \times 225}$ .

Since  $B$  is a  $112 \times 225$  matrix the homog. system  $BX = 0$  has  $\infty$ -ly many solutions. (Why?) Let  $X_0 \neq 0$  be a non-zero solution of  $BX = 0$ . Then  $BX_0 = 0$ .

Using  $AB = I$

$$(AB)X_0 = I X_0$$

$$\stackrel{||}{A(BX_0)} = X_0$$

$$\stackrel{||}{A \cdot 0} = X_0$$

$0 = X_0$  gives a contradiction.

Thus our assumption is not true. There are no such matrices.

b) (10 pts.) True or false: If true, give a proof, if false give a counter example.

If  $A$  and  $B$  are  $2 \times 2$  matrices then  $(A+B)^2 = A^2 + 2AB + B^2$ .

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$$

$$\Rightarrow A^2 + 2AB + B^2 \Leftrightarrow AB = BA.$$

Therefore the statement is not true.

e.g.  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$A^2 = A, B^2 = 0_{2 \times 2}, AB = B, BA = 0_{2 \times 2}$$

$$\text{then } (A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A+B$$

but

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq (A+B)^2$$