2014-2015- Fall

SOLUTIONS TO MDT 2

1) Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ by **using its adjoint matrix.** **Verify your solution.**

$\det A = -1$ & $A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = M_{11} = -5$ $\quad A_{12} = -(-4) = 4$ $\quad A_{13} = 7$

$A_{21} = -(-1) = 1$ $\quad A_{22} = -1$ $\quad A_{23} = -2$

$A_{31} = 1$ $\quad A_{32} = -1$ $\quad A_{33} = -1$

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} -5 & 1 & 1 \\ 4 & -1 & -1 \\ 7 & -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A) = \begin{bmatrix} 5 & -1 & -1 \\ -4 & 1 & 1 \\ -7 & 2 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 & -1 \\ -4 & 1 & 1 \\ -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5-4 & -1+1 & -1+1 \\ 15-8-7 & -3+2+2 & -3+2+1 \\ 5-12+7 & -1+3-2 & -1+3-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 5 & -1 & -1 \\ -4 & 1 & 1 \\ -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 5-3-1 & 5-2-3 & -1+1 \\ -4+3+1 & -4+2+3 & 1-1 \\ -7+6+1 & -7+4+3 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: $A_{ij} \neq a_{ij} \cdot (-1)^{i+j} M_{ij}$

If you wrote the above eqn. then you lost very many points!!!

2) a) **(10 pts.)** Show that $S = \{(x, y, z) \in R^3 \mid y - 2z = 0\}$ is a subspace of R^3.

$(0,0,0) \in S \Rightarrow S \neq \emptyset$

Let $u_1 = (x_1, y_1, z_1)$ & $u_2 = (x_2, y_2, z_2)$ be in S, $c \in \mathbb{R}$.

1) $u_1 \in S \Rightarrow y_1 - 2z_1 = 0$

$u_2 \in S \Rightarrow y_2 - 2z_2 = 0$

$$(y_1 + y_2) - 2(z_1 + z_2) = 0$$

Since $u_1 + u_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ & $(y_1 + y_2) - 2(z_1 + z_2) = 0$

$\Rightarrow u_1 + u_2 \in S$.

2) $u_1 \in S \Rightarrow y_1 - 2z_1 = 0 \Rightarrow cy_1 - 2cz_1 = 0 \Rightarrow cu_1 \in S$.

$\therefore$ S is a subspace of $\mathbb{R}^3$.

b) **(7 pts.)** Find a basis **B** for S. What is the dimension of S?

$S = \{(x, y, z) \in \mathbb{R}^3 \mid y = 2z\} = \{(x, 2z, z) \in \mathbb{R}^3 \mid x, z \in \mathbb{R}\}$

$= \{x\underbrace{(1, 0, 0)}_{v_1} + z\underbrace{(0, 2, 1)}_{v_2} \mid x, z \in \mathbb{R}\} = \text{span}\{v_1, v_2\}$.

$c_1 v_1 + c_2 v_2 = 0_V \Rightarrow c_1(1, 0, 0) + c_2(0, 2, 1) = (0, 0, 0)$

$$\left.\begin{aligned} c_1 &= 0 \\ 2c_2 &= 0 \\ c_2 &= 0 \end{aligned}\right\} c_1 = c_2 = 0$$

Thus $B = \{v_1, v_2\}$ is a basis for S.

c) **(8 pts.)** Extend **B** to find a basis for R^3 <u>**using the standard basis vectors**</u> of R^3.

$\{i, j, k\}$.

$$\begin{array}{ccccc} v_1 & v_2 & i & j & k \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

lin ind.

$\therefore$ $\{v_1, v_2, j\}$ are lin. ind. & they form a basis.

<u>NOTE</u>: $v_1 = i$. You can perform the same process for only v_1, v_2, j, k.

3) Let $A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$.

a) **(8 pts.)** Find the **reduced echelon** form of A.

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{-R_2+R_3 \\ -R_2+R_4}]{-2R_2+R_1} \begin{bmatrix} 0 & 2 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 \end{bmatrix} \xrightarrow[-2R_3+R_4]{-2R_3+R_1} \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{-R_1 \\ -R_4}]{R_1+R_2} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_4+R_1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_i \leftrightarrow R_j} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R.$$

lin. ind.

b) **(5 pts.)** Find a basis for the row space of A.

$\mathcal{B}_R = \{[1\ 0\ 0\ 0], [0\ 1\ 0\ 0], [0\ 0\ 1\ 0], [0\ 0\ 0\ 1]\}$

c) **(5 pts.)** Find a basis for the column space of A.

$$\mathcal{B}_C = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

d) **(5 pts.)** Find a basis for the null space of A.

$x_1 = x_2 = x_3 = x_4 = 0$ is the unique solution of $AX = 0$.

Thus, $Null(A) = \{0_V\}$. Since 0_V is lin. dependent,

$\mathcal{B}_N = \emptyset$.

e) **(2 pts.)** What is the rank of A and nullity of A?

$rank(A) = 4$

$nullity(A) = 0$

4) **a) (20 pts.)** Let $S = \{v_1, v_2, \ldots, v_n\}$ and $T = \{u_1, u_2, \ldots, u_m\}$ be two sets in a vector space V such that S is a subset of $span(T)$. Prove that $span(S)$ is also a subset of $span(T)$.

$span(T) = \{c_1u_1 + c_2u_2 + \cdots + c_mu_3 \mid c_i \in \mathbb{R}, i = 1, \ldots, m\}$.

$S \subset span(T) \Rightarrow v_i = c_{i1}u_1 + c_{i2}u_2 + \cdots + c_{im}u_m$ for every $i = 1, \ldots, n$.

Let $w \in span(S)$. Then

$w = d_1v_1 + d_2v_2 + \cdots + d_nv_n \quad ; \quad d_i \in \mathbb{R}, \quad i = 1, \ldots, n.$

$= d_1(c_{11}u_1 + c_{12}u_2 + \cdots + c_{1m}u_m) + d_2(c_{21}u_1 + c_{22}u_2 + \cdots + c_{2m}u_m)$

$+ \cdots + d_n(c_{n1}u_1 + c_{n2}u_2 + \cdots + c_{nm}u_m)$

$= (\overbrace{d_1c_{11} + d_2c_{21} + \cdots + d_nc_{n1}}^{K_1})u_1 + (\overbrace{d_1c_{12} + d_2c_{22} + \cdots + d_nc_{n2}}^{K_2})u_2$

$+ \cdots + (\underbrace{d_1c_{1m} + d_2c_{2m} + \cdots + d_nc_{nm}}_{K_m})u_m$

$= K_1u_1 + K_2u_2 + \cdots + K_mu_m \in span(T) \; ; \; K_i \in \mathbb{R}, \; i = 1, \ldots, m.$

$\therefore \; span(S) \subset span(T)$

NOTES:

1) $span(S) \not\subset S$
2) You cannot talk about the dimensions of S & T!!!
3) We don't know whether S & T are lin. ind. or not.

b) (5 pts.) Is $W = \{A \in M_{3x3} \mid \det A \neq 0\}$ a subspace of M_{3x3}? If so, find a basis for W. If not so, explain why not.

$0_{3\times3} \notin W$ because $\det(0_{3\times3}) = 0$.

Hence, W is not a subspace of $M_{3\times3}$.