6 January 2015
Time: 9:00-11:00

MATH 225
FINAL

SURNAME	SOLUTIONS
NAME	
STUDENT NO	
SIGNATURE	
Department	

IMPORTANT:

- This exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding question. Be neat.
- **Show all your work. Correct answers without sufficient explanation might not get full grades.**
- Calculators and dictionaries are **not** allowed.
- Close your cellular phones.

1	2	3	4	5	TOTAL
20	20	20	20	20	100

$\frac{\partial M}{\partial y} = x^3 - 2y$ ⇒ The d.e. is not exact

$\frac{\partial N}{\partial x} = -4x^3 - y$

$\overbrace{x^3y - y^2}^{M}$

1) Consider the differential equation $y(x^3 - y)dx - x(x^3 + y)dy = 0$.

a) **(5 pts.)** Find a real number number α so that $\rho(x,y) = \frac{x^\alpha}{y^3}$ is an integrating factor of this equation. **Show all your work. Correct answers without sufficient explanation might not get full grades.**

$$\frac{x^\alpha}{y^3}(x^3y - y^2)\,dx + \frac{x^\alpha}{y^3}(-x^4 - xy)\,dy = 0$$

$$\underbrace{\left(\frac{x^{3+\alpha}}{y^2} - \frac{x^\alpha}{y}\right)}_{\tilde{M}}dx + \underbrace{\left(\frac{-x^{4+\alpha}}{y^3} - \frac{x^{\alpha+1}}{y^2}\right)}_{\tilde{N}}dy = 0$$

$$\frac{\partial \tilde{M}}{\partial y} = -\frac{2x^{3+\alpha}}{y^3} + \frac{x^\alpha}{y^2} = \frac{\partial \tilde{N}}{\partial x} = \frac{-(4+\alpha)x^{3+\alpha}}{y^3} - \frac{(\alpha+1)x^\alpha}{y^2}$$

$\Longleftrightarrow$ $-2 = -(4+\alpha)$ & $-(\alpha+1) = 1$

$\boxed{\alpha = -2}$

b) (10 pts.) For this α, solve the given differential equation which satisfies $y(1) = -1$. **Show all your work. Correct answers without sufficient explanation might not get full grades.**

When $\alpha = -2$: $\underbrace{\left(\frac{x}{y^2} - \frac{1}{x^2y}\right)}_{M}dx + \underbrace{\left(\frac{-x^2}{y^3} - \frac{1}{xy^2}\right)}_{N}dy = 0$ $(*)$

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^3} + \frac{1}{x^2y^2} = \frac{\partial N}{\partial x} = \frac{-2x}{y^3} + \frac{1}{x^2y^2} \Rightarrow (*) \text{ is exact.}$$

$$F(x,y) = \int\left(\frac{x}{y^2} - \frac{1}{x^2y}\right)dx = \frac{x^2}{2y^2} + \frac{1}{xy} + g(y)$$

$$\frac{\partial F}{\partial y} = -\frac{x^2}{y^3} - \frac{1}{xy^2} + g'(y) = N(x,y) = -\frac{x^2}{y^3} - \frac{1}{xy^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

$\therefore$ $\frac{x^2}{2y^2} + \frac{1}{xy} + C = 0$

$y(1) = -1 \Rightarrow \frac{1}{2} - 1 + C = 0 \Rightarrow \boxed{C = \frac{1}{2}}$

$$\frac{x^2}{2y^2} + \frac{1}{xy} + \frac{1}{2} = 0.$$

2) (**20 pts.**) Find a $4x5$ **reduced echelon** matrix R **with all the following properties**:

a) $rank(R) = 3$,

b) the leading entries of the first two rows appear in the first two columns,

c) the $4x4$ non homogeneous system with augmented matrix R is **inconsistent**,

d) $(2,1,1,0,0)$ and $(-1,0,2,-1,0)$ are solutions of the homogeneous system with coefficient matrix R.

THERE IS NO PARTIAL CREDIT FOR THIS QUESTION.YOU WILL GET EITHER 0 OR 20.

Show all your work. Correct answers without sufficient explanation might not get full grades.

Using a), b) & c) we see

$$R = \begin{bmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To find a, b, c & d, use d)

$$\begin{bmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+a \\ 1+c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = -2 \\ c = -1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & b & 0 \\ 0 & 1 & -1 & d & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1-4-b \\ -2-d \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b = -5 \\ d = -2 \end{matrix}$$

$$\therefore R = \begin{bmatrix} 1 & 0 & -2 & -5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3) **a) (10 pts.)** Is there a $2x5$ matrix A whose null space is

$$Null(A) = \left\{ (a,b,c,d,e) \in R^5 \mid a = 3b\,,\, c = d = e \right\}$$? Explain your answer.

Show all your work. Correct answers without sufficient explanation might not get full grades.

$Null(A) = \{(3b, b, c, c, c) \mid b, c \in \mathbb{R}\} = \{b\underbrace{(3,1,0,0,0)}_{v_1} + c\underbrace{(0,0,1,1,1)}_{v_2} \mid b, c \in \mathbb{R}\}$

$= span\{v_1, v_2\}$ & v_1, v_2 are lin ind.

Thus $\mathcal{B} = \{v_1, v_2\}$ is a basis for $Null(A)$ & nullity $= 2$.

$rank(A) + null(A) = 5 \Rightarrow rank(A) = 3$ but rank of a 2×5 matrix can be at most 2.

$\therefore$ There is no such a matrix A.

b) (10 pts.) Let A be an nxn matrix, and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be a collection of linearly independent vectors in R^n. Let $\beta_1, \beta_2, \ldots, \beta_k$ be vectors which satisfy $\beta_j = A\alpha_j$ for $j = 1,2,\ldots,k$. Show that the β's must be linearly independent if A is non-singular.

Show all your work. Correct answers without sufficient explanation might not get full grades.

Write $c_1\beta_1 + c_2\beta_2 + \cdots + c_k\beta_k = 0_V$ (I)

Aim: To show that $c_1 = c_2 = \cdots = c_k = 0$ is the only solution.

We're given $\beta_j = A\alpha_j$, $j = 1, \ldots, k$. Hence eqn. (I) becomes,

$c_1(A\alpha_1) + c_2(A\alpha_2) + \cdots + c_k(A\alpha_k) = 0_V$

$\Rightarrow A(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k) = 0_V$. Since A is invertible, we multiply both sides by A^{-1} to get

$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k = 0_V \Rightarrow c_1 = c_2 = \cdots = c_k = 0$ is the only solution since $\alpha_1 \ldots \alpha_k$ are lin. independent.

Thus, $\beta_1 \ldots \beta_k$ are lin. indep.

Note: Here you cannot say that either $A = 0$ or $c_1\alpha_1 + \cdots + c_k\alpha_k = 0$ because for two matrices A & B, AB may be zero but A may not be zero & B may not be zero.

(5 pts. each) Suppose A is a $4x4$ matrix and v_1, v_2, v_3, v_4 are non-zero vectors in R^4 so that $Av_1 = 2v_1$, $Av_2 = -v_2$, $Av_3 = v_3$, $Av_4 = -2v_4$. Answer the following questions.

Show all your work. Correct answers without sufficient explanation might not get full grades.

$\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 1$, $\lambda_4 = -2$ are the eigenvalues of A.

a) Find the characteristic polynomial of A.

$p(\lambda) = (\lambda-2)(\lambda-1)(\lambda+1)(\lambda+2) = (\lambda^2-4)(\lambda^2-1) = \lambda^4 - 5\lambda^2 + 4$

b) Is A diagonalizable? If so, find a diagonal matrix D such that $P^{-1}AP = D$. If not so, explain why not.

Yes, A is diag'ble because it's a 4x4 matrix with 4 distinct eigenvalues.

$D = \operatorname{diag}(2, -1, 1, -2)$.

c) Is A invertible? If so, find A^{-1} in terms of A. If not so, explain why not.

Yes, A is invertible because $\det A = \lambda_1\lambda_2\lambda_3\lambda_4 = 4 \neq 0$

By Cayley Hamilton Thm, A satisfies its charact. polynomial i.e.

$A^4 - 5A^2 + 4I = 0 \Rightarrow A(A^3 - 5A) = -4I \Rightarrow A(-\frac{1}{4}A^3 + \frac{5}{4}A) = I$

Thus, $A^{-1} = -\frac{1}{4}A^3 + \frac{5}{4}A$.

d) Find $A^4(2v_2 - 3v_3)$ in terms of v_2 and v_3.

$Av_2 = -v_2 \Rightarrow A^2v_2 = -(Av_2) = v_2 \Rightarrow A^4v_2 = A^2v_2 = v_2$

$Av_3 = v_3 \Rightarrow A^4v_3 = v_3$

$A^4(2v_2 - 3v_3) = 2(A^4v_2) - 3(A^4v_3) = 2v_2 - 3v_3$.

5) Decode the message **VOJEWRROVTWBYE** given that it is a Hill cipher with enciphering matrix $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$.

Use the following tables, if necessary.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	0

$A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ & $\det A = 5$ & $(\det A)^{-1} \equiv 21 \mod 26$

$A^{-1} = 21 \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 0 \\ 0 & 105 \end{bmatrix} \equiv \begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix} \mod 26$

VO JE WR RO VT WB YE

22 15 10 5 23 18 18 15 22 20 23 2 25 5

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 22 \\ 15 \end{bmatrix} = \begin{bmatrix} 462 \\ 15 \end{bmatrix} \equiv \begin{bmatrix} 20 \\ 15 \end{bmatrix} \pmod{26}$ 20 15 T O

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 210 \\ 5 \end{bmatrix} \equiv \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ 2 5 B E

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 23 \\ 18 \end{bmatrix} = \begin{bmatrix} 483 \\ 18 \end{bmatrix} \equiv \begin{bmatrix} 15 \\ 18 \end{bmatrix}$ 15 18 O R

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 18 \\ 15 \end{bmatrix} = \begin{bmatrix} 168 \\ 15 \end{bmatrix} \equiv \begin{bmatrix} 14 \\ 15 \end{bmatrix}$ 14 15 N O

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 22 \\ 20 \end{bmatrix} = \begin{bmatrix} 462 \\ 20 \end{bmatrix} \equiv \begin{bmatrix} 20 \\ 20 \end{bmatrix}$ 20 20 T T

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 23 \\ 2 \end{bmatrix} = \begin{bmatrix} 483 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} 15 \\ 2 \end{bmatrix}$ 15 2 O B

$\begin{bmatrix} 21 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 25 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \equiv \begin{bmatrix} E \\ E \end{bmatrix}$ E E dummy letter

The plaintext is "TO BE OR NOT TO BE"