Math 225

2014-2015 Fall

Solutions to Mdt 1

1) Let $\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{6y(y-2)} \\ y(2) = 1 \end{cases}$ be the given initial value problem.

a) (10 pts.) Using Existence and Uniqueness Theorem show that the initial value problem has a unique solution.

b) (7 pts.) Find this unique solution.

c) (8 pts.) Determine the interval in which the solution is valid.

Show your work.

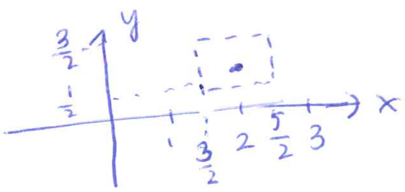
a) $f(x,y) = \dfrac{x}{6y(y-2)}$ is cont if $y \neq 0$ & $y \neq 2$.

Hence, $f(x,y)$ is cont. in $R = \{(x,y) \mid |x-2| < \frac{1}{2}, |y-1| < \frac{1}{2}\}$

$\therefore$ By E & U Thm. there exists at least one solution.

$f_y = -\dfrac{x(y-1)}{3y^2(y-2)}$ is also cont. in R.

$\therefore$ By E & U Thm. the IVP has a unique solution.



b) $\int 6y(y-2)\,dy = \int x\,dx$

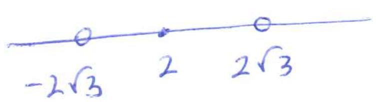
$2y^3 - 6y^2 = \frac{x^2}{2} + C$

$y(2) = 1 \Rightarrow C = -6$ & $2y^3 - 6y^2 = \frac{x^2}{2} - 6$.

c) $f(x,y)$ is not cont. when $y=0$ & $y=2$. Find the corresponding x-values:

$y = 0 \Rightarrow x^2 = 12 \Rightarrow x = \mp 2\sqrt{3}$

$y = 2 \Rightarrow x^2 = -4 \Rightarrow$ There is no solution.



$I = (-2\sqrt{3}, 2\sqrt{3})$

2) Solve the initial value problem $\begin{cases} \dfrac{dy}{dx} = \dfrac{y+x}{x} & if \quad x \geq 2 \\ \dfrac{dy}{dx} = \dfrac{y+x}{2} & if \quad x < 2 \end{cases}$, y (e) =0.

Show your work.

$x \geq 2$:

$\frac{dy}{dx} - \frac{y}{x} = 1$ lin in y.

$\rho(x) = e^{\int -\frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$

$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}}_{} = \frac{1}{x}$

$\frac{d}{dx}\left(\frac{1}{x}\cdot y\right) = \frac{1}{x} \Rightarrow \frac{1}{x}\cdot y = \ln x + C \Rightarrow y(x) = x(\ln x + C)$.

$y(e) = 0 \Rightarrow e(\underbrace{\ln e}_{1} + C) = 0 \Rightarrow \boxed{C = -1}$

$y(x) = x(\ln x - 1)$.

$x < 2$:

$\frac{dy}{dx} - \frac{y}{2} = \frac{x}{2}$ lin in y.

$\rho(x) = e^{-\int \frac{dx}{2}} = e^{-\frac{x}{2}}$

$\underbrace{e^{-x/2}\frac{dy}{dx} - \frac{1}{2}e^{-x/2}y}_{} = e^{-x/2}\cdot\frac{x}{2}$

$\frac{d}{dx}\left(y\cdot e^{-x/2}\right) = e^{-x/2}\cdot\frac{x}{2} \Rightarrow y e^{-x/2} = \frac{1}{2}\int e^{-x/2}\cdot x\,dx$

$= -e^{-x/2}\cdot x - 2e^{-x/2} + D$.

$\therefore\ y(x) = -x - 2 + De^{+x/2}$.

To find D, we use the cont. of $y(x)$ at $x=2$:

$\lim_{x\to 2^+} y(x) = \lim_{x\to 2^-} y(x) \Rightarrow 2(\ln 2 - 1) = -2 - 2 + De$

$D = 2(\ln 2 + 1)e^{-1}$

$y(x) = \begin{cases} x(\ln x - 1) & \text{if } x \geq 2 \\ -x - 2 + 2(\ln 2 + 1)e^{-x/2 + 1} \end{cases}$

3) **Using the substitution** $u = \ln y$ solve $x\dfrac{dy}{dx} - 4x^2y + 2y\ln y = 0$.

Show your work.

$$u=\ln y \Rightarrow \frac{du}{dx} = \frac{1}{y}\cdot\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y\cdot\frac{du}{dx} = e^u\cdot\frac{du}{dx}$$

$$\Downarrow$$

$$y=e^u$$

$$x\frac{dy}{dx} - 4x^2y+2y\ln y=0 \overset{x\neq 0}{\Rightarrow} \frac{dy}{dx} = 4xy - 2\frac{y}{x}\ln y$$

$$\cancel{e^u}\frac{du}{dx} = 4x\cancel{e^u} - \frac{2\cancel{e^u}}{x}\cdot u$$

$$\frac{du}{dx}+\frac{2}{x}u = 4x \quad \text{lin in } u.$$

$$\rho(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\underbrace{x^2\frac{du}{dx}+2xu} = 4x^3$$

$$\frac{d}{dx}(x^2 u) = 4x^3$$

$$x^2u = x^4+C$$

$$u = x^2 + Cx^{-2}$$

$$\|$$

$$\ln y$$

$$y(x) = e^{x^2+Cx^{-2}}$$

4) For what values of k will the system

$$x + y + kz = 1$$
$$x + ky + z = 1$$
$$kx + y + z = -2$$

have

a) no solution,
b) a unique solution,
c) infinitely many solutions.

DON'T WRITE THE SOLUTION SET. SHOW YOUR WORK.
SOLUTIONS WITHOUT ANY EXPLANATION WILL NOT BE GRADED.

$k=0$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

leading

When $k=0$, there is a unique solution.

$k \neq 0$:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \xrightarrow[-kR_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -k-2 \end{array}\right] = E$$

when $k=1$:

$$E = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right] \to \text{"Bad" row \& no solution.}$$

when $k \neq 1$:

$$E \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & \underbrace{-k^2-k+2}_{-(k+2)(k-1)} & -k-2 \end{array}\right] \xrightarrow[\frac{R_3}{-(k+2)(k-1)},\ \frac{R_2}{k-1}]{k\neq -2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{k-1} \end{array}\right]$$

leading

when $k \neq 1$, $k \neq -2$ then there is a unique solution

$k=-2$:

$$E = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

leading & ∞-ly many solt.

Result:

$k=0$ & $k \neq 1$, $k \neq -2 \Rightarrow$ Unique sol.

$k=-2 \Rightarrow \infty$-ly many solts.

$k=1 \Rightarrow$ No solution.