2013-2014
Spring
Mdt 1 Solutions

1) Consider the differential equation $\frac{dy}{dx} = -2x + 2\sqrt{x^2 + y}$. $(*)$

a) (15 pts.) Solve this differential equation by using the substitution $u = x^2 + y$.

$$u = x^2 + y \Rightarrow \frac{du}{dx} = 2x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2x$$

$(*)$ becomes $\frac{du}{dx} - \cancel{2x} = -\cancel{2x} + 2\sqrt{u}$

$$\frac{du}{2\sqrt{u}} = dx$$

$$\sqrt{u} = x + C$$

$$u = (x+C)^2$$

$$\shortparallel$$

$$\cancel{x^2} + y = \cancel{x^2} + 2xC + C^2$$

$$\boxed{y = 2xC + C^2} \quad (**)$$

b)(10 pts.) Verify that the solution $y_1(x) = 1 - 2x$ can be obtained from the general solution of the differential equation given in part a). What about the other solution $y_2(x) = -x^2$? Explain your answer.

If we put $C = -1$ in $(**)$, then we obtain $y = 1 - 2x$. Hence, it's a particular solution.

First of all $y_2(x) = -x^2$ is a sol. of $(*)$:

$$\frac{dy_2}{dx} = -2x \overset{\checkmark}{=} -2x + 2\sqrt{\cancel{x^2} - \cancel{x^2}}$$

Secondly note that $y_2(x) = -x^2$ cannot be obtained from the general solution. Therefore it's a singular solution of this d.e.

2) Find the values for the constants a and b for which
$\underbrace{\left(x+ye^{bxy}\right)}_{M}dx+\underbrace{axe^{2xy}}_{N}dy=0$ is exact. Then find an explicit solution that satisfies $y(1)=0$.

$$\frac{\partial M}{\partial y}=e^{bxy}+bxye^{bxy} \qquad \frac{\partial N}{\partial x}=ae^{2xy}+2axye^{2xy}$$

The given d.e. is exact $\Leftrightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$\Leftrightarrow \underbrace{e^{bxy}+bxye^{bxy}}=ae^{2xy}+2axye^{2xy}$$

$$e^{bxy}(1+bx)=ae^{2xy}(1+2xy)$$

Thus, $b=2$, $a=1$

In this case we've

$\underbrace{\left(x+ye^{2xy}\right)}_{M}dx+\underbrace{xe^{2xy}}_{N}dy=0$ & $\frac{\partial M}{\partial y}=e^{2xy}+2xye^{2xy}$

$$\parallel$$

$$\frac{\partial N}{\partial x}=e^{2xy}+2xye^{2xy}$$

∴ The d.e. is exact.

$$F(x,y)=\int\left(x+ye^{2xy}\right)dx=\frac{x^2}{2}+\frac{e^{2xy}}{2}+h(y)=0$$

$$F_y=\cancel{xe^{2xy}}+h'(y)=\cancel{xe^{2xy}} \Rightarrow h(y)=C_1$$

∴ $\boxed{x^2+e^{2xy}=C \ ; \text{ where } C=-2C_1}$

3) Solve the initial value problem $x^2 y'' + 2xy' = 1$, $y(1) = 2$ and $y'(1) = 0$.

It's "y" missing type.

$y' = p, \quad y'' = p'$

$\underbrace{x^2 p' + 2xp}_{} = 1$

$\frac{d}{dx}(x^2 \cdot p) = 1 \Rightarrow x^2 p = x + C_1$

$\frac{dy}{dx} = \frac{1}{x} + \frac{C_1}{x^2}$

$y'(1) = 0 \Rightarrow 0 = 1 + C_1 \Rightarrow \boxed{C_1 = -1}$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2}$

$y(x) = \ln x + \frac{1}{x} + C_2$

$y(1) = 2 \Rightarrow 2 = \cancel{\ln 1}^{0} + 1 + C_2$

$\boxed{C_2 = 1}$

$y(x) = \ln x + \frac{1}{x} + 1$

4) Solve the given linear system of equations by using **Gauss-Jordan elimination . Show all your work. Your notation is very important.**

$$\begin{aligned} 10y-4z+w&=1\\ x+4y-z+w&=2\\ 3x+2y+z+2w&=5\\ -2x-8y+2z-2w&=-4\\ x-6y+3z\qquad&=1 \end{aligned}$$

$$[A:b]=\left[\begin{array}{cccc:c} 0 & 10 & -4 & 1 & 1\\ 1 & 4 & -1 & 1 & 2\\ 3 & 2 & 1 & 2 & 5\\ -2 & -8 & 2 & -2 & -4\\ 1 & -6 & 3 & 0 & 1 \end{array}\right] \xrightarrow[\substack{2R_2+R_4\\ -R_2+R_4}]{-3R_2+R_3} \left[\begin{array}{cccc:c} 0 & 10 & -4 & 1 & 1\\ 1 & 4 & -1 & 1 & 2\\ 0 & -10 & 4 & -1 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & -10 & 4 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[-R_3+R_4]{R_3+R_1} \left[\begin{array}{cccc:c} 0 & 0 & 0 & 0 & 0\\ 1 & 4 & -1 & 1 & 2\\ 0 & -10 & 4 & -1 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2\leftrightarrow R_3]{R_1\leftrightarrow R_2} \left[\begin{array}{cccc:c} 1 & 4 & -1 & 1 & 2\\ 0 & -10 & 4 & -1 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{R_2}{10}} \left[\begin{array}{cccc:c} 1 & 4 & -1 & 1 & 2\\ 0 & 1 & -\frac{2}{5} & \frac{1}{10} & \frac{1}{10}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-4R_2+R_1} \begin{array}{c} \begin{array}{ccccc} x & y & & & \end{array}\\ \left[\begin{array}{cccc:c} 1 & 0 & -\frac{13}{5} & \frac{3}{5} & \frac{8}{5}\\ 0 & 1 & -\frac{2}{5} & \frac{1}{10} & \frac{1}{10}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

leading var.

$x=(13/5)s-(3/5)t+8/5 \quad ; s,t\in\mathbb{R}.$

$y=(2/5)s-(1/10)t+1/10$

$z=s$

$w=t$