2013-2014 Spring

Solutions to Final Exam Questions

1) Consider the following IVP.

$$y' + \frac{2x+2}{x^2+2x-15}\, y = -\frac{4}{x^2+2x-15}, \quad y(4) = 10 .$$

a) **(10 pts.)** Find the **largest possible interval** on which the given IVP has a unique solution.

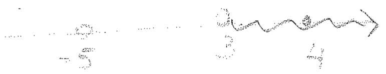
b) **(10 pts.)** Find this unique solution.

We use the Existence & Uniq. Thm for the linear d.e.:

a) $p(x) = \dfrac{2x+2}{x^2+2x-15}$ & $Q(x) = -\dfrac{4}{x^2+2x-15}$ are continuous

except at the points $x=3$ & $x=-5$

The largest possible interval is



$I = (3, +\infty)$

b) It's a 1st order lin. d.e.

$$\rho(x) = e^{\int \frac{2x+2}{x^2+2x-15}dx} = e^{\ln(x^2+2x-15)} = x^2+2x-15$$

$$\underbrace{(x^2+2x-15)\,y' + (2x+2)\,y} = -4$$

$$\frac{d}{dx}\left[y\cdot(x^2+2x-15)\right] = -4 \Rightarrow y\cdot(x^2+2x-15) = -4x + C$$

$$y = \frac{-4x+C}{x^2+2x-15}$$

$$y(4) = 10 \Rightarrow 10 = \frac{-16+C}{16+8-15} = \frac{C-16}{9} \Rightarrow C = 90+16 = 106$$

$$y(x) = \frac{-4x+106}{x^2+2x-15}$$

2) Let $S = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}$,

$T = \{(x_1, x_2, x_3, x_4) \in R^4 \mid 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0\}$,

$U = \{(x_1, x_2, x_3, x_4) \in R^4 \mid 5x_1 + 4x_2 + 3x_3 + 2x_4 = 0\}$.

a) (8 pts.) Find $S \cap T \cap U$.

b) (5 pts.) Find a basis for $S \cap T \cap U$.

c) (7 pts.) Find a basis for the orthogonal complement $(S \cap T \cap U)^{\perp}$ of $S \cap T \cap U$.

a) $S \cap T \cap U = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 \mid x_1 + 2x_2 + 3x_3 + 4x_4 = 0,\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$ and $5x_1 + 4x_2 + 3x_3 + 2x_4 = 0\}$

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \\ 5x_1 + 4x_2 + 3x_3 + 2x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \\ 5 & 4 & 3 & 2 & 0 \end{array}\right] \xrightarrow[-5R_1+R_3]{-5R_1+R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -4 & -8 & -12 & 0 \\ 0 & -6 & -12 & -18 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{6}R_3]{\frac{1}{4}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow[-2R_2+R_3]{-2R_2+R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & t & s \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = t + 2s$
$x_2 = -2t - 3s$
$x_3 = t$
$x_4 = s$

$$S \cap T \cap U = \{(t+2s, -2t-3s, t, s) \mid s, t \in \mathbb{R}\} = \operatorname{span}\{\underbrace{(1,-2,1,0)}_{v_1}, \underbrace{(2,-3,0,1)}_{v_2}\}$$

b) $S \cap T \cap U$ is spanned by the vectors v_1 & v_2. They're lin. independent. Hence they form a basis for $S \cap T \cap U$.

c) $(S \cap T \cap U)^{\perp} = \{\underbrace{(x_1, x_2, x_3, x_4)}_{u} \in \mathbb{R}^4 \mid u \cdot v_i = 0,\ i = 1, 2\}$.

$u \cdot v_1 = 0 \Rightarrow x_1 - 2x_2 + x_3 = 0$

$u \cdot v_2 = 0 \Rightarrow 2x_1 - 3x_2 + x_4 = 0$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 2 & -3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{2R_2+R_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & s & t \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \end{array}$$

$x_1 = 3s - 2t$
$x_2 = 2s - t$
$x_3 = s$
$x_4 = t$; $s, t \in \mathbb{R}$

$$(S \cap T \cap U)^{\perp} = \{(3s-2t, 2s-t, s, t) \mid s, t \in \mathbb{R}\} = \operatorname{span}\{\underbrace{(3,2,1,0)}_{u_1}, \underbrace{(-2,-1,0,1)}_{u_2}\}$$

& u_1, u_2 lin. indep.

Thus $\{u_1, u_2\}$ forms a basis for $(S \cap T \cap U)^{\perp}$.

3) Consider the subspace $W = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 - 2x_2 + x_3 - x_4 = 0\}$.

a) **(10 pts.)** Find a *3x4* matrix A with non-zero entries such that $Null(A) = W$. What is the rank of A?

b) **(10 pts.)** Find a *4x4* matrix B with $Col(B) = W$. What is the rank of B?

a) $A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & -1 \end{bmatrix}$. This matrix is not unique!

Rank$(A) = 1$

b) $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 2x_2 - x_3 + x_4\}$

$= \{(2x_2 - x_3 + x_4, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_2, x_3, x_4 \in \mathbb{R}\}$

$= \{x_2 \underbrace{(2,1,0,0)}_{v_1} + x_3 \underbrace{(-1,0,1,0)}_{v_2} + x_4 \underbrace{(1,0,0,1)}_{v_3} \mid x_2, x_3, x_4 \in \mathbb{R}\}$

$B = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, B is not unique.

Rank$(B) = 3$.

4) **(5 pts. each)** Let $A = \begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

7/ **a)** Find all eigenvalues of A.
9/ **b)** Find all eigenvectors of A.
2/ **c)** Give a reason that A is diagonalizable.
2/ **d)** Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

a) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ -1 & -2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)(-2-\lambda)+3\right]$

$-4-\cancel{2\lambda}+\cancel{2\lambda}+\lambda^2+3$

$= (2-\lambda)(\lambda^2-1) = -(\lambda+1)(\lambda-1)(\lambda-2) = 0$

$\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$ are the eigenvalues of A.

b) For $\lambda_1 = 1$:

$(A - I)X = 0$

$\begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = -3t \\ x_2 = t \\ x_3 = 0 \end{matrix}$; $t \in \mathbb{R}$. $\Big\}$ $v_1 = (-3, 1, 0)$

For $\lambda_2 = -1$:

$(A+I)X = 0$

$\begin{bmatrix} 3 & 3 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = -t \\ x_2 = t \\ x_3 = 0 \end{matrix}$; $t \in \mathbb{R}$ $\Big\}$ $v_2 = (-1, 1, 0)$

For $\lambda_2 = 2$:

$(A - 2I)X = 0$

$\begin{bmatrix} 0 & 3 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = s \end{matrix}$; $s \in \mathbb{R}$ $\Big\}$ $v_3 = (0, 0, 1)$

c) A is diag'ble because it's a 3x3 matrix having 3 eigenvectors.

d) $P = [v_1\ v_2\ v_3] = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $D = \text{diag}(1, -1, 2)$ (They're not unique).

5) Decode the message VWDVQFCQRX given that it is a Hill 2-cipher with enciphering matrix $A = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$. Use the following tables, if necessary :

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	0

and the reciprocals modulo 26:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

Use modular arithmetic. Other solutions will not be accepted.

$\det A = 3 - 8 = -5 \equiv 21 \pmod{26}$

$A^{-1} = (21)^{-1} \cdot \begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = 5 \begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -40 \\ -5 & 5 \end{bmatrix} \equiv \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} \pmod{26}$

VW DV QF CQ RX

22 23 4 22 17 6 3 17 18 24

$\begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix}\begin{bmatrix} 22 \\ 23 \end{bmatrix} = \begin{bmatrix} 606 \\ 577 \end{bmatrix} \overset{(26)}{\equiv} \begin{bmatrix} 8 \\ 5 \end{bmatrix} \longrightarrow$ 8 5 H E

$\begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 22 \end{bmatrix} = \begin{bmatrix} 324 \\ 194 \end{bmatrix} \overset{(26)}{\equiv} \begin{bmatrix} 12 \\ 12 \end{bmatrix} \longrightarrow$ 12 12 L L

$\begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix}\begin{bmatrix} 17 \\ 6 \end{bmatrix} = \begin{bmatrix} 327 \\ 387 \end{bmatrix} \overset{(26)}{\equiv} \begin{bmatrix} 15 \\ 23 \end{bmatrix} \longrightarrow$ 15 23 O W

$\begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 249 \\ 148 \end{bmatrix} \overset{(26)}{\equiv} \begin{bmatrix} 15 \\ 18 \end{bmatrix} \longrightarrow$ 15 18 O R

$\begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix}\begin{bmatrix} 18 \\ 24 \end{bmatrix} = \begin{bmatrix} 558 \\ 492 \end{bmatrix} \overset{(26)}{\equiv} \begin{bmatrix} 12 \\ 4 \end{bmatrix} \longrightarrow$ 12 4 L D

"HELLO WORLD"

Note! $A^{-1} \neq -\frac{1}{5}\begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 8/5 \\ 1/5 & -1/5 \end{bmatrix}$

since $-\frac{3}{5} \notin \mathbb{Z}_{26}$, $\frac{8}{5} \notin \mathbb{Z}_{26}$, $\frac{1}{5} \notin \mathbb{Z}_{26}$, $-\frac{1}{5} \notin \mathbb{Z}_{26}$

Note that $\mathbb{Z}_{26} = \{0, 1, \ldots, 25\}$ & it doesn't contain rational numbers!