2013 - 2014

Spring

Mdt 2 / Solutions

1) **a)(5 pts.)** Let A be an nxn matrix. Prove that $\det(adjA) = (\det A)^{n-1}$ where $adjA$ is the adjoint matrix of A.

$A^{-1} = \frac{1}{\det A} \cdot adj A \Rightarrow adj A = (\det A) \cdot A^{-1}$

$\det(adj A) = \det(\det A \cdot A^{-1})$

$= (\det A)^n \cdot (\det A)^{-1}$

$= (\det A)^{n-1}$

b) (4 pts. each) The product of two $3x3$ matrices A and B is known to be

$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and the adjoint matrix $adjA$ has determinant 4.

Find the determinants

i) $\det(A^{-1})$.

ii) $\det(A)$.

iii) $\det(B)$.

iv) $\det(A^2B^3)$.

Show your work.

i) $\det(adj A) = (\det A)^2 = 4 \Rightarrow \det A = \mp 2 \Rightarrow \det A^{-1} = \mp \frac{1}{2}$

ii) $\det A = \mp 2$

iii) $\det(AB) = 24$ & $\det A = \mp 2 \Rightarrow \det B = \mp 12$

iv) $\det(A^2B^3) = (\det A)^2 \cdot (\det B)^3 = \mp 4 \cdot (12)^3$.

2) **(25 pts.)** Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ be a subset of R^3 containing 5 vectors. Let $A = [v_1 v_2 v_3 v_4 v_5]$ be the 3x5 matrix obtained by writing the vectors in the set S as its columns. Suppose that by applying elementary row operations one may reduce the matrix A to an echelon matrix $E = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 3 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$.

Let W be the subspace of R^3 spanned by the set S.

a) Find a basis for W consisting entirely of vectors from the set S.

$$E = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 3 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \xrightarrow{\frac{1}{6}R_3} \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 3 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{-5R_3+R_2 \\ -\frac{1}{3}R_2}]{-4R_3+R_1} \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R$$

lin. ind.

v_1, v_2 & v_5 are lin. independent, since they are lead 1's in R.

$\therefore \mathcal{B}_W = \{v_1, v_2, v_5\}$

b) Is there a vector in R^3 that is not in W? **Explain your answer.**

No! Because $\dim W = 3 = \dim \mathbb{R}^3 \Rightarrow W = \mathbb{R}^3$

c) Are the vectors v_1, v_2, v_3 linearly independent? If not, express one of them as a linear combination of the others.

No, they're <u>not</u> lin. indep.

$v_3 = 2v_1 + v_2$.

3) **(10 pts. each)**

a) Let A be a $2x5$ matrix such that the linear system $AX = b$ is consistent for all $2x1$ column vectors b. What is the nullity of A? **Explain your result.**

Since the linear system $AX=b$ is consistent for all 2×1 column vector b, every 2×1 column vector belongs to the column space of A so that the dimension of the column space is 2. Thus, $rank(A) = 2$.

$rank(A) + null(A) = 5 \Rightarrow null(A) = 3$.

b) Let A be a $4x5$ matrix whose nullity is 1. Are the rows of A linearly independent? **Explain your answer.**

$rank(A) + \underbrace{null(A)}_{1} = 5 \Rightarrow rank(A) = 4$

$rank(A) = \#$ linearly independent rows of A

and it is equal to the number of rows of A

Thus, the rows of A are linearly independent

c) Let A be a $3x5$ matrix whose columns span R^3. What is the nullity of A?

As the columns of A span $\mathbb{R}^3$, the dimension of the column space is 3 i.e. $rank(A) = 3$

$rank(A) + null(A) = 5 \Rightarrow null(A) = 2$

4) Let $\{v_1, v_2, \ldots, v_k\}$ be the basis for the proper subspace W (i.e. $W \neq \{0_V\}$ and $W \neq V$) of the vector space V, and suppose that the vector v of V is **not in** W. Prove that $v_1, v_2, \ldots, v_k, v$ are linearly independent. **Show your work.**

$v \notin W$

Suppose that $v_1, v_2, \ldots, v_k, v$ are lin. dependent.

Then there are constants $c_1, c_2, \ldots, c_k, c \in \mathbb{R}$, not all zero s.t.

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + c v = 0.$$

If $c = 0$, then $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ for some constants $c_1, \ldots, c_k$ not all zero but it's a contradiction since $\{v_1, \ldots, v_k\}$ is a basis for W

Then $c \neq 0 \Rightarrow v = \left(-\frac{c_1}{c}\right) v_1 + \cdots + \left(-\frac{c_k}{c}\right) v_k$

i.e. $v \in \operatorname{span}\{v_1, \ldots, v_k\} = W$

This is also a contradiction.

$\therefore$ $v_1, v_2, \ldots, v_k, v$ are lin independent.

Exercise 29 in 4.4