4 January 2014
Time: 12:30- 14:30

MATH 225
2013-2014 Fall Semester
Final Exam

SURNAME	
NAME	
STUDENT NO	
SIGNATURE	
Department	

IMPORTANT:

- This exam consists of *5* questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding question. Be neat.
- **Show all your work. Correct answers without sufficient explanation might not get full grades.**
- Calculators and dictionaries are **not** allowed.
- Close your cellular phones.

1	2	3	4	5	TOTAL
20	20	20	20	20	100

1) Let $P_3 = \{ax^3 + bx^2 + cx + d \mid a,b,c,d \in R\}$ be the vector space of all polinomials of degree less than or equal to 3 and $U = \{q(x) \in P_3 \mid q(1) = 0 \; and \; q(2) = 0\}$ be a subspace of P_3. Find a basis for U. What is the dimension of U? **Explain your answer.**

Let $q(x) = ax^3 + bx^2 + cx + d$ be in P_3. It is in U if $q(1) = 0$ & $q(2) = 0$ i.e.

$$\begin{cases} a+b+c+d=0 \\ 8a+4b+2c+d=0 \end{cases}$$ It's a homog. system of eqns.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 0 \end{array}\right] \xrightarrow{-8R_1+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -4 & -6 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{+\frac{1}{4}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -\frac{3}{2} & -\frac{7}{4} & 0 \end{array}\right] \xrightarrow{+R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1/2 & -3/4 & 0 \\ 0 & -1 & -3/2 & -7/4 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1/2 & -3/4 & 0 \\ 0 & 1 & 3/2 & 7/4 & 0 \end{array}\right]$$

$a = t/2 + 3s/4$; $t,s \in \mathbb{R}$.
$b = -3t/2 - 7s/4$
$c = t$
$d = s$

Thus, $U = \{ax^3 + bx^2 + cx + d \mid a,b,c,d \in \mathbb{R}\}$

$= \{(\frac{t}{2} + \frac{3s}{4})x^3 + (-\frac{3t}{2} - \frac{7s}{4})x^2 + tx + s \mid s,t \in \mathbb{R}\}$

$= \{(\frac{x^3}{2} - \frac{3}{2}x^2 + x)t + (\frac{3}{4}x^3 - \frac{7}{4}x^2 + 1)s \mid s,t \in \mathbb{R}$

$= \text{span}\{\underbrace{x^3 - 3x^2 + 2x}_{P_1}, \underbrace{3x^3 - 7x^2 + 4}_{P_2}\}$

and P_1 & P_2 are lin. independent because

$$\begin{array}{cc} P_1 & P_2 \end{array}$$
$$\underbrace{\begin{bmatrix} 1 & 3 \\ -3 & -7 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}}_{\text{lin. ind}} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{leading 1's}}$$

$\therefore$ $\mathcal{B} = \{P_1, P_2\}$ is a basis for U & $\dim U = 2$.

2) Solve the given linear system of differential equations

$$\frac{dx_1}{dt} = 3x_1 - x_2$$

$$\frac{dx_2}{dt} = -x_1 + 2x_2 - x_3$$

$$\frac{dx_3}{dt} = -x_2 + 3x_3$$

Show your work in full details.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad \& \quad |A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right] + (\lambda - 3)$$

$$= (3-\lambda)\,[\underbrace{(2-\lambda)(3-\lambda)-1-1}_{\lambda^2 - 5\lambda + 4}]$$

$$= -(\lambda-3)(\lambda-1)(\lambda-4) = 0$$

$\lambda_1 = 3,\ \lambda_2 = 1\ \&\ \lambda_3 = 4$ are the eigenvalues of A.

For $\lambda_1 = 3$: Solve $(A - 3I)v = 0$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -t \\ x_2 = 0 \\ x_3 = t \end{array} \quad ; t \in \mathbb{R}$$

$S_1 = \text{span}\{\underbrace{(-1, 0, 1)}_{v_1}\}$

For $\lambda_2 = 1$: Solve $(A - I)v = 0$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} x_1 = s \\ x_2 = 2s \\ x_3 = s \end{array}\right\} S_2 = \text{span}\{\underbrace{(1, 2, 1)}_{v_2}\}$$

$s \in \mathbb{R}$

For $\lambda_3 = 4$: Solve $(A - 4I)v = 0$

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} x_1 = t \\ x_2 = -t \\ x_3 = t \end{array}\right\} ; t \in \mathbb{R} \quad S_3 = \text{span}\{\underbrace{(1, -1, 1)}_{v_3}\}$$

$\therefore$ A is diagble with $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ & $D = \text{diag}(3, 1, 4)$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} K_1 e^{3t} \\ K_2 e^{t} \\ K_3 e^{4t} \end{bmatrix} = \begin{bmatrix} -K_1 e^{3t} + K_2 e^{t} + K_3 e^{4t} \\ 2K_1 e^{t} - K_3 e^{4t} \\ K_1 e^{3t} + 2K_2 e^{t} + K_3 e^{4t} \end{bmatrix}$$

$\therefore\ x_1(t) = -K_1 e^{3t} + K_2 e^{t} + K_3 e^{4t}, \quad x_2(t) = 2K_1 e^{t} - K_3 e^{4t}, \quad x_3(t) = K_1 e^{3t} + 2K_2 e^{t} + K_3 e^{4t}$

3) Let $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$.

a) **(10 pts.)** Find **the reduced echelon form (not an echelon form!)** R of A.

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \xrightarrow[\substack{R_1+R_2 \\ -2R_1+R_2}]{-3R_1+R_2} \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \xrightarrow[\substack{-\frac{1}{4}R_3 \\ -\frac{1}{5}R_4}]{-\frac{1}{14}R_2} \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

lin. ind.

$$\xrightarrow[\substack{-R_2+R_3 \\ -R_2+R_4}]{-4R_2+R_1} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

leading 1's

$x_1 = -t_1 - 2t_2 - t_3$; $t_1, t_2, t_3 \in \mathbb{R}$

$x_2 = -t_1 - t_2 - 2t_3$

$x_3 = t_1$

$x_4 = t_2$

$x_5 = t_3$

b) **(2 pts.)** Find a basis for the row space of A.

$\mathcal{B}_R = \{ (1,0,1,2,1), (0,1,1,1,2) \}$

c) **(2 pts.)** Find a basis for the column space of A.

$\mathcal{B}_C = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix} \right\}$

d) **(4 pts.)** Find a basis for the null space of A.

Solution set of $AX = 0$ is

$S = \{ (-t_1 - 2t_2 - t_3, -t_1 - t_2 - 2t_3, t_1, t_2, t_3) \mid t_1, t_2, t_3 \in \mathbb{R} \}$

$= \text{span}\{ \underbrace{(-1,-1,1,0,0)}_{v_1}, \underbrace{(-2,-1,0,1,0)}_{v_2}, \underbrace{(-1,-2,0,0,1)}_{v_3} \}$ & $\mathcal{B}_N = \{v_1, v_2, v_3\}$

e) **(2 pts.)** Write the rank and nullity of A.

Rank of $A = 2$

Nullity of $A = 3$.

4) a) (**12 pts.**) A certain *4x4* real matrix A is known to have these properties:

1. Two of the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$.
2. The number *4* is an eigenvalue of the matrix $(A+3I)$.
3. det A= *-8*.

Use this information to answer the following questions about the matrix A.

i) What are the other two eigenvalues of A?
ii) What is the characteristic polynomial of A?
iii) What is the characteristic polynomial of A^{-1}?
iv) Is A diagonalizable? Why?

i) 4 is an eig-value of $(A+3I) \Rightarrow (A+3I)v = 4v$
$\Rightarrow Av = v \Rightarrow \boxed{\lambda_3 = 1}$ is an eig. value of A.

$\det A = \lambda_1\lambda_2\lambda_3\lambda_4 = (-1)(2)(1)\cdot\lambda_4 = 8 \Rightarrow \boxed{\lambda_4 = -4}$

ii) $p(\lambda) = (\lambda+1)(\lambda-2)(\lambda-1)(\lambda+4)$

iii) λ is an eigvalue of $A \Leftrightarrow \lambda^{-1}$ is an eig. value of A^{-1}

Thus, $-1, \frac{1}{2}, 1$ & $-\frac{1}{4}$ are the eig. values of A^{-1}.

$p(\lambda) = (\lambda+1)(\lambda-\frac{1}{2})(\lambda-1)(\lambda+\frac{1}{4})$ is the char. polyn. of A^{-1}.

iv) Yes. Because A is a 4x4 matrix having 4 distinct eigenvalues.

b) (8 pts.) Let A be a *3x3* matrix such that rank(A) =1 and rank($A+I$) = 2. Is A diagonalizable?**Explain your answer.**

rank$(A) = 1 \Rightarrow$ nullity $A = 2$ i.e. the hom. system $AX = 0$ has non-trivial solutions but it means that A is not invertible. Hence 0 is an eigenvalue of A having 2 eigenvectors

rank$(A+I) = 2 \Rightarrow$ nullity $A = 1$ i.e. the hom. system $(A+I)X = 0$ has non-trivial solutions & $AX = -X \Rightarrow \lambda = -1$ is an eigenvalue of A having one eigenvector.

Thus, A is a 3x3 matrix having 3 eigenvectors & it's diagonalizable.

5)a) **(6 pts.)** Let A be a $4x5$ matrix whose columns span R^4. What is the nullity of A? **Explain your answer. (There will be no points if there is no explanation).**

As the columns span $\mathbb{R}^4$, the dimension of its column space is 4.
i.e. rank$(A)=4$. Thus nullity of $A = 5-4=1$

b) (7 pts.) Let V be a vector space . Suppose that v_1, v_2, v_3 are linearly independent vectors of V. If there is a vector v_4 such that the vectors v_1, v_2, v_3, v_4 span V, then what can you say about the dimension of V? **Explain your answer. (There will be no points if there is no explanation).**

Let V be a v.s. with $\dim V = r$. If there are n-lin. indep. vectors, then $n \leq r$. Moreover, if there are m-vectors whose span is V, then $m \geq r$.
Consequently, the dimension of V is 3 or 4.

c) (7 pts.) Let V be a vector space which can be spanned by 7 **linearly dependent** vectors. If V contains some 4 linearly independent vectors whose span is not equal to V, what are the possibilities for the dimension of V? **Explain your answer. (There will be no points if there is no explanation).**

Let $\dim V = n$. Since 7 lin. dependent vectors span V,
i.e. $\dim V < 7$.
In addition to this, 4 lin. ind. vectors doesn't span V
i.e. $\dim V \geq 5$.
Thus $\dim V$ can be 5, 6.