December 6, 2012
**Time:** 17:45- 19: 45

SOLUTIONS

**MATH 225**
**MIDTERM 2**

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|---|---|
| **SURNAME** | |
| **NAME** | |
| **STUDENT NO** | |
| **SIGNATURE** | |
| **DEPARTMENT** | |

**IMPORTANT:**

- This exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding question. Be neat.
- **Show all your work. Correct answers without sufficient explanation might not get full grades.**
- Calculators and dictionaries are **not** allowed.
- **Close your cellular phones.**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 20 | 20 | 20 | 20 | 20 | 100 |

1) Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & x & 5 \\ -2 & -4 & 4 \end{bmatrix}$, where $x$ is a variable.

a) **(10 pts)** Find det ($A$).

b) **(5 pts.)** For what value(s) of $x$ will $A$ have an inverse?

c) **(5 pts.)** For those $x$ such that $A^{-1}$ exists, find det ($A^{-1}$).

a) $\begin{vmatrix} 1 & 3 & -1 \\ 0 & x & 5 \\ -2 & -4 & 4 \end{vmatrix} = (4x+20) - 3(0+10) - (0+2x) = 2x - 10$

b) A has an inverse if $\det A \neq 0$ i.e. if $2x - 10 \neq 0$

$x \neq 5$.

c) $\det(A^{-1}) = \dfrac{1}{\det A} = \dfrac{1}{2x-10}$ if $x \neq 5$.

Note: a) If you use Sarrus' rule and find a wrong result, then your grade is $\underline{0}$.

b) In part c), you're not asked to find $A^{-1}$!!!

2) Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 1 & 2 & -1 & 3 \end{bmatrix}$.

a) **(10 pts.)** Find the reduced echelon form $R$ of $A$?

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 1 & 2 & -1 & 3 \end{bmatrix} \xrightarrow[R_2+R_1]{-R_1+R_3} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & -1 & 2 \end{bmatrix} \xrightarrow[-R_2]{R_2+R_3} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Note: You're asked to find the reduced echelon form & it's not enough to find the echelon form!

$x_1 = -s+t$ ; $s,t \in \mathbb{R}$
$x_2 = s-2t$
$x_3 = s$
$x_4 = t$

Note: $S = \{1, 0, 1, -1\}$ is a set with elements $1, 0, 1, -1$.
Use $(1, 0, 1, -1)$ to denote a vector!

b) **(2 pts.)** Find a basis for the Row ($A$).

$\mathcal{B}_R = \{[1\ 0\ 1\ -1], [0\ 1\ -1\ 2]\}$

c) **(2 pts.)** Find a basis for the Col ($A$).

$\mathcal{B}_C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

d) **(4 pts.)** Find a basis for the Null ($A$). $X \in Null(A)$ if and only if

$X = (x_1, x_2, x_3, x_4) = (-s+t, s-2t, s, t) = s\underbrace{(-1, 1, 1, 0)}_{u_1} + t\overbrace{(1, -2, 0, 1)}^{u_2}$
$= \text{span}\{u_1, u_2\}$.

In addition to this, $\{u_1, u_2\}$ is lin. independent.

$\therefore$ $B_N = \{u_1, u_2\}$.

e) **(2 pts.)** What is the rank of $A$? What is the nullity of $A$?

$\text{rank}(A) = \text{nullity}(A) = 2$.

Note: $\text{rank}(A) + \text{nullity}(A) = 2$ & If your result doesn't satisfy this equation then you get zero.

3) Let $S=\{(x,y,z)\in R^3 \mid 2x-y+z=0\}$ and $T=\{(x,y,z)\in R^3 \mid x+3y=0\}$ be two subsets of the vector space $R^3$.

a) **(10 pts.)** Show that $S$ and $T$ are subspaces of $R^3$.

$S\neq\emptyset$ since $(0,0,0)\in S$.

1) $u=(x_1,y_1,z_1)$ & $v=(x_2,y_2,z_2)$ are in $S$

Then $2x_1-y_1+z_1=0$ &

$2x_2-y_2+z_2=0$.

$\Rightarrow 2(x_1+x_2)-(y_1+y_2)+(z_1+z_2)=0$

gives $u+v\in S$.

2) $c\in\mathbb{R}$, $u=(x_1,y_1,z_1)\in S$.

Then $2x_1-y_1+z_1=0$

$c(2x_1-y_1+z_1)=0$

$\Rightarrow cu\in S$.

$\therefore$ $S$ is a subspace of $\mathbb{R}^3$

$T\neq\emptyset$ since $(0,0,0)\in T$.

$T=\{(-3y,y,z)\in\mathbb{R}^3 \mid y,z\in\mathbb{R}\}$

1) $u=(-3y_1,y_1,z_1)\in T$ & $v=(-3y_2,y_2,z_2)\in T$

$u+v=(-3(y_1+y_2),\ y_1+y_2,\ z_1+z_2)\in T$.

2) $c\in\mathbb{R}$, $u\in T$

$cu=(-3(cy_1),\ cy_1,\ cz_1)\in T$.

$\therefore$ $T$ is a subspace.

NOTE:

1) Don't write:

$u=2x-y+z=0\in S$ !!!

$u=(x,y,z)$ is a vector.

2) $u=2x-y+z=0\in S$ means that $0\in S$ but $0\in\mathbb{R}$ but $S\subseteq\mathbb{R}^3$!

3) Don't take the vectors $u$ & $v$ from $\mathbb{R}^3$! We already know that $\mathbb{R}^3$ is a vector space.

b) **(8 pts.)** Find the subspace $S\cap T$ and describe the geometric configuration in $R^3$.

$S\cap T=\{(x,y,z)\in\mathbb{R}^3 \mid 2x-y+z=0 \ \& \ x+3y=0\}=\{t(-\frac{3}{7},\frac{1}{7},1) \mid t\in\mathbb{R}\}$

$\begin{cases}2x-y+z=0\\ x+3y=0\end{cases}$ $\begin{bmatrix}2&-1&1\\1&3&0\end{bmatrix}\dashrightarrow\begin{bmatrix}1&0&3/7\\0&1&-1/7\end{bmatrix}$ = line through the origin.

c) **(2 pts.)** Find a basis for $S\cap T$.

$S\cap T=\text{span}\{(-\frac{3}{7},\frac{1}{7},1)\}$ & $(-\frac{3}{7},\frac{1}{7},1)$ is a non-zero vector so it's lin. independent.

$\therefore$ $\mathcal{B}_{S\cap T}=\{(-\frac{3}{7},\frac{1}{7},1)\}$.

4) **(20 pts.)** Find all values of "$a$" such that the poynomial $q(x) = x^2 + 3x + 5$ of $P_2 = \{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in R\}$ is **not** in the subspace spanned by the polynomials $q_1(x) = 3x^2 + 2x + 5$, $q_2(x) = 2x^2 + 4x + 7$, $q_3(x) = 5x^2 + 6x + a$.

If $q(x) \in \text{span}\{q_1, q_2, q_3\}$, then we can find $c_1, c_2, c_3 \in \mathbb{R}$ s.t. $q(x) = c_1 q_1 + c_2 q_2 + c_3 q_3$ i.e.

$$x^2 + 3x + 5 = c_1(3x^2 + 2x + 5) + c_2(2x^2 + 4x + 7) + c_3(5x^2 + 6x + a)$$
$$= (3c_1 + 2c_2 + 5c_3)x^2 + (2c_1 + 4c_2 + 6c_3)x + (5c_1 + 7c_2 + ac_3)$$

and we obtain a non-hom. system

$$(*) \begin{cases} 3c_1 + 2c_2 + 5c_3 = 1 \\ 2c_1 + 4c_2 + 6c_3 = 3 \\ 5c_1 + 7c_2 + ac_3 = 5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 5 & 1 \\ 2 & 4 & 6 & 3 \\ 5 & 7 & a & 5 \end{array}\right] \xrightarrow[\substack{-3R_2+R_1 \\ -5R_2+R_3 \\ R_1 \leftrightarrow R_2 \\ -3R_2+R_3}]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3/2 \\ 0 & -1 & -1 & -7/8 \\ 0 & 0 & a-12 & 1/8 \end{array}\right]$$

$(*)$ is consistent if $a \neq 12$.

$\therefore$ $q(x)$ is in $\text{span}\{q_1, q_2, q_3\} \iff a = 12$.

**5) ( 20 pts.)** Suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for the vector space $V$. Let $c_1, c_2, \ldots, c_n$ be scalars with $c_1 \neq 0$ and $S = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n, v_2, v_3, \ldots, v_n\}$. Show that $S$ is a basis for $V$.

$\{v_1, .., v_n\}$ is a basis for $V$. Hence $\dim V = n$.

Therefore it's enough to show that either $S$ is lin. indep.

or $S$ spans $V$.

S is linearly independent:

$d_1(c_1v_1 + c_2v_2 + \cdots + c_nv_n) + d_2v_2 + \cdots + d_nv_n = 0_V$

$(d_1c_1)v_1 + (d_1c_2 + d_2)v_2 + (d_1c_3 + d_3)v_3 + \cdots + (d_1c_n + d_n)v_n = 0_V$

Since $v_1, v_2, .., v_n$ is lin. independent,

$d_1c_1 = 0$ & $c_1 \neq 0 \Rightarrow d_1 = 0$

$d_1c_2 + d_2 = 0 \Rightarrow d_2 = 0$

$\vdots \qquad \vdots$

$d_1c_n + d_n = 0 \Rightarrow d_n = 0$

$\therefore$ $d_1 = d_2 = \cdots = d_n = 0$ & $S$ is lin. independent.

OR

S spans V:

Let $w \in V$. Then $w = a_1v_1 + a_2v_2 + \cdots + a_nv_n$; $a_1, .., a_n \in \mathbb{R}$

& this representation is unique.

If $w = d_1(c_1v_1 + \cdots + c_nv_n) + d_2v_2 + \cdots + d_nv_n$, then show that

we can always find $d_1, d_2, .., d_n \in \mathbb{R}$, satisfying this eqn.

$w = d_1c_1v_1 + (d_1c_2 + d_2)v_2 + (d_1c_3 + d_3)v_3 + \cdots + (d_1c_n + d_n)v_n$

From uniqueness of the represention for $w$:

$a_1 = d_1c_1 \Rightarrow \quad d_1 = \dfrac{a_1}{c_1}$ since $c_1 \neq 0$.

$a_2 = d_1c_2 + d_2 \Rightarrow d_2 = a_2 - d_1c_2 = a_2 - \dfrac{a_1c_2}{c_1}$

$a_3 = d_1c_3 + d_3 \Rightarrow d_3 = a_3 - d_1c_3 = a_3 - \dfrac{a_1c_3}{c_1}$

$\vdots \qquad \vdots \qquad \vdots$

$a_n = d_1c_n + d_n \Rightarrow d_n = a_n - d_1c_n = a_n - \dfrac{a_1c_n}{c_1}$

$\therefore$ S spans V.