Math 225
2012-2013 Fall
Md 1 Solutions

1) Let $\dfrac{dy}{dx}=\dfrac{x}{6y(y-2)}$, $y(2)=1$ be given.

a) **(10 pts.)** Using Existence and Uniqueness Theorem, show that the given initial value problem has a unique solution.

b) **(8 pts.)** Solve the given initial value problem and find this unique solution.

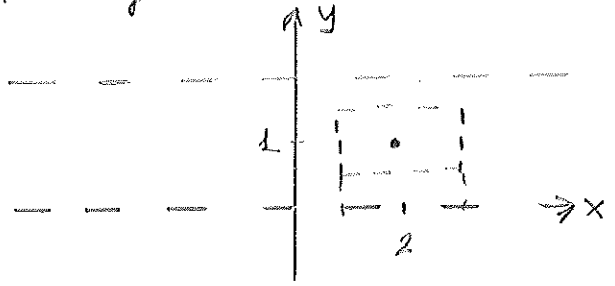
c) **(7 pts.)** Determine the interval in which the solution is valid.

a) $f(x,y)=\dfrac{x}{6y(y-2)}$ & $f_y=-\dfrac{x}{(6y(y-2))^2}\cdot(12y-12)$

are cont. if $y\neq 0$ & $y\neq 2$.

Then f & f_y are cont. in the rectangle

$R=\{(x,y)\mid |x-2|<1, |y-1|<\frac{1}{2}\}$



$\therefore$ The given d.e. has a unique solution around $x_0=2$.

b) $\dfrac{dy}{dx}=\dfrac{x}{6y(y-2)} \Rightarrow (6y^2-12y)\,dy=x\,dx$

$\Rightarrow 2y^3-6y^2=\dfrac{x^2}{2}+C$

$y(2)=1 \Rightarrow \boxed{C=-6} \Rightarrow 2y^3-6y^2-\dfrac{x^2}{2}=-6.$

c) At the points $y=0$ & $y=2$, $f(x,y)$ is undefined

$\therefore$ There are points x_1 & x_2 s.t. at the points $(x_1,0)$ & $(x_2,2)$ s.t. the slopes of the tangent lines are undefined.

$y=0 \Rightarrow -\dfrac{x^2}{2}=-6 \Rightarrow x=\mp\sqrt{12}$

$y=2 \Rightarrow -x^2=4$ has no real values.

$\therefore I=(-\sqrt{12},\sqrt{12})$.

2) Solve the initial value problem $\overbrace{x^2y}^{M}\,dx + (\overbrace{yx^3 + e^{-3y}y\sin y}^{N})\,dy = 0$, $y(0) = \pi$.

1st solution: $\dfrac{\partial M}{\partial y} = x^2 \neq \dfrac{\partial N}{\partial x} = 3yx^2$, the d.e. is not exact.

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{x^2 - 3x^2y}{yx^3 + e^{-3y}y\sin y} \quad \text{is not only a function of } x.$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = 3 - \frac{1}{y} \quad \text{is a function of } y.$$

$$\rho(y) = e^{\int (3 - \frac{1}{y})dy} = \frac{e^{3y}}{y}$$

Multiply the d.e. by $\dfrac{e^{3y}}{y}$:

$$\underbrace{e^{3y}\cdot x^2}_{\tilde{M}}\,dx + (\underbrace{e^{3y}x^3 + \sin y}_{\tilde{N}})\,dy = 0 \quad (*) \quad \& \quad \frac{\partial \tilde{M}}{\partial y} = 3e^{3y}x^2 = \frac{\partial \tilde{N}}{\partial x} \Rightarrow (*) \text{ is exact.}$$

$$F(x,y) = \int e^{3y}x^2\,dx = e^{3y}\frac{x^3}{3} + g(y) = 0$$

$$\frac{\partial F}{\partial y} = e^{3y}\cancel{x^3} + g'(y) = \tilde{N} = e^{3y}\cancel{x^3} + \sin y \Rightarrow g(y) = -\cos y + C$$

$$\therefore F(x,y) = e^{3y}\frac{x^3}{3} - \cos y + C = 0, \quad y(0) = \pi \Rightarrow C = -1 \text{ \& we get}$$

$$\boxed{\frac{e^{3y}x^3}{3} - \cos y = 1}$$

2nd solution:

$$\frac{dy}{dx} = \frac{-x^2y}{yx^3 + e^{-3y}y\sin y} \Rightarrow \frac{dx}{dy} = \frac{yx^3 + e^{-3y}y\sin y}{-x^2y} = -x - e^{-3y}\sin y\cdot x^{-2}$$

$$\frac{dx}{dy} + x = -e^{-3y}\sin y\cdot x^{-2}, \text{ is Bernoulli with } n = -2.$$

$$x^2\frac{dx}{dy} + x^3 = -e^{-3y}\sin y \quad \& \quad u = x^3, \quad \frac{du}{dy} = 3x^2\frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{1}{3x^2}\frac{du}{dy}$$

$$\cancel{x^2}\cdot\frac{1}{3\cancel{x^2}}\frac{du}{dy} + u = -e^{-3y}\sin y \Rightarrow \frac{du}{dy} + 3v = -3e^{-3y}\sin y$$

$$\rho(y) = e^{\int 3dy} = e^{3y}$$

$$\frac{d}{dy}(e^{3y}\cdot v) = -3\sin y \Rightarrow e^{3y}\cdot v = 3\cos y + C$$

$$x^3e^{3y} = 3\cos y + C$$

$$y(0) = \pi \Rightarrow C = 3$$

$$\boxed{\frac{e^{3y}x^3}{3} - \cos y = 1}$$

3) a)(10 pts.) Find the general solution of the differential equation $\frac{dy}{dx} = -\frac{3x^2+2y^2}{4xy} = -\frac{3+2\left(\frac{y}{x}\right)^2}{4\frac{y}{x}}$

$\frac{dy}{dx} = -\left(\frac{3}{4}\frac{x}{y} + \frac{1}{2}\frac{y}{x}\right)$ hom. $\frac{y}{x} = u \Rightarrow y = ux$ & $\frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$x\frac{du}{dx} + u = -\frac{3}{4u} - \frac{u}{2} \Rightarrow x\frac{du}{dx} = -\frac{3+6u^2}{4u} \Rightarrow -\int \frac{4u}{3+6u^2} = \int\frac{dx}{x}$

$-\frac{1}{3}\ln(\overbrace{3+6u^2}^{>0}) = \ln|x| + C_1$

$(3+6u^2)^{-\frac{1}{3}} = C\cdot|x|,\ C = e^{C_1}$

Put $u = \frac{y}{x}$:

$\left(3+6\frac{y^2}{x^2}\right)^{-\frac{1}{3}} = C\cdot|x| \Rightarrow \left(\frac{x^2}{3+6y^2}\right)^{\frac{1}{3}} = C\cdot|x|$

b) (15 pts.) Solve $y' + 4xy = 4x^3y^{\frac{1}{2}}$. Bernoulli $n = \frac{1}{2}$

$\underbrace{y^{-\frac{1}{2}}y' + 4xy^{\frac{1}{2}} = 4x^3}$, $u = y^{\frac{1}{2}} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{y}}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2\sqrt{y}\frac{du}{dx}$

$\frac{1}{\cancel{\sqrt{y}}}\left(2\cancel{\sqrt{y}}\cdot\frac{du}{dx}\right) + 4x\,u = 4x^3$

$\frac{du}{dx} + 2xu = 2x^3$ linear in u.

$\rho(x) = e^{\int 2x\,dx} = e^{x^2}$

$\underbrace{e^{x^2}\frac{du}{dx} + 2xe^{x^2}u} = 2e^{x^2}\cdot x^3$

$\frac{d}{dx}\left(e^{x^2}u\right) = 2xe^{x^2}\cdot x^2$

$\int 2xe^{x^2}x^2\,dx = \int e^u\cdot u\,du = ue^u - \int e^u du = ue^u - e^u + C$

$u = x^2$, $du = 2x\,dx$; $t = u$, $dt = du$; $ds = e^u du$, $s = e^u$; $= e^{x^2}(x^2-1) + C$

$\to e^{x^2}u = e^{x^2}(x^2-1) + C \Rightarrow \underset{y^{1/2}}{\underset{\|}{u}} = (x^2-1) + Ce^{-x^2} \Rightarrow y = \left[(x^2-1) + Ce^{-x^2}\right]^{1/2}$

4) For which values of a does the system

$$bx_1 + bx_2 + ax_3 = 1$$
$$bx_1 + ax_2 + bx_3 = 1 \qquad \text{have}$$
$$ax_1 + bx_2 + bx_3 = 1$$

a) (5 pts.) no solution?
b) (10 pts.) infinitely many solutions? Find the solution set, if possible.
c) (10 pts.) a unique solution? Find the solution set, if possible.

$$\left[\begin{array}{ccc|c} b & b & a & 1 \\ b & a & b & 1 \\ a & b & b & 1 \end{array}\right] \xrightarrow[\substack{-\frac{a}{b}R_1+R_3 \\ b\neq 0}]{-R_1+R_2} \left[\begin{array}{ccc|c} b & b & a & 1 \\ 0 & a-b & b-a & 0 \\ 0 & b-a & b-\frac{a^2}{b} & -\frac{a}{b}+1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} b & b & a & 1 \\ 0 & a-b & b-a & 0 \\ 0 & 0 & 2b-a-\frac{a^2}{b} & 1-\frac{a}{b} \end{array}\right] \xrightarrow{bR_3} \left[\begin{array}{ccc|c} b & b & a & 1 \\ 0 & a-b & b-a & 0 \\ 0 & 0 & \underbrace{2b^2-ab-a^2}_{(2b+a)(b-a)} & b-a \end{array}\right] = E$$

a) If $a=-2b$, then there is no solution.

b) If $a=b$, then there are ∞-ly many solutions.

In this case

$$E = \left[\begin{array}{ccc|c} b & b & a & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{b}R_1} \left[\begin{array}{ccc|c} 1 & 1 & \frac{a}{b} & \frac{1}{b} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -s - \frac{a}{b}t - \frac{1}{b}$; $s,t \in \mathbb{R}$.
$x_2 = s$
$x_3 = t$

c) If $2b^2-ab-a^2 \neq 0$, then there is a unique solution.

$\boxed{x_3 = \frac{1}{2b+a}}$ using back substitution:

$(a-b)x_2 + (b-a)x_3 = 0 \Rightarrow x_2 = \frac{\cancel{a-b}}{\cancel{a-b}}x_3 \Rightarrow \boxed{x_2 = \frac{1}{2b+a}}$

$bx_1 + bx_2 + ax_3 = 1$

$$\boxed{x_1 = \frac{1}{b}\left[1 - bx_2 - ax_3\right] = \frac{1}{b}\left[1 - \frac{b}{2b+a} - \frac{a}{2b+a}\right] = \frac{1}{b}\left(\frac{b-2a}{2b+a}\right)}$$