1) **a)** Determine the of the following differential equations. **DO NOT SOLVE.**

Differential Equation	**TYPE**
i) $(y^2 - x) + xyy' = 0.$	Bernoulli, $n = -1$
ii) $(y^2 - x^2) + xyy' = 0.$	Homog. or Bernoulli, $n = -1$
iii) $(y^2 - x^3) + (2xy + y^3)y' = 0.$	Exact
iv) $y(2x - y) + y' = 1 + x^2.$	Riccati
v) $(y^2 + x) + (xy + y^3)y' = 0.$	Seperable or Bernoulli, $n = -1$

b) Solve equation **ii)**.

$(y^2 - x^2) + xyy' = 0$ becomes $y' + \frac{y}{x} = \frac{x}{y} \cdot$ (1)

Say $z = \frac{y}{x} \Rightarrow y = xz$ & $y' = z + x \cdot \frac{dz}{dx}$

(1) becomes $z + x\frac{dz}{dx} + z = \frac{1}{z} \Rightarrow \frac{z}{-2z^2+1}\,dz = \frac{dx}{x}$ seperable.

Integrating both sides we get

$-\frac{1}{4}\ln|1-2z^2| = \ln|x| + \ln|c|$

$|1-2z^2| = |cx|^{-4} = \frac{D}{x^4}$, $D = c^{-4}$ constant; $c \neq 0$.

Assuming $D > 0$ & $1 - 2z^2 > 0$

$z^2 = \frac{1}{2} - \frac{D}{2x^4}$ & $\frac{y^2}{x^2} = \frac{x^4 - D}{2x^4} \Rightarrow y^2 = \frac{x^4 - D}{2x^2}.$

2) Solve the differential equation $\left(\sec^2 y\right)y' + \frac{\tan y}{1+x} = \frac{1}{1+x}$ by using **a suitable transformation. Show all your work.**

$t = \tan y$ & $\frac{dt}{dx} = \sec^2 y \cdot y'$. Then the given equation becames

(1) $\frac{dt}{dx} + \frac{t}{1+x} = \frac{1}{1+x}$ linear in t. (It's also seperabl

$$\rho(x) = e^{\int \frac{dx}{1+x}} = 1+x$$

Multiply (1) by $1+x$:

$$\underbrace{(1+x)\frac{dt}{dx} + t}_{} = 1$$

$$\frac{d}{dx}\left[(1+x)\cdot t\right] = 1 \Rightarrow (1+x)\cdot t = x + C$$

$$t = \frac{x+C}{x+1}$$

$$\tan y = \frac{x+C}{x+1}$$

$$y = \arctan\left(\frac{x+C}{x+1}\right).$$

3) Solve the initial value problem $y' = \frac{1-e^y}{x}$ where $y(2) = 0$. **Explain your solution.**

The function $f(x,y) = \frac{1-e^y}{x}$ is continuous

except $x=0$. and $f_y(x) = \frac{-e^y}{x}$ is also continuous

except $x=0$. Thus we can find a rectangle R

around the point $(2,0)$, e.g. $R = \{(x,y) \mid |x-2|<1, |y|<1$

s.t. $f(x,y)$ & $f_y(x,y)$ are continuous in R.

Thus by Existence & Uniqueness Theorem there exists a unique solution.

This unique solution is the singular solution

$y=0$.

4) Find the conditions on a, b and c (if any) such that the system

$$x + z = -1$$
$$2x - y = 2$$
$$y + 2z = -4$$
$$ax + by + cz = 3$$

a) has no solution.
b) has infinitely many solutions. Find the solution set , if possible.
c) has a unique solution. Find this unique solution, if possible.
Show all your work.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ a & b & c & 3 \end{bmatrix} \xrightarrow[\substack{R_2+R_3 \\ -R_2 \\ R_3 \leftrightarrow R_4 \\ -bR_2+R_3}]{\substack{-2R_1+R_2 \\ -aR_1+R_4}} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & c-a-2b & 3+a+4b \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

a) If $c-a-2b=0$ but $3+a-4b \neq 0$ then there is no solution.

b) If $c-a-2b \neq 0$ then

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & c-2b-a & 3+a+4b \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-2R_3+R_2]{\substack{\frac{R_3}{c-a-2b} \\ -R_3+R_1}} \begin{bmatrix} 1 & 0 & 0 & -2b-c-3/c-a-2 \\ 0 & 1 & 0 & -6a-8b+4c-14/c-a \\ 0 & 0 & 1 & 3+a+4b/c-a-2b \\ 0 & 0 & 0 & \end{bmatrix}$$

there is a unique solution

$$x_1 = \frac{-2b-c-3}{c-a-2b}, \quad x_2 = \frac{-6a-8b+4c-14}{c-a-2b}, \quad x_3 = \frac{3+a+4b}{c-a-2b}.$$

c) If $c-a-2b=0$ & $3+a+4b=0$ then

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we've infinitely many solutions ;

$x_1 = -1-t$; $t \in \mathbb{R}$.
$x_2 = -4-2t$
$x_3 = t$

5) A) Each of them is 2 pts.

Indicate whether the statement is always true **(T)** or sometimes false **(F)**.

If it is true, prove it. If it is false, justify your answer by giving your reason or giving a counter example.

--F-- **a)** If the reduced row echelon form of the augmented matrix for a non-homogenous linear system has a row of zeros, then the system must have infinitely many solutions.

$\begin{cases} x+y=1 \\ x-y=1 \\ 2x+2y=2 \end{cases}$ has an aug. matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and it has a unique solution.

--F--**b)** If a non-homogenous linear system of n equations in n unknowns has n leading entry in the reduced row echelon form of its augmented matrix, then the system has exactly one solution.

The system may be inconsistent for example,

$\begin{cases} x+y=1 \\ x+y=0 \end{cases}$ gives $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ & no solutic

T-----**c)** If u and v are two solutions to $AX = B$ then u-v is a solution to the corresponding homogenous system $AX=0$.

u is a solution $\Rightarrow Au = B$

v " " $\Rightarrow Av = B$

Then $Au - Av = 0$ gives $A(u-v) = 0$ & $u-v$ is a solution of $AX=0$.

F----**d)** If Gaussian elimination on the augmented matrix of a non-homogenous system of linear equation leads to the matrix $\begin{bmatrix} 0 & 1 & 3 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ then the system has a unique solution.

It's inconsistent because of the last row.

F---**e)** $\pi x_1 + \sqrt{3}x_2 + 5\sqrt{x_3} = \ln 7$ is a linear equation.

It's ~~not~~ <u>not</u> linear because of $\sqrt{x_3}$.

5) B) Each of them is 2 pts.

a) Find *2x2* matrix A with $A \neq 0$ with $A^2 = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \& \quad A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) Find *2x2* matrix A with $A \neq 0$ with $A^2 = 2A$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \& \quad A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2A$$

c) Find *2x2* matrix A with $A \neq 0$ and $A \neq I$ with $A^2 = I$.

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \& \quad A^2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

d) Find *2x2* matrix A with $A \neq 0$ with $A^2 = -I$.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \& \quad A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

e) Find *2x2* matrices A and B with $A \neq 0$, $B \neq 0$ satisfying $A^2 + B^2 = 0$.

Take $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ where $A^2 = I$ & $B^2 = -I$.