A new notion of rank for finite supersolvable groups and free linear actions on products of spheres

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Abstract

For a finite supersolvable group G, we define the *saw rank* of G to be the minimum number of sections $G_k - G_{k-1}$ of a cyclic normal series G_* such that $G_k - G_{k-1}$ owns an element of prime order. The *axe rank* of G, studied by Ray [10], is the minimum number of spheres in a product of spheres admitting a free linear action of G. Extending a question of Ray, we conjecture that the two ranks are equal. We prove the conjecture in some special cases, including that where the axe rank is 1 or 2. We also discuss some relations between our conjecture and some questions about Bieberbach groups and free actions on tori.
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1 Introduction

What is the minimum number a such that a given group has a free linear action on a product of a spheres? Not all finite groups have a free linear action on a product of spheres but any supersolvable finite group does have such an action. For finite supersolvable groups, we seek an abstract group-theoretic characterization of the number a. We also discuss, in the final section, some related cohomological invariants.

Throughout, let G be a finite supersolvable group. An element $g \in G$ is said to **act freely** on a $\mathbb{C}G$-module X provided no non-zero element of X is fixed by g. Given a set $\mathcal{X}$ of $\mathbb{C}G$-modules, then G is said to **act freely** on $\mathcal{X}$ provided each non-trivial element of G acts freely on at least one of the elements of $\mathcal{X}$. Imposing a G-invariant inner product on X, we get a G action on the unit sphere $S(X)$ in X. Hence, G acts on the product $\prod_{X \in \mathcal{X}} S(X)$. When an action of G on a product of spheres can be constructed in this way, we say that the action is **linear**. Plainly, G acts freely on $\mathcal{X}$ if and only if G acts freely on $\prod_{X \in \mathcal{X}} S(X)$.

Ray [10] defined a **good group** to be a finite group that has a free linear action on a product of spheres. She proved that any non-abelian simple factor of a good group is isomorphic to A_5 or A_6. The solvable group A_4 is not good since its involutions do not act freely on any $\mathbb{C}A_4$-module. Ray observed that any finite supersolvable group is good. The proof is easy: Consider a normal cyclic subgroup C of G. Let Y be a faithful $\mathbb{C}C$-module. The non-trivial elements of C act freely on the induced $\mathbb{C}G$-module $\mathrm{Ind}_C^G(Y)$. Meanwhile, by an inductive argument on $|G|$, each element of $G - C$ acts freely on the inflation of some $\mathbb{C}(G/C)$-module. As required, we have shown that each non-trivial element of G acts freely on some $\mathbb{C}G$-module.

We define $\mathrm{axe}(G)$, called the **axe rank** of G, to be the minimum size of a set of $\mathbb{C}G$-modules upon which G acts freely. In other words, $\mathrm{axe}(G)$ is the minimum number of spheres in a product of spheres admitting a free linear action.

A group element of prime order is called a **Cauchy element**. Consider a normal series

$$G_* \ : \ 1 = G_0 \lhd ... \lhd G_t = G$$

whose factors G_k/G_{k-1} are cyclic. For $1 \leq k \leq t$, the factor G_k/G_{k-1} is said to be **Cauchy** provided the set $G_k - G_{k-1}$ owns a Cauchy element. The number of Cauchy factors in G_* is called the **rank** of G_*, denoted $\mathrm{rk}(G_*)$. We define the **saw rank** of G, denoted $\mathrm{saw}(G)$, to be the minimum $\mathrm{rk}(G_*)$ as G_* runs over the normal series with cyclic factors. Note that, when G is a p-group, the factor G_k/G_{k-1} is Cauchy if and only if the sequence $1 \to G_{k-1} \to G_k \to G_k/G_{k-1} \to 1$ splits.

In the case where G has prime exponent p and order p^s, Ray [10, Section 3] asked whether $\mathrm{axe}(G)$ is equal to s. In this case, $s = \mathrm{saw}(G)$. We extend the question and, in view of the evidence we shall accumulate, we pose it as a conjecture:

Conjecture 1.1. *For a finite supersolvable group, the axe rank is equal to the saw rank.*

Thus, we are proposing $\mathrm{saw}(G)$ as an abstract group-theoretic characterization of $\mathrm{axe}(G)$. Note that there are examples where the other notions of rank fail to coincide with the axe rank. For instance, when G is the non-abelian p-group of order p^3 and exponent p, the number of generators of G and the rank of maximal elementary abelian subgroups in G are both 2, whereas $\mathrm{saw}(G) = \mathrm{axe}(G) = 3$. The sectional rank of G (the largest number s such that every subgroup $H \leq G$ is generated by s elements) is also 2. Another interesting example is given in [10] where G is the 3-group which is not meta-cyclic, but $\mathrm{axe}(G) = \mathrm{saw}(G) = 2$. So, the axe rank can be strictly less than the minimum number of cyclic sections.

In the cases where we have resolved the conjecture affirmatively, the equality $\mathrm{axe}(G) = \mathrm{saw}(G)$ may be of interest in its own right. The resolved cases and the reductions we have obtained seem to be sufficiently diverse to justify the conjectural status of the equality. Proposition 2.5 says that the conjecture holds for all finite abelian groups and, in this case, the axe rank and saw rank are both equal to the minimum size of a generating set. As a special case of Corollary 2.6, the conjecture holds for all p-central groups. Theorem 2.8 asserts that $\mathrm{axe}(G) = 1$ if and only if $\mathrm{saw}(G) = 1$. Theorem 4.1 asserts that, when G is a p-group, $\mathrm{axe}(G) = 2$ if and only if $\mathrm{saw}(G) = 2$. Theorem 5.9 asserts the conjectured equality in the case where some maximal subgroup of G is elementary abelian.

It is an immediate consequence of Corollary 2.4 that, if the conjecture holds for all groups of prime power order, then it holds for all finite nilpotent groups. Proposition 5.5 asserts that if the conjecture holds for all finite groups with exponent p, then it holds for all finite regular p-groups. Half of Conjecture 1.1 is almost trivial: Lemma 2.2 says that $\mathrm{axe}(G) \leq \mathrm{saw}(G)$.

The axe rank and saw rank are, in different ways, the sizes of minimal covers of the Cauchy elements. When G acts freely on $\mathcal{X}$, each element $X \in \mathcal{X}$ sweeps out a ragged swath consisting of those Cauchy elements that act freely on X. The axe rank is the minimum number of swipes needed to reap all the Cauchy elements. On the other hand, a subnormal series G_* for G cuts the set of non-trivial elements neatly into slices $G_1 - G_0, ..., G_t - G_{t-1}$. A Cauchy factor G_k/G_{k-1} accounts for precisely those Cauchy elements that belong to $G_k - G_{k-1}$.

Although it is rather difficult to characterize the groups with a fixed saw rank, some observations can be made in the case of saw rank 2. A brief discussion of classification of p-groups with saw rank 2 can be found in Section 4. For more general characterizations, it would be desirable to have a way of constructing a cyclic normal series G_* with the minimal rank $\mathrm{rk}(G_*) = \mathrm{saw}(G)$. We must leave that as an open problem.

In Section 6, we discuss relations between free linear actions on products of spheres and free actions on tori. The main observation is that, given a free linear action on a product of k spheres, there is a natural way to construct a free action on a torus, where the group acts on the homology of the torus by permuting a basis with k orbits. When G acts freely on a torus $X = T^n$, the fundamental group of orbit space X/G is a Bieberbach group. So there is an associated Bieberbach group for every free linear action on a product of spheres. Using these associations we show that the axe-saw conjecture would follow from affirmative answers to certain questions about Bieberbach groups and free actions on tori.

Finally, we would like to point out that the saw rank conjecture has applications to product actions on products of spheres. Given G-spaces $X_1, \dots, X_k$, the diagonal G-action on $X_1 \times \cdots \times X_k$ is called a product action. Notice that free linear actions are product actions where G acts on each sphere through a complex representation. Dotzel and Hamrick proved in [6] that when G is a p-group, G-actions on mod p-homology spheres resemble linear actions. The resemblance is explained through dimension functions. In particular, their result implies that if G acts on a mod p homology sphere, it acts on a sphere linearly with exactly same group elements acting freely. So, given a free product action on products of k-spheres, there is a free linear action on same number of spheres. In this way, for p-groups, the saw-axe conjecture applies to product actions as well.

2 Some general properties and easy consequences

We begin with two easy but very useful lemmas.

Lemma 2.1. *Given $H \le G$, then $\mathrm{axe}(H) \le \mathrm{axe}(G)$ and $\mathrm{saw}(H) \le \mathrm{saw}(G)$.*

Proof. For the first inequality, observe that if G acts freely on a set $\mathcal{X}$, then H also acts freely by restriction. For the second, let G_* be a normal cyclic-factor series for G with $\mathrm{rk}(G_*) = \mathrm{saw}(G)$, and let H_* be the series obtained by intersecting each term with H. Then $\mathrm{rk}(H_*) \le \mathrm{rk}(G_*) = \mathrm{saw}(G)$. □

Lemma 2.2. $\mathrm{axe}(G) \le \mathrm{saw}(G)$*.*

Proof. Let G_* be a normal cyclic-factor series for G with $\mathrm{rk}(G_*) = \mathrm{saw}(G)$. For each Cauchy factor G_k/G_{k-1}, let Y_k be a $\mathbb{C}G_k$-module such that $\ker Y_k = G_{k-1}$. Let $X_k = \mathrm{Ind}_{G_k}^{G} Y_k$. Then every Cauchy element of $G_k - G_{k-1}$ acts freely on X_k. Hence, G acts freely on the set $\{X_k\}$. □

We shall consider some special cases. First, we need a lemma:

Lemma 2.3. *For a direct product $G = G' \times G''$, we have $\mathrm{axe}(G) \le \mathrm{axe}(G') + \mathrm{axe}(G'')$ and $\mathrm{saw}(G) \le \mathrm{saw}(G') \times \mathrm{saw}(G'')$. Furthermore, if $|G'|$ and $|G''|$ are coprime, then $\mathrm{axe}(G) = \max(\mathrm{axe}(G'), \mathrm{axe}(G''))$ and $\mathrm{saw}(G) = \max(\mathrm{saw}(G'), \mathrm{saw}(G''))$.*

Proof. The inequality for axe rank holds by considering induction from G' and G'' to G. The inequality for saw rank is obvious. Now suppose that $|G'|$ and $|G''|$ are coprime. By Lemma 2.1, we need only show that $\mathrm{axe}(G) \le \max(\mathrm{axe}(G'), \mathrm{axe}(G''))$ and similarly for $\mathrm{saw}(G)$. If G' and G'' act freely on $\{X'_1, ..., X'_\alpha\}$ and $\{X''_1, ..., X''_\alpha\}$, then G acts freely on $\{X'_1 \otimes X''_1, ..., X'_\alpha \otimes X''_\alpha\}$. If G'_* and G''_* are normal cyclic-factor series for G' and G'', then there exists a normal cyclic-factor series G_* for G such that the j-th Cauchy factor of G_* is isomorphic to the direct product of the j-th Cauchy factor of G'_* and the j-th Cauchy factor of G''_*. □

As an immediate consequence:

Corollary 2.4. *Suppose that G is nilpotent. Then the axe rank of G is the maximum axe rank of a Sylow subgroup of G. The saw rank of G is the maximum saw rank of a Sylow subgroup of G.*

Proposition 2.5. *If G is abelian, then* $\mathrm{axe}(G) = \mathrm{rk}(G) = \mathrm{saw}(G)$.

Proof. By the previous two results, $\mathrm{axe}(G) \leq \mathrm{saw}(G) \leq \mathrm{rk}(G)$. Suppose that G acts freely on a set of irreducibles $\{X_1, ..., X_a\}$, and let $\chi_1, ..., \chi_a$ be the corresponding characters. The function $g \mapsto (\chi_1(g), ..., \chi_a(g))$ is a group monomorphism, so $\mathrm{rk}(G) \leq \mathrm{axe}(G)$. □

Alternatively, for abelian G, we can obtain the inequality $\mathrm{axe}(G) \leq \mathrm{rk}(G)$ by the following counting argument. Let p be a prime with maximal multiplicity r in $|G|$, and let E be the maximal elementary abelian p-subgroup of A. Then $\mathrm{rk}(E) = r = \mathrm{rk}(G)$. By Lemma 2.1, we may assume that $E = G$. Hence, the kernel of any irreducible $\mathbb{C}G$-module has index p in G. The intersection of a such kernels has index at most p^a. Again, we have shown that $\mathrm{axe}(G) \leq \mathrm{rk}(G)$.

Corollary 2.6. *If all the Cauchy elements of G are central, then* $\mathrm{axe}(G) = \mathrm{axe}(Z(G)) = \mathrm{saw}(Z(G)) = \mathrm{saw}(G)$.

Proof. By considering induction and restriction, it is easy to see that $\mathrm{axe}(G) = \mathrm{axe}(Z(G))$. By extending a normal cyclic factor series for $Z(G)$, we deduce that $\mathrm{saw}(G) = \mathrm{saw}(Z(G))$. The middle equality holds by Proposition 2.5. □

Let us compare the saw rank with some other ranks. Recall that, for a finite group H, the rank of H, written as $\mathrm{rk}(H)$, is defined to be the largest r such that $(\mathbb{Z}/p)^r \leq H$ for some prime p. The minimal number of generators of H is denoted by $\mathrm{d}(H)$. The sectional rank, $\mathrm{srk}(H)$, is defined to be the maximal $\mathrm{d}(K)$ over all subgroups $K \leq H$.

Proposition 2.7. *We have* $\mathrm{rk}(G) \leq \mathrm{saw}(G)$. *If G is a p-group with $p > 2$, then* $\mathrm{d}(G) \leq \mathrm{srk}(G) \leq \mathrm{saw}(G)$.

Proof. The first inequality follows from Lemma 2.1 and Proposition 2.5. When G is a p group, with $p > 2$, Laffey [9] proves that

$$\mathrm{d}(G) \leq \max\{ \log_p |K| \ : \ K \trianglelefteq G, \ [K, K] \leq Z(K), \ \exp(K) = p \}.$$

Since $\log_p |K| = \mathrm{saw}(K)$ when K has exponent p, and $\mathrm{saw}(K) \leq \mathrm{saw}(G)$ for all subgroups $K \leq G$, we get $\mathrm{d}(G) \leq \mathrm{saw}(G)$. Applying this to each subgroup, we get $\mathrm{srk}(G) \leq \mathrm{saw}(G)$. □

In Section 4, we shall see that the following result is an easy consequence of some material in Section 3. Let us also give a direct proof.

Theorem 2.8. *The following conditions are equivalent:*

(a) $\mathrm{saw}(G) = 1$,
(b) $\mathrm{axe}(G) = 1$,
(c) *Every subgroup whose order is a product of two primes is cyclic,*
(d) *The Cauchy elements generate a cyclic subgroup.*

Proof. By Lemma 2.2, (a) implies (b). A special case of Wolf [13, Theorem 5.3.1] says that (b) implies (c). Supposing that (d) holds, consider a normal series G_* with cyclic quotients and such that G_1 is the subgroup generated by the Cauchy elements. Then $\mathrm{rk}(G_*) = 1$, and condition (a) holds.

Before showing that (c) implies (d), let us recall a general property of finite supersolvable groups: Any maximal normal abelian subgroup A of G is maximal as an abelian subgroup of G. To see this, suppose, for a contradiction, that A is a maximal normal abelian subgroup such that $A < C_G(A)$. The normal series $1 \trianglelefteq A \trianglelefteq C_G(A) \trianglelefteq G$ refines to a chief series G_*. Writing $A = G_{k-1}$, then G_k is a normal abelian subgroup of G. This contradicts the maximality of A.

Now assume (c). Let A be a maximal normal abelian subgroup of G. By the hypothesis, the Sylow subgroups of A are cyclic. To deduce (d), we may assume, for a contradiction, that an element g of prime order p belongs to $G - A$. Consider the conjugation action of g on A. For each prime divisor q of $|A|$, the conjugation action stabilizes the Sylow q-subgroup Q of A, and also stabilizes the subgroup $Q_0 \leq Q$ generated by the Cauchy elements. Notice that since Q is cyclic, Q_0 is cyclic of order q. If $q = p$, we obtain a contradiction by observing that g and Q_0 generate an elementary abelian p-group of rank 2. Supposing now that $q \neq p$, the hypothesis implies that $\langle g \rangle Q_0$ is cyclic. If a non-trivial automorphism of a cyclic q-group has order coprime to q, then it restricts to a non-trivial automorphism of the subgroup of order q. Therefore g must centralize Q. We have shown, in fact, that g centralizes A. This contradicts the condition that A is maximal as an abelian subgroup. □

Recall that the quaternion groups and the cyclic groups of prime-power order are the only finite groups with a unique subgroup of prime order. (See, for instance, Ashbacher [2, Exercise 8.4]). So, when G is a p-group, the axe rank and saw rank of G are unity if and only if G is quaternion or cyclic.

3 Swaths

We have noted that the definition of the saw rank is purely group theoretic. In order to relate the axe rank and the saw rank, it will help to have a purely group theoretic description of the Cauchy elements that act freely on a suitable $\mathbb{C}G$-module. Given subgroups K and H of G such that $K \trianglelefteq H \trianglelefteq G$ and H/K is cyclic, the subset

$$H -_{\mathrm{G}} K := H - \bigcup_{g \in G} K^g$$

is called a **swath** of G. Given a $\mathbb{C}G$-module X, we write $\mathcal{C}(X)$ for the set of Cauchy elements that act freely on X.

Recall that a $\mathbb{C}G$-module X is said to be **monomial** provided X is induced from a 1-dimensional module of a subgroup.

Lemma 3.1. *Let $H -_G K$ be a swath. Let Y be any 1-dimensional $\mathbb{C}H$-module with kernel K, and let X be the monomial $\mathbb{C}G$-module $\mathrm{Ind}_H^G(Y)$. Then $\mathcal{C}(X)$ is the set of Cauchy elements belonging to $H -_G K$.*

Proof. As a $\mathbb{C}H$-module by restriction, X is the sum of the 1-dimensional modules of the form $Y \otimes g$ where $g \in G$. The elements of $H - K^g$ are precisely the elements $x \in G$ such that x (piecewise) stabilizes $Y \otimes g$ and x does not (pointwise) fix the elements of $Y \otimes g$. □

Lemma 3.2. *Let X be a monomial $\mathbb{C}G$-module. Then there exists a swath $H -_G K$ such that $|H : K|$ is square-free and $\mathcal{C}(X) \subseteq H -_G K$.*

Proof. Let $X = \mathrm{Ind}_H^G Y$ where Y is a 1-dimensional $\mathbb{C}H$-module. Any Cauchy element not in the core of H must permute the spaces $Y \otimes g$ non-trivially, and therefore fixes a non-zero vector. So any Cauchy element acting freely on X must belong to the core of H'. Replacing H with its core and X with its restriction to the core, we can assume that X is a monomial $\mathbb{C}G$ induced from normal H.

Let K be the kernel of Y. By Lemma 3.1, $\mathcal{C}(X)$ is contained in $H -_G K$. Assume that H is minimal with this property. Let L be the subgroup of H such that $K \leq L \leq H$ and L/K is the subgroup of H/K generated by the Cauchy elements in H/K. Plainly, $|L : K|$ is square-free. Any Cauchy element $x \in H -_G K$ also belongs to $L -_G K$. The conjugates of x, being Cauchy elements in $H -_G K$, also belong to $L -_G K$. Therefore

$$x \in \bigcap_{g \in G} (L -_G K)^g = \mathrm{core}_G(L) -_G (\mathrm{core}_G(L) \cap K).$$

By the minimality assumption, $H = L$. Therefore $|H : K|$ is square-free. □

If G acts freely on a set $\mathcal{X}$ of $\mathbb{C}G$-modules, then each element $X \in \mathcal{X}$ can be replaced with an irreducible summand of X; the action will still be free. It is well-known (see Serre [11, Theorem 16]) that every irreducible complex representation of a finite supersolvable group is monomial. So, we can apply the above two lemmas to the free actions on set of arbitrary $\mathbb{C}G$-modules, and obtain the following alternative characterization for the axe rank.

Proposition 3.3. *The axe rank $\mathrm{axe}(G)$ is the minimum number a such that all the Cauchy elements belong to a union $\bigcup_{j=1}^{a} H_j -_G K_j$ of a swaths of G. Furthermore, we may assume that each index $|H_j : K_j|$ is square free.*

Proof. The assertion now follows from Lemmas 3.1 and 3.2. □

Lemma 3.4. *Suppose that G is a non-trivial 2-group. Given a monomial $\mathbb{C}G$-module X, then $\mathcal{C}(X)$ is contained in a swath $H -_G K$ such that $|H : K| = 2$ and $K \trianglelefteq G$.*

Proof. We may assume that $\mathcal{C}(X)$ is non-empty. Let K be the kernel of X, and let H be the subgroup generated by K and $\mathcal{C}(X)$. The elements of $\mathcal{C}(X)$ act on X as multiplication by -1, so the product of any two of them belongs to K, so $|H : K| = 2$. □

When G is a p-group with p odd, $\mathcal{C}(X)$ need not be contained in a swath $H -_G K$ with $K \trianglelefteq G$. Indeed, let G be the wreath product of $C_3 \wr C_3$, let H be the normal subgroup $C_3 \times C_3 \times C_3$, and put $X = \mathrm{Ind}_H^G(Y)$ where Y has kernel $C_3 \times C_3 \times 1$. Writing the elements of H as vectors (x, y, z) over the field of order 3, then $\mathcal{C}(X)$ consists of the 8 vectors whose coordinates x, y, z are all non-zero. Let K be the subgroup of H consisting of the vectors whose coordinates sum to zero. Then K is the unique index p subgroup of H such that $K \trianglelefteq G$. But $H - K$ does not contain $\mathcal{C}(X)$.

Lemma 3.5. *Suppose that G is a p-group with p odd. Let $H' -_G K'$ be a swath of G such that K' is cyclic. Then the set of Cauchy elements in $H' -_G K'$ is contained in a swath $H -_G K$ of G such that $H \cong C_p \times C_p$ and $K \cong C_p$.*

Proof. Let H be the subgroup of H' generated by the Cauchy elements, and let $K = H \cap K'$. Ashbacher [2, 23.4] says that, for p-groups with class at most 2, the Cauchy elements generate an elementary abelian subgroup. In particular, H is elementary abelian. Since H'/K' and K' are cyclic, and $1 < K < H$, we have $|K| = p$ and $|H| = p^2$. □

4 Supersolvable groups of low axe and saw rank

Using swaths, the rank 1 case of Conjecture 1.1 is very easy. Indeed, we can now give a quicker proof of Proposition 2.8. Trivially, (d) implies (c). By Lemma 2.1, (c) implies (a). As noted before, (a) implies (b) by Lemma 2.2. Assume (b). By Proposition 3.3, the Cauchy elements of G all belong to some swath $H -_{\mathrm{G}} K$. Since K is trivial, the normal subgroup H of G is cyclic. We have deduced (d), and the argument is complete.

Theorem 4.1. *Suppose that G is a p-group. Then* $\mathrm{axe}(G) = 2$ *if and only if* $\mathrm{saw}(G) = 2$.

Proof. By Theorem 2.8, it suffices to show that $\mathrm{axe}(G) \leq 2$ if and only if $\mathrm{saw}(G) \leq 2$. One direction is immediate from Lemma 2.2. For the other direction, suppose that $\mathrm{axe}(G) \leq 2$. Let $\mathcal{C}$ be the set of Cauchy elements of C. By Proposition 3.3, we can write

$$\mathcal{C} \subseteq (H_1 -_{\mathrm{G}} K_1) \cup (H_2 -_{\mathrm{G}} K_2)$$

where $|H_1 : K_1| = p = |H_2 : K_2|$.

First, let us assume that $p \neq 2$. If $|K_1| = 1 = |K_2|$, then H_1 and H_2 are normal cyclic groups of order p, and $\mathcal{C} \cup \{1\} = H_1 \cup H_2$. Hence G is cyclic and $\mathrm{saw}(G) = 1$. Suppose that $|K_1| = 1 \neq |K_2|$. Then $|H_1| = p$. All the Cauchy elements of K_2 belong to H_1, so H_1 is the unique subgroup of K_2 with order p. So K_2 is cyclic. (See the comment at the end of Section 2). By Lemma 3.5, we may assume that $H_2 \cong C_p \times C_p$. The normal series $1 \lhd H_1 \lhd H_2 \unlhd G$ refines to a chief series with rank 2. So $\mathrm{saw}(G) \leq 2$.

Now consider the case where both K_1 and K_2 are non-trivial. The intersection $\mathcal{C} \cap K_1 \cap K_2$ is empty, so $K_1 \cap K_2$ is trivial. The subgroup $H_1 \cap K_2$ owns all the Cauchy elements of K_2 and is isomorphic to a subgroup of the cyclic group H_1/K_1. Therefore K_2 has a unique subgroup of order p. Again K_2 is cyclic. Similarly, K_1 is cyclic. As before, we may assume that H_1 and H_2 are isomorphic to $C_p \times C_p$. But K_1 and K_2 are both contained in H_1 and are both contained in H_2, so $H_1 = K_1K_2 = H_2$. The normal series $1 \lhd Z(G) \cap H_1 \lhd H_1 \unlhd G$ refines to a chief series with rank 2. The case $p \neq 2$ is finished.

Now assume that $p = 2$. By Lemma 3.4, we may assume that K_1 and K_2 are normal in G. When both K_1 and K_2 are trivial, the argument is the same for odd p. Supposing that only K_1 is trivial, then H_1 is the unique subgroup of order 2 in K_2. It follows that the normal series $1 \lhd H_1 \unlhd K_1 \lhd H_2 \unlhd G$ refines to a chief series with rank 2.

Finally, supposing that both K_1 and K_2 are non-trivial, then they own central involutions c_1 and c_2, respectively. Furthermore, c_1 and c_2 are distinct because K_1 and K_2 intersect trivially. By Lemma 2.1, G does not contain an elementary abelian subgroup of rank 3. So $\mathcal{C} \subseteq \langle c_1, c_2 \rangle$ and, once again, $\mathrm{saw}(G) = 2$. □

In the above proof, the separation of the cases $p > 2$ and $p = 2$ was necessary because the Cauchy elements do not need to generate a subgroup of exponent 2 when G is a 2-group. For example, the group $G = D_8$, the dihedral group of order 8, is such a group and has axe and saw rank 2.

Let us now discuss the p-groups with saw rank 2. By Lemma 2.7, such groups have rank 2, and when $p > 2$ all of its subgroups are generated by 2 elements. Using Blackburn's work on these groups, we prove the following:

Proposition 4.2. *For odd p, a p-group of saw rank 2 is either meta-cyclic or a 3-group of maximal class.*

Proof. Theorem 4.2 in [4] tells us immediately that, given a p-group G of order greater than or equal to p^6 such that each subgroup of order p^4 is generated by two elements, then G is either meta-cyclic or a 3-group of maximal class. For smaller groups, we use the fact that the groups of saw rank 2 cannot include a subgroup of order p^3 and exponent p. For groups of order p^5, this observation disposes of the exceptional cases given in [4, Theorem 4.2]. For p-group G with saw rank 2 of order less than p^5, [4, Theorem 3.2] implies that if G is not meta-cyclic, then G is a group of order 3^4 and of maximal class. (This is the group mentioned in the introduction as an example of a non-meta-cyclic group of saw rank 2). □

For 2-groups, the situation is more complicated. A simple example, $G = Q_8$, the quaternion group of order 8, shows that the number of generators may be more than the saw rank. (In this case saw rank is 1, whereas the number of generators is 2.)

By Proposition 2.7 and Theorem 2.8, we know that the groups with saw rank 2 must have $\mathrm{rk}(G) = 2$. On the other hand, there are groups with rank 2 with saw rank strictly bigger than 2. We give an example of such group after the following proposition.

Proposition 4.3. *Let G be a 2-group with $\mathrm{saw}(G) = 2$. Then the subgroup generated by the Cauchy elements has an index 2 subgroup that is cyclic or generalized quaternion.*

Proof. Let G be a 2-group of saw rank 2, and let $\Omega(G)$ denote the subgroup generated by the Cauchy elements in G. By Lemma 2.1, $\Omega(G)$ has saw rank 1 or 2. If $G_* : 1 \lhd G_1 \lhd \cdots G_{n-1} \lhd G_n = \Omega(G)$ is a cyclic series for $\Omega(G)$ with $\mathrm{rk}(G_*) = \mathrm{saw}(\Omega(G))$, then the top section must include a Cauchy element, so G_{n-1} must be cyclic or generalized quaternion. □

Example 4.4. *Let G be the central product of D_8 with Q_8. This is the quotient group of $D_8 \times Q_8$ with kernel $\langle c_1 c_2 \rangle$ where c_1 and c_2 are central elements of order 2 in D_8 and Q_8. Let a and b be involutions generating D_8, and let c and d be generators of Q_8. The elements a, b, abc, abd are Cauchy elements and they generate G. On the other hand, every subgroup $H \leq G$ of index 2 fits into a central extension of the form $0 \to \mathbb{Z}/2 \to H \to (\mathbb{Z}/2)^3 \to 0$ which shows that H has $\mathrm{rk}(H) \geq 2$. Hence, no index 2 subgroup of G is cyclic or generalized quaternion.*

The classification of groups with saw rank two seems to be manageable problem. This might serve as a first step for the classification of groups with rank 2, which is recognized as a difficult problem.

5 The exponent p case, and related cases

In Proposition 5.4, we return to the case of Conjecture 1.1 originally raised (as a question) by Ray, namely the case where G has exponent p. But that case is more general than it appears to be, since Proposition 5.5 says that the case of a regular p-group reduces to the exponent p case. In Theorem 5.9, we show that the conjecture holds for a certain class of non-regular p-groups.

Throughout this section, we let G be a p-group, and we write

$$a = \mathrm{axe}(G), \qquad s = \mathrm{saw}(G).$$

Our conjectured equality is $a = s$. Lemma 2.2 already tells us that $a \leq s$. We seek to prove the reverse inequality.

Suppose that $\exp(G) = p$. Then $|G| = p^s$. Proposition 5.4, below, implies that if $s \leq p+2$, then $a = s$. To prove it, we first need a technical definition and some lemmas. Let us say that G is **inductible** provided, whenever G acts freely on a set consisting of a irreducible $\mathbb{C}G$-modules, at least of them is 1-dimensional.

Lemma 5.1. *Suppose that* $\exp(G) = p$. *If G is inductible and* $\mathrm{axe}(H) = \mathrm{saw}(H)$ *for every maximal subgroup H of G, then $a = s$.*

Proof. Let G act freely on a set $\mathcal{X}$ consisting of a irreducible $\mathbb{C}G$-modules one of which, say X, is 1-dimensional. The kernel K of X has order p^{s-1}. If $s \neq 1$, then K acts freely on the set of restrictions of $\mathcal{X} - \{X\}$. We have $s - 1 = \mathrm{saw}(K) = \mathrm{axe}(K) \leq a - 1$. □

Lemma 5.2. *Suppose that* $\exp(G) = p$. *If every subgroup of G is inductible, then $a = s$.*

Proof. This follows from Lemma 5.1 via an inductive argument. □

Lemma 5.3. *Suppose that* $\exp(G) = p$. *If $a \leq p + 1$, then G is inductible.*

Proof. It suffices to show that whenever G acts freely on a set $\mathcal{X}$ of monomial $\mathbb{C}G$-modules all of dimension greater than unity, we have $|\mathcal{X}| \geq p + 2$. Consider an element $X \in \mathcal{X}$, and let $H -_{\mathrm{G}} K$ be a swath such that $|H : K| = p$ and $\mathcal{C}(X) \subseteq H -_{\mathrm{G}} K$. Since $H \neq G$, we have

$$|\mathcal{C}(X)| \leq p^{s-1} - p^{s-2}.$$

On the other hand, $\bigcup_{X \in \mathcal{X}} \mathcal{C}(X) = G - \{1\}$, hence

$$|\mathcal{X}|(p^{s-1} - p^{s-2}) \geq p^s - 1.$$

But G is non-abelian, so $s \geq 3$. Therefore $|\mathcal{X}|(p - 1) \geq p^2$, that is to say, $|\mathcal{X}| \geq p + 2$. □

Proposition 5.4. *Suppose that* $\exp(G) = p$. *If $a \neq s$, then $p + 1 < a < s$.*

Proof. This is immediate from Lemmas 5.2 and 5.3. □

Recall that a p-group is said to be **regular** provided, for all elements x and y, and any $n = p^\alpha$, we have $(xy)^n = x^n y^n s$ where s is a product of n-th powers of elements of the derived group of $\langle x, y \rangle$. Regular p-groups are discussed in Hall [7, Section 12.4]. We mention that every p-group with nilpotency class less than p is regular. In particular, every p-group of order at most p^p is regular.

As noted in Hall [7, Theorem 12.4.5], the Cauchy elements of a regular p-group G, together with the identity element, comprise a normal subgroup G_p of G.

Proposition 5.5. *Suppose that G is regular. If* $\mathrm{axe}(G_p) = \mathrm{saw}(G_p)$, *then $a = s$.*

Proof. The normal series $1 \trianglelefteq G_p \trianglelefteq G$ refines to a chief series with rank $\mathrm{saw}(G_p)$. Hence $s \leq \mathrm{saw}(G_p)$. By Lemma 2.1, $\mathrm{axe}(G_p) \leq a$. □

Therefore, if Conjecture 1.1 holds for all groups of exponent p, then it holds for all regular p-groups. Furthermore, Propositions 5.4 and 5.5 imply:

Corollary 5.6. *If $|G| \leq p^p$, then $a = s$.*

An example of a non-regular p-group is the wreath product $C_p \wr C_p$. Indeed, $C_p \wr C_p$ is generated by two Cauchy elements but, on the other hand, observing that $C_p \wr C_p$ is isomorphic to the Sylow p-subgroups of the symmetric group S_{p^2}, we see that $\exp(C_p \wr C_p) = p^2$. As a special case of Theorem 5.9, below, $\mathrm{axe}(C_p \wr C_p) = p = \mathrm{saw}(C_p \wr C_p)$. Again, we put part of the proof in some preliminary lemmas.

Lemma 5.7. *Suppose that G is a semidirect product EC where $|C| = p$ and the normal subgroup E is elementary abelian. Then any irreducible $\mathbb{C}G$-module of dimension greater than unity is induced from an irreducible $\mathbb{C}E$-module.*

Proof. It is well-known that the assertion still holds when E is replaced by any abelian subgroup of index p. We give a short proof for completeness. Let X be a simple $\mathbb{C}G$ module of dimension greater than unity, and let Y be a 1-dimensional summand of $\mathrm{Res}^G_E(X)$. By Frobenius Reciprocity, X must be summand of $\mathrm{Ind}^G_E(Y)$. But the dimension of X is divisible by p, hence $X \cong \mathrm{Ind}^G_E(Y)$. □

Lemma 5.8. *Suppose that $G = EC$ as in the previous lemma. Regard E as a vector space over the field $\mathbb{F}_p$ of order p, and hence regard E as an $\mathbb{F}_pC$-module. Then G has exponent p if and only if no direct summand of the $\mathbb{F}_pC$-module E is free.*

Proof. We may assume that E is indecomposable as an $\mathbb{F}_pC$-module. It is well-known that the free $\mathbb{F}_pC$-module of rank unity has a unique composition series $0 < M_1 < ... < M_p = \mathbb{F}_pC$ where $\dim(M_d) = d$. Furthermore, each M_d is indecomposable, and every indecomposable $\mathbb{F}_pC$-module is isomorphic to one of the modules M_d (see, for instance, Landrock [8, Section I.8]). Write $E \cong M_d$, and identify G with the subgroup M_dC of M_pC. We are to show that $\exp(G) = p$ if and only if $d < p$.

The elements of M_pC can be written in the form $(x_1, ..., x_p)g^\alpha$ with $x_k \in \mathbb{F}_p$; the group operation is given by

$$(x_1, ..., x_p)g^\alpha(y_1, ..., y_p)g^\beta = (x_1 + y_{1+\beta}, ..., x_p + y_{p+\beta})g^\alpha g^\beta.$$

Here, the subscripts are interpreted modulo p. The Frattini subgroup of M_pC is the abelian group M_{p-1}, which consists of the elements of the form $(x_1, ..., x_p)$ such that $x_1 + ... + x_p = 0$. We have

$$((x_1, ..., x_p)g^\alpha)^p = (x, ..., x),$$

where $x = x_1 + ... + x_p$. So $M_pC - M_{p-1}C$ is precisely the set of elements of order p^2. In particular, the group $E = M_dC$ has exponent p if and only if $d < p$. □

Theorem 5.9. *Suppose that G has an elementary abelian p-subgroup with index p. Write $|G| = p^n$. If $\exp(G) = p$, then $a = n = s$, otherwise $a = n - 1 = s$.*

Proof. Write $n = e + 1$. Let E be an elementary abelian subgroup of G with $|E| = p^e$. By Lemmas 2.2, 2.1 and Proposition 2.5,

$$e + 1 \geq s \geq a \geq \mathrm{axe}(E) = e = \mathrm{saw}(E).$$

Our task is to show that if $\exp(G) = p$ then $a = e + 1$, otherwise $s = e$. We may assume that $G - E$ owns a Cauchy element g, since otherwise $s = e$. Writing C for the subgroup generated by g, then $G = EC$ as a semidirect product.

Suppose that $\exp(G) = p$. The element g does not act freely on any $\mathbb{C}G$-module induced from E. By Lemma 5.7, G is inductible. By an inductive argument on e, we have $\mathrm{axe}(H) = \mathrm{saw}(H)$ for every proper subgroup H of G. Lemma 5.1 now yields $a = s = e + 1$.

Now suppose that $\exp(G) \neq p$. As in Lemma 5.8, we regard E as an $\mathbb{F}_pC$-module. First, consider the case where E is indecomposable. Lemma 5.8 tells us that $E \cong M_p$. (Thus, we are now dealing with the case $G \cong M_pC \cong C_p \wr C_p$.) Let f be any element of G with order p^2. Using the above formula for the group operation, an easy calculation shows that the conjugacy class of f is the coset $M_{p-1}f$. So every element of $\langle M_{p-1}f \rangle - M_{p-1}$ has order p^2, and

$$\mathrm{rk}(M_1 \lhd \dots \lhd M_{p-1} \lhd \langle M_{p-1}f \rangle \lhd M_pC) = p.$$

We have shown that $s \leq p$ when E is indecomposable. In fact, we must have equality $s = a = p$.

For the general case $G = EC$, Lemma 5.8 allows us to write $E = M \oplus N$ as a direct sum of $\mathbb{F}_pC$-modules, where M is free of rank unity. As we saw above, the quotient group G/N has a chief series where one of the factors is non-Cauchy. So the normal series $1 \unlhd N \lhd G$ refines to a chief series with rank e. Again, we have shown that $s \leq e$ and, again, we must have equality $s = a = e$. □

6 Special classes associated to free linear actions

Let G be a finite group and M a $\mathbb{Z}G$-lattice (a $\mathbb{Z}$-free $\mathbb{Z}G$-module). Let $H^*(G, M)$ denote the cohomology of G in twisted coefficients M. In particular, $H^2(G, M)$ denotes the equivalence classes of factor sets $f : G \times G \to M$. Recall that, for every subgroup $H \leq G$, the inclusion map gives rise to the restriction map $\mathrm{Res}^G_H : H^*(G, M) \to H^*(H, M)$.

Definition 6.1. *A cohomology class $\alpha \in H^2(G, M)$ is called as a special class if $\mathrm{Res}^G_H \alpha \neq 0$ for all cyclic subgroups H in G.*

Special classes appear in many contexts. Given a group extension of the form

$$0 \to M \to \Gamma \to G \to 1,$$

it is known that Γ is torsion free if and only if the associated cohomology class $\alpha \in H^2(G, M)$ is a special class (see, for instance, [14]). These types of extensions appear as short exact sequences of fundamental groups associated to a free action on a torus.

The most common appearance of special classes is in the study of compact flat manifolds (Riemannian manifolds with zero curvature). It is well known that Γ is isomorphic to the fundamental group of such a manifold if and only if it fits into an extension $0 \to M \to \Gamma \to G \to 1$ where G is finite and M is a free abelian and maximal abelian in Γ (see Charlap [5]). Such a group Γ is called a **Bieberbach group**. The group G is the holonomy group of the corresponding manifold. The condition that M is a maximal abelian subgroup is equivalent to M being a faithful $\mathbb{Z}G$-lattice. In fact, given an arbitrary $\mathbb{Z}G$-lattice M with kernel $K \leq G$ and a special extension (extension with associated class special) $0 \to M \to \Gamma \to G \to 1$, the group extension

$$0 \to M \to A \to K \to 1$$

is necessarily abelian (see, for instance, Theorem 5 in [14]), so Γ fits into an extension

$$0 \to A \to \Gamma \to G/K \to 1$$

where A is now maximal abelian in Γ. Therefore, for any $\mathbb{Z}G$-lattice M, the extension group Γ of a special extension is a Bieberbach group.

In fact, we have:

Proposition 6.2. *Let G be a finite group and M a $\mathbb{Z}G$-lattice. Then the following are equivalent:*
(i) *There is a special class $\alpha \in H^2(G, M)$.*
(ii) *There is an extension $0 \to M \to \Gamma \to G \to 1$ where Γ is torsion free.*
(iii) *The group G acts freely on a torus $X = T^n$ where $H_1(X, \mathbb{Z}) \cong M$ as a $\mathbb{Z}G$-module.*

Proof. For (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (i), see [14]. We only need to show (ii) $\Rightarrow$ (iii). By the above discussion, Γ is a Bieberbach group, so Γ imbeds into the group of isometries of $\mathbb{R}^n$ where $n = \dim M$ (see [3]). Since M acts as translations, $\mathbb{R}^n/M = T^n$, and the group $G = \Gamma/M$ acts freely on $\mathbb{R}^n/M$. □

Unlike the case of free actions on products of spheres, for every finite group G, we can find a free G-torus. In other words, for every group G, there is a suitable M such that $H^2(G, M)$ has a special class. In fact, if we take M as the direct sum of all induced modules $\operatorname{Ind}_C^G \mathbb{Z}$ over all cyclic subgroups $C \leq G$, we have

$$H^2(G, M) \cong \bigoplus_C H^2(C, \mathbb{Z}) \cong \bigoplus_C H^1(C, \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_C \operatorname{Hom}(C, \mathbb{Q}/\mathbb{Z}).$$

So, by picking nontrivial homomorphisms for each cyclic subgroup, we can form a special class in $H^2(G, M)$. But this is not the most efficient way to get such a class, since M is usually very big. In general, it is a difficult problem to find the minimal dimension of M for a given holonomy group G.

In the rest of this section we show that axe-saw conjecture is related to a form of this minimal dimension problem. We consider the case where M is a permutation module. Recall that a module is a permutation module if it is a direct sum of modules of the form $\operatorname{Ind}_H^G \mathbb{Z}$. We write $M \cong \bigoplus_{i=1}^k \operatorname{Ind}_{H_i}^G \mathbb{Z}$. Notice that the $\mathbb{Z}$-rank of M^G is equal to k, the number of summands in M.

Question 6.3. *Let G and M be as above. If there is a special class in $H^2(G, M)$, does it follow that* $\operatorname{saw}(G) \leq k$ *?*

It is known that the answer is affirmative when G is abelian (see [1], [14]). We shall show below that an affirmative answer to this question implies the axe-saw conjecture.

Let G be a finite group and H a subgroup of G. Let $Y \in \operatorname{Hom}(H, \mathbb{C}^\times)$ be a 1-dimensional representation of H and let $X = \operatorname{Ind}_H^G Y$. Recall that there is an isomorphism

$$\operatorname{Hom}(H, \mathbb{C}^\times) \xrightarrow{\cong} H^2(H, \mathbb{Z})$$

which maps each 1-dimensional representation to its first Chern class (see page of 67 of [12]). Let γ denote the first Chern class of Y. Note that the above isomorphism commutes with restrictions. Hence, $\operatorname{Res}_C^H \operatorname{ch}_1(Y) = \operatorname{ch}_1(\operatorname{Res}_C^H(Y))$ for any subgroup C of H. In fact, this last identity holds more generally for higher dimensional representations as well.

We want to define a class in $H^2(G, \operatorname{Ind}_H^G \mathbb{Z})$ associated to a given γ in $H^2(H, \mathbb{Z})$. This can be done using Shapiro's Lemma, which states that

$$H^*(H, \mathbb{Z}) \xrightarrow{\cong} H^*(G, \operatorname{Ind}_H^G \mathbb{Z})$$

for every subgroup H in G. Let α be the image of γ under Shapiro's isomorphism.

Lemma 6.4. *If g is a Cauchy element which acts freely on X, then* $\mathrm{Res}^G_{<g>}\alpha \neq 0$.

Proof. Let us denote the cyclic subgroup generated by g by C. By Lemma 3.1, the element g lies in the core of H, so $C \cap H^x = C$ for every $x \in G$. Now, consider the following double coset formula, where E denotes the set of coset representatives:

$$\mathrm{Res}^G_C\ \mathrm{Ind}^G_H\ \mathbb{Z} = \bigoplus_{x \in E} \mathrm{Ind}^C_{C\cap H^x}\ \mathrm{Res}^{H^x}_{C\cap H^x}\mathbb{Z}$$

where the right-hand side simplifies to $\bigoplus_{x\in E} \mathrm{Res}^{H^x}_C \mathbb{Z}$. Hence, we can write

$$\mathrm{Res}^G_C\alpha = \bigoplus_{x\in E} \mathrm{Res}^{H^x}_C \gamma^x.$$

Notice that $\mathrm{Res}^H_C\alpha$ is nonzero if one of the components is zero. By considering the component corresponding to $x = 1$, we see that it suffices to show $\mathrm{Res}^H_C\gamma \neq 0$.

Since C acts freely on Y, the restricted module $\mathrm{Res}^G_C Y$ is non-trivial, hence $\mathrm{ch}_1(\mathrm{Res}^H_C Y)$ is non-trivial. By the naturality of Chern isomorphism, we conclude that

$$\mathrm{Res}^H_C\gamma = \mathrm{Res}^H_C[\mathrm{ch}_1(Y)] = \mathrm{ch}_1[\mathrm{Res}^H_C Y] \neq 0.$$

□

Observe that the converse of the lemma does not hold in general. To see this, observe that, for a prime order cyclic subgroup C, the restriction $\mathrm{Res}^G_C\alpha$ is non-zero if at least one of the terms in the double coset formula is non-zero. The terms in the double coset formula correspond to first Chern classes of summands of $\mathrm{Res}^G_C X$, and we have

$$\mathrm{Res}^G_C X = \mathrm{Res}^G_C\ \mathrm{Ind}^G_H\ Y = \bigoplus_{x\in E} \mathrm{Ind}^C_{C\cap H^x}\ \mathrm{Res}^{H^x}_{C\cap H^x}\ Y^x.$$

For C to act freely on X, all these summands must be non-trivial. So, in general $\mathrm{Res}^G_C\alpha \neq 0$ does not imply that C acts freely on X.

Now, let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a set of monomial representations and $\{H_1, \ldots, H_k\}$ the set of subgroups such that $X_i = \mathrm{Ind}^G_{H_i} Y_i$ for some $Y_i \in \mathrm{Hom}(G, \mathbb{C}^\times)$. For each i, let γ_i denote the first Chern class of Y_i in $H^2(H_i, \mathbb{Z})$, and let $\alpha_i \in H^2(G, \mathrm{Ind}^G_{H_i}\mathbb{Z})$ be the image of γ_i under Shapiro's isomorphism. Putting these together we get an element

$$\alpha_{\mathcal{X}} = (\alpha_1, \ldots, \alpha_k) \in H^2(G, \bigoplus_{i=1}^{n} \mathrm{Ind}^G_{H_i}\mathbb{Z}).$$

We now come to the main result of this section:

Proposition 6.5. *If G acts freely on the set $\mathcal{X}$, then $\alpha_{\mathcal{X}}$ is a special class.*

Proof. Given a cyclic subgroup C in G, we can pick a Cauchy element g in C. Since G acts freely on $\mathcal{X}$, there is at least one X_i such that g acts freely on X_i. By Lemma 6.4, we have $\mathrm{Res}^G_{<g>}\alpha_i \neq 0$. So, α restricts to $< g >$ non-trivially. Since $\mathrm{Res}^G_{<g>} = \mathrm{Res}^C_{<g>}\mathrm{Res}^G_C$, it restricts non-trivially to C as well. □

Corollary 6.6. *If the answer to Question 6.3 is affirmative, then the axe-saw conjecture holds.*

Let us note some other consequences of Proposition 6.5 for Bieberbach groups.

Corollary 6.7. *Let G be a supersolvable group with* $\mathrm{saw}(G) = s$*. Then there is a Bieberbach group Γ which fits into an extension of the form $0 \to M \to \Gamma \to G \to 0$ such that the $\mathbb{Z}$-rank of the M^G is s.*

Proof. By Lemma 2.2, we can find a free linear action on a set $\mathcal{X}$ with $|\mathcal{X}| = s$. Associated with this set there is a group extension $0 \to M \to \Gamma \to G \to 1$, where M is a permutation module with s summands such that the extension class is special. □

Notice that in the corollary, the permutation module M need not be faithful. In other words, the holonomy group of Γ need not be G. However, replacing M with $M' = M \oplus \mathrm{Ind}_1^G \mathbb{Z}$ and Γ with Γ', where Γ' is the extension group of M' and G with the extension class $\alpha' = (\alpha, 0)$ in $H^2(G, M')$, we obtain a Bieberbach group Γ' with holonomy group G, and where the $\mathbb{Z}$-rank of the center $Z(\Gamma') = M'$ is $s + 1$. From this it follows:

Corollary 6.8. *Let G be a supersolvable group with* $\mathrm{saw}(G) = s$*. Then there is a flat Riemannian manifold X with holonomy group G such that the holonomy representation is a permutation module and the first Betti number in rational coefficients is equal to $s + 1$.*

The converses of Proposition 6.5 and Corollary 6.6 may fail as did the converse of Lemma 6.4. As an example, let $G = D_8$, the dihedral group of order 8, and let $\mathcal{X} = \{X_C \mid |C| = 2\}$ where $X_C = \mathrm{Ind}_C^G Y_C$ and Y_C is the non-trivial one dimensional representation of C. It is clear that $\alpha_{\mathcal{X}}$ is a special class. But G does not act freely on $\mathcal{X}$ because if g is a non-central Cauchy element of G, then it will fix a point on all $X \in \mathcal{X}$.

One can try to get a converse to these results by assuming that each X in $\mathcal{X}$ is induced from a normal subgroup. But then, writing $X = \mathrm{Ind}_H^G Y$ and $K = \ker Y$, we see that a Cauchy element acts freely on X if $g \in H - K^x$ for all x, whereas the restriction of the corresponding cohomology class is non-zero if $g \in H - K^x$ for some x. So, the converse still fails. However, if we assume that both H and K are normal, then the two conditions are equivalent. We call an induced representation $X = \mathrm{Ind}_H^G Y$ a **dinormal representation** if both H and $\ker Y$ are normal subgroups of G. Similarly, we call a class $\alpha \in H^2(G, \mathrm{Ind}_H^G \mathbb{Z})$ a **dinormal class** provided H is normal and the kernel of the associated class $\gamma \in H^2(H, \mathbb{Z})$ is normal.

Proposition 6.9. *The following are equivalent:*
(i) *G acts freely on $\mathcal{X} = \{X_1, \dots, X_k\}$ where each X_i is a dinormal representation.*
(ii) *There is a special class $\alpha = (\alpha_1, \dots, \alpha_k)$ in $H^2(G, \bigoplus_{i=1}^k \mathrm{Ind}_{H_i}^G \mathbb{Z})$ where each α_i is a dinormal class.*
(iii) *There exist normal subgroups H_i, K_i for $i = 1, \dots, k$ such that the quotients H_i/K_i are non-trivial and cyclic, and every Cauchy element is in* $\bigcup_{i=1}^{k} (H_i - K_i)$.

Proof. It follows from the discussion above. □

When G is a p-group of exponent p, part (iii) says that G is covered by index p sections where each H_i and K_i are normal. It is an interesting group theoretical question as to whether the non-trivial elements of a p-group of order p^s can be covered using less than s sections $H_i - K_i$ such that $|H_i : K_i| = p$ (no longer assuming that the H_i and K_i are normal). More generally one can ask:

Question 6.10. *Let G be a p-group of order p^s. For $i = 1, \dots, k$, let H_i and K_i be subgroups such that $|H_i : K_i| = p$. If $\bigcup_{i=1}^{k} (H_i - K_i) = G - \{1\}$, then does it follow that*

$$k \geq \frac{p^t - 1}{p - 1} + s - t$$

where t is the minimum of $\log_p |G : K_i|$ over all i?

This question is related to earlier questions and conjectures only by its form. When the H_i and K_i are not normal, there seem to be no implications between possible answers to these questions.

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