On Nilpotent ideals in the cohomology ring of a finite group

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Abstract

In this paper we find upper bounds for the nilpotency degree of some ideals in the cohomology ring of a finite group by studying fixed point free actions of the group on suitable spaces. The ideals we study are the kernels of restriction maps to certain collections of proper subgroups. We recover the Quillen-Venkov lemma and the Quillen F-injectivity theorem as corollaries, and discuss some generalizations and further applications.

We then consider the essential cohomology conjecture, and show that it is related to group actions on connected graphs. We discuss an obstruction to constructing a fixed point free action of a group on a connected graph with zero "k-invariant" and study the class related to this obstruction. It turns out that this class is a "universal essential class" for the group and controls many questions about the groups essential cohomology and tranfers from proper subgroups.

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1 Introduction

A well known "localization" result in the theory of transformation groups states that if G is an elementary abelian p-group and X is a finite-dimensional G-complex then the G action on X has a fixed point ($X^G \neq \emptyset$) if and only if the map $H^*_G(pt) \to H^*_G(X)$ is injective. Although the "only if" part of this result is true for all groups, the "if" part fails in general when G is not elementary abelian. We interpret this failure as follows: The injectivity of the above map still gives us a restriction on the action in terms of the cohomology of the group, but when G is not elementary abelian this restriction no longer implies that G has a fixed point.

In particular, in this paper, given a G-CW-complex X, we will define the obstruction ring $O_G(X)$ as the quotient $I_G(X)/K_G(X)$ where $I_G(X)$ is the ideal of classes which restrict to zero on every isotropy subgroup of X, and $K_G(X)$ is the kernel of the map $H^*_G(pt) \to H^*_G(X)$. We prove the following:

Theorem 1.1. *Let X be a finite dimensional G-CW-complex. Then, the nilpotency degree [see Definition 2.1] of $O_G(X)$ is less than or equal to $dim(X) + 1$.*

This is proved as Corollary 3.4 in the paper. Although a form of this result already occurs in Quillen's seminal work "The spectrum of the Equivariant Cohomology Ring" [Qu1], we provide an elementary proof of this result based on an equivariant version of the classical argument bounding the Lusternik-Schnirelmann category of a space via its cup product length.

We do this for completeness and because this presentation of the result is more convenient for our applications.

The majority of the paper deals with showing how one can derive many conclusions on the structure of the nilpotent elements in the cohomology of a group G by constructing G-actions on suitable finite dimensional complexes and using the above nilpotency result.

Among the classical theorems we recover are:

(1) The Quillen-Venkov Lemma [QuVe], which follows by considering the classical action of $\mathbb{Z}/p$ on the circle S^1 via the pth roots of unity.

(2) Quillen's F-injectivity theorem [Qu1], which follows by considering the G action on the projective space of some irreducible complex representation for G of complex dimension ≥ 2.

(Note: This proof does not use Serre's theorem on Bocksteins [Se] and indeed since that is an obvious corollary, this provides another elementary proof of Serre's theorem.)

(3) The localization result mentioned at the beginning of this section.

Besides recovering these classical results, we derive many more similar results by considering various natural actions. In particular, we prove a more general version of the Quillen-Venkov lemma:

Theorem 1.2. *Let K, L be subgroups of G such that $|K : L| = p$ and let $x_L \in H^1(K; \mathbb{F}_p)$ be a one dimensional class such that $\ker(x_L) = L$. If $u_1, ..., u_{2|G:K|} \in H^*(G; \mathbb{F}_p)$ such that $\operatorname{res}^G_H u_i = 0$ for every $H \subseteq G$ which satisfies $H^g \cap K \subseteq L$ for some $g \in G$, then $u_1 \cdots u_{2|G:K|} \in \left(\mathcal{N}^G_K(\beta(x_L))\right)$ where $\mathcal{N}$ is the Evens norm map.*

Fixing a ring of coefficients k, the essential cohomology of G, denoted by $ess(G)$, is defined as the ideal of classes in $H^*(G; k)$ which restrict to zero on every proper subgroup. The following has been conjectured to hold for the essential cohomology of a finite group (see [Mar], [Mu]).

Conjecture 1.3 (essential cohomology conjecture). *If G is a finite group which is not elementary abelian, then $ess(G)^2 = 0$.*

We study this conjecture using the main theorem stated above. We observe that if X is a connected one dimensional G-CW-complex, (i.e., a connected graph) with no fixed points then $(ess(G))^2 \subseteq K_G(X)$. So, if K_G is the intersection of $K_G(X)$ for all connected, fixed point free G-graphs, then $(ess(G))^2 \subseteq K_G$. We use this to conclude:

Theorem 1.4. *If the essential cohomology conjecture is not true for a p-group G, then for any connected G-graph X the following is true: X has a fixed point if and only if the map $H^*_G(pt) \to H^*_G(X)$ is injective.*

Thus the essential cohomology conjecture is in some sense a converse statement to the localization lemma mentioned in the introduction. The above result appears as Theorem 6.2 in the paper.

In the second part of the paper, we study group actions on graphs using homological arguments. We observe that a fixed point free G-graph X gives rise to a short exact sequence of kG-modules

$$0 \to H_1(X) \to C_1(X) \to C_0(X) \to k \to 0$$

where $C_1(X)$ and $C_0(X)$ are permutation modules with no trivial summands. We call the corresponding extension class $\gamma_X \in Ext^2(k, H_1(X))$, the extension class associated to the G-graph X.

As a special situation we ask when one can find a fixed point free G-graph X with zero extension class. Such a graph will have $K_G(X) = 0$ yet have $X^G = \emptyset$. It turns out that there is a single obstruction for the existence of such a graph:

Theorem 1.5. *There exists a fixed point free action of G on a connected graph X with zero extension class if and only if $\alpha' \cdot \alpha = 0$.*

Here $\alpha \in Ext^1(k, N)$ [all extensions are over the ring kG] is the extension class for the extension

$$0 \to N \to kX \to k \to 0$$

where X is the set of cosets of maximal subgroups of G considered as a G-set with the usual multiplication action. The map on the right is the augmentation map, and $\alpha' \in Ext^1(N^*, k)$ is the Yoneda dual of α.

The class α is an essential class and it is a universal class in the following sense:

Theorem 1.6. *(1) A class in $Ext^*(k, k)(= H^*(G, k))$ is essential if and only if it is a Yoneda multiple of α.*
(2) A class β in $Ext^*(k, k)$ is a sum of transfers from proper subgroups if and only if the Yoneda product $\beta \cdot \alpha$ is zero.
(3) $ess(G) \subseteq tr(G)$ if and only if the map induced by Yoneda multiplication with $\alpha' \cdot \alpha$ from $Ext^*(N, k) \to Ext^{*+2}(N^*, k)$ is zero. (Here $tr(G)$ is the ideal of $H^*(G; k)$ generated by transfers from proper subgroups.)

We show for certain 2-groups, the product $\alpha' \cdot \alpha$ is zero:

Proposition 1.7. *If G is a 2-group, such that there are two nonzero elements x and y in $H^1(G; \mathbb{F}_2)$ such that $xy = 0$, then for this group, $\alpha' \cdot \alpha = 0$.*

Finally, we show that $\alpha' \cdot \alpha = 0$ does not hold in general: (Note this does not necessarily mean that (3) in Theorem 1.6 does not hold since it is still possible that multiplication by $\alpha' \cdot \alpha$ is zero even though $\alpha' \cdot \alpha$ is not.)

Proposition 1.8. *Let G be a 2-group such that $\Phi(K) = \Phi(G)$ for every subgroup $K \subseteq G$ of index 4, then $\alpha' \cdot \alpha \neq 0$. (Here $\Phi(G)$ is the Frattini subgroup of G.)*

Notice that extra-special 2-groups which have no elementary abelian subgroup of index 4 will satisfy the condition of this proposition, and hence $\alpha' \cdot \alpha \neq 0$ for these groups.

2 The Obstruction Ring

In this section we give proper definitions for concepts mentioned in the introduction. Then we make some basic observations.

Definition 2.1. *If $R^* = \oplus_{i=0}^{\infty} R^i$ is a graded ring, we define $R^+ = \oplus_{i=1}^{\infty} R^i$ and $n(R) = min\{n \in \mathbb{N} | x_1 \dots x_n = 0$ whenever $x_i \in R^+\}$. $n(R)$ is called the nilpotency degree of R. Thus we have for example*
(a) $n(R) = 1$ if and only if $R^+ = 0$.
(b) $n(R) = \infty$ if R^+ has a non-nilpotent element.

Definition 2.2. *Fix a ring of coefficients k. A space X is k-acyclic if $H^*(X; k)$ is the same as $H^*(pt, k)$.*

We start by recalling the following basic result (see page 332 of [Br1]):

Proposition 2.3. *Let X be a space and k a ring of coefficients and suppose that X can be covered by N k-acyclic open sets. Then $n(H^*(X; k)) \leq N$.*

We start out by generalizing this result to an equivariant one which also allows for more general open sets in the cover. From this point on, G will always stand for a discrete group and all cohomology will be with coefficients in a fixed commutative ring of coefficients k, which possesses an identity.

Definition 2.4. *Let X be a G-space. An open G-set U is an open subset of X which has $GU = U$. A G-equivariant open cover of X is an open cover of X by open G-sets.*

Definition 2.5. *If X is a G-space, $H^*_G(X) = H^*(EG \times_G X)$ is the equivariant cohomology of X. Here $EG \times_G X$ is the usual Borel construction on X.*

Proposition 2.6. *If X is a G-space with a finite equivariant open cover $U_1, \dots, U_k$ then*

$$n(H^*_G(X)) \leq \sum_{i=1}^{k} n(H^*_G(U_i)).$$

Proof. Let $n_i = n(H^*_G(U_i))$ and $N = \sum_{i=1}^k n_i$. Without loss of generality, $N < \infty$. Take N elements $x_1, \dots, x_N$ from $H^*_G(X)^+$.

Notice that the natural inclusion map $Id \times_G j : EG \times_G U_1 \to EG \times_G X$, induces a map $H^*_G(X) \to H^*_G(U_1)$ which takes the product $x_1 \dots x_{n_1}$ to zero as $n_1 = n(H^*_G(U_1))$. Thus looking at the long exact sequence,

$$\dots \to H^*_G(X, U_1) \to H^*_G(X) \to H^*_G(U_1) \to \dots$$

we see that $x_1 \dots x_{n_1}$ lifts to an element of $H^*_G(X, U_1)$.

Similarly, the product of the next n_2 x_i's lifts to $H^*_G(X, U_2)$ and so on. Thus using the same relative cup product argument as in the nonequivariant case (note that the $EG \times_G U_i$ are indeed open in $EG \times_G X$) we see that $x_1 \dots x_N$ is an image of $H^*_G(X, X)$ and hence must be zero.

This completes the proof. □

Notice that if one sets $G = 1$ in Proposition 2.6, one recovers Proposition 2.3. We will now proceed to extract a more useable version of Proposition 2.6.

Definition 2.7. *If X is a G-space, and $\pi : EG \times_G X \to BG$ is the canonical projection of the Borel construction, then we define the K-ideal of X, denoted $K_G(X)$, to be the kernel of $\pi^* : H^*(G) \to H^*_G(X)$. Notice $K_G(X)$ is in particular a graded ring (without identity).*

Remark 2.8. *Notice if X is a G-space where $X^G = \{x \in X \mid gx = x, \forall g \in G\}$ is nonempty, then $\pi : EG \times_G X \to BG$ has a section given by mapping $[y] \to (y, x_0)$ where x_0 is a fixed point from X^G. Thus π^* is injective and the K-ideal $K_G(X)$ is zero.*

Recall that, if X is a G-space and $x_0 \in X$ then the isotropy group at x_0, G_{x_0} is defined as $G_{x_0} = \{g \in G \mid gx_0 = x_0\}$. The elements of the collection of subgroups $\{G_x \mid x \in X\}$ are referred to as the isotropy groups of X. An isotropy subgroup G_x of X is called a maximal isotropy subgroup of X if it is maximal under inclusion in the collection of all isotropy subgroups of X.

Definition 2.9. *If X is a G-space, the isotropy ideal of X, denoted $I_G(X)$, is defined to be the kernel of*

$$i^* : H^*(G) \to \prod H^*(G_i)$$

where the product ranges over all the isotropy groups G_i of X and the map i^* is induced by restriction.

Notice that if we had formed the product only over a set of G-conjugacy representatives of the maximal isotropy subgroups of X, the kernel of i^* would be unchanged. Also notice that if $X^G \neq \emptyset$, then G is itself an isotropy group and hence i^* is injective and $I_G(X) = 0$.

Proposition 2.10. *If X is a G-space, $K_G(X) \subseteq I_G(X)$.*

Proof. Fix an isotropy group G_0, thus there is some point $x_0 \in X$ with $G_0 x_0 = x_0$. Notice we can take EG as EG_0 by just restricting the G action to G_0. Then one has

$$EG \times_{G_0} x_0 \to EG \times_{G_0} X \to EG \times_G X \xrightarrow{\pi} BG.$$

Notice that $EG \times_{G_0} x_0 = BG_0$ and the composite map is just $Bi : BG_0 \to BG$ where i is the inclusion homomorphism $i : G_0 \to G$. Since π appears in the composition, it follows immediately that the K-ideal $K_G(X)$ lies in the kernel of $Bi^* : H^*(G) \to H^*(G_0)$. Since this is true for all isotropy groups G_0, the lemma follows. □

Definition 2.11. *If $\mathfrak{A}$ is a collection of subgroups of G such that the restriction*

$$i^* : H^*(G) \to \prod_{H \in \mathfrak{A}} H^*(H)$$

is injective, we say $H^*(G)$ is detected on $\mathfrak{A}$.

Without loss of generality, we will always take our collections to be closed under taking subgroups and conjugates. Notice that this does not affect the question of whether $H^*(G)$ is detected on the collection or not.

Definition 2.12. *If $\mathfrak{A}$ and $\mathfrak{B}$ are two collections of subgroups of G, we say $\mathfrak{A}$ contains $\mathfrak{B}$ if $\mathfrak{B} \subseteq \mathfrak{A}$.*

The following corollary is immediate:

Corollary 2.13. *Let G be a group and $\mathfrak{A}$ be a collection of subgroups on which $H^*(G)$ is detected. Then for any G-space X whose isotropy groups contain $\mathfrak{A}$, one has $K_G(X) = 0$, i.e., $\pi^* : H^*(G) \to H^*_G(X)$ is injective.*

Conversely, if X is a G-space whose K-ideal is nonzero, then $H^*(G)$ is not detected on the isotropy subgroups of X.

Definition 2.14. *Let G be a group and $\mathfrak{M}$ be the collection of all proper subgroups of G, then the kernel of the restriction*

$$i^* : H^*(G) \to \prod_{H \in \mathfrak{M}} H^*(H)$$

is called the essential cohomology of G and is denoted $ess(G)$.

Corollary 2.15. *If G is a group with $ess(G) = 0$ (for example if $char(\mathbb{F}) = p$ and G is not a p-group) then every G-space X whose isotropy groups contain $\mathfrak{M}$, has zero K-ideal.*

Definition 2.16. *For any G-space X, we define the obstruction ring $O_G(X)$, to be the quotient graded ring $I_G(X)/K_G(X)$. Notice in fact, $O_G(X)$ inherits a graded $H^*(G)$-module structure.*

2.1 G-CW-complexes

Let X be a CW-complex with cellular G-action. Suppose further that the G action permutes the cells of X, and if $g \in G$ fixes a cell setwise, it also fixes it pointwise. We will refer to this type of action as an *admissible* action. Note that the G action on a G-CW-complex obviously satisfies this condition, and conversely for any CW-complex with an admissible action we can find a G-CW-complex cell decomposition.

Now let X be a simplicial complex with a simplicial G-action. This means that the action of each element g induces a simplicial map on X. As before we call a simplicial action *admissible* if it has the property that if $g \in G$ fixes a simplex, it also fixes all its vertices.

Recall that any simplicial complex can be made into an admissible one by taking its barycentric subdivision. Note that an admissible simplicial complex is also an admissible CW-complex.

Also recall that, by Illman's Theorem, for any smooth action of a finite group G on a smooth manifold X, one can triangulate X to get an admissible simplicial action.

For a complete account of these results we refer the reader to a book on this subject by Allday and Puppe [AlPu].

Remark 2.17. *For our purposes admissible G-actions are enough, but we want to remark that these type of actions in general don't satisfy $|X|/G = |X/G|$, where $|X|$ denotes the realization of the simplicial complex X. Bredon (pg 116 in [Br2]), calls a simplicial action regular if it satisfies the following property for every subgroup $H \subset G$: If $h_0, ..., h_n$ are elements of H and $(v_0, ..., v_n)$ and $(h_0v_0, ..., h_nv_n)$ are both simplices of X, then there exists an element $h \in H$ such that $hv_i = h_iv_i$ for all i. A regular action clearly satisfies the above orbit property. It is well known that taking the barycentric subdivision of an admissible complex makes it regular.*

3 Nilpotency Degree of $O_G(X)$

In this section we will prove that the nilpotency degree of $O_G(X)$ is less than or equal to $dim(X)+1$ under reasonable conditions on X. We start by defining two sequences of ideals that extend the definition of $K_G(X)$ and $I_G(X)$ in the case that X is a G-CW-complex.

Let X be a G-CW-complex. Let $X^{(n)}$ denote the n-skeleton. Then the inclusion $i : X^{(n)} \to X$ is a G-map which induces a map $EG \times_G X^{(n)} \to EG \times_G X$. Using this map, it is easy to see that $K_G(X) \subseteq K_G(X^{(n)})$. In general, if $\dim(X) = N$, we have a sequence of nested ideals:

$$K_G(X) = K_G(X^{(N)}) \subseteq K_G(X^{(N-1)}) \subseteq \cdots \subseteq K_G(X^{(1)}) \subseteq K_G(X^{(0)}) = I_G(X).$$

Here the last identity $K_G(X^{(0)}) = I_G(X)$ can be obtained by considering the spectral sequence for $H^*(EG \times_G X^{(0)})$ which collapses at the $E_2^{*,*}$ term since it is concentrated on the row $q = 0$. Since $X^0 = \sqcup_{i\in I} G/G_i$ where the G_i range over isotropy subgroups, it is easy to check using Shapiro's lemma, that $H^*(EG \times_G X^{(0)}) = \prod_{i\in I} H^*(G_i)$, and that the map

$$H^*(G) \to H^*_G(X^{(0)}) = \prod_{i\in I} H^*(G_i)$$

is given by restriction. Thus $K_G(X^{(0)}) = I_G(X)$.

We also need the following definition:

Definition 3.1. *Let X be a G-CW-complex. Let $Iso(k)$ denote the collection of all isotropy groups of k-cells of X.*

It is easy to see that when X is a G-CW-complex, the collection $Iso(k)$ is contained in the collection $Iso(k-1)$ for all $1 \leq k \leq \dim(X)$.

Furthermore, $Iso(0)$ is the collection of all isotropy subgroups of X.

Definition 3.2. *If X is a G-CW-complex, let $I_G(X, m)$ denote the ideal of elements of $H^*(G)$ which restrict to zero in every group in $Iso(m)$.*

Thus $I_G(X,0) = I_G(X)$ and we have in general a sequence of nested ideals:

$$I_G(X) = I_G(X,0) \subseteq I_G(X,1) \subseteq \cdots \subseteq I_G(X, \dim(X)) \subseteq H^*(G).$$

(We will define by convention, $I_G(X,m) = H^*(G)$ for $m > \dim(X)$.)

We are now ready, to prove one of the main tools of this paper:

Theorem 3.3. *Let X be a G-CW-complex. Then*

$$I_G(X,m)K_G(X^{(k-1)}) \subseteq K_G(X^{(k)})$$

whenever $m \leq k$.

Proof. Since $X^{(k)}$ is a G-CW-complex, we can cover it by two open G-sets: $U = X^{(k)} - X^{(k-1)}$ and V, where V is an open neighborhood of $X^{(k-1)}$ in $X^{(k)}$ which (deformation) retracts to $X^{(k-1)}$ through radial projections from centers of the k-cells.

Let $\alpha \in I_G(X,m)$ and $\beta \in K_G(X^{(k-1)})$ and let $\bar{\alpha}$ and $\bar{\beta}$ be their images in $H^*_G(X^{(k)})$. We need to show $\bar{\alpha}\bar{\beta} = 0$.

If we look at the long exact sequence for the pair $(EG \times_G X^{(k)}, EG \times_G V)$ we have:

$$\cdots \to H^*_G(X^{(k)}, V) \to H^*_G(X^{(k)}) \xrightarrow{j^*} H^*_G(V) \to \dots$$

and by assumption, $j^*(\bar{\beta})$ is zero. (Since $H^*_G(V) = H^*_G(X^{(k-1)})$ naturally.)

Thus $\bar{\beta}$ comes from $H^*_G(X^{(k)}, V)$.

Similarly, looking at the long exact sequence for the pair $(EG \times_G X^{(k)}, EG \times_G U)$, we have $\bar{\alpha}$ maps to zero in $H^*_G(U) = \oplus_{H \in iso(k)} H^*(H)$. Thus $\bar{\alpha}$ comes from $H^*_G(X^{(k)}, U)$. (Since when $m \leq k$, the collection $iso(k)$ is contained in the collection $iso(m)$.)

Thus by the usual relative cup product argument, we see $\bar{\alpha}\bar{\beta}$ is the image of an element in $H^*_G(X^{(k)}, U \cup V = X^{(k)})$ and hence is zero as required.

Thus the proof is complete. □

Corollary 3.4. *Let X be a finite dimensional G-CW-complex or a smooth manifold on which G acts smoothly, then $n(O_G(X)) \leq \dim(X) + 1$.*

Proof. As mentioned before, the case of a smooth action reduces to the case of a G-CW-complex by Illman's theorem, thus we may assume that X is a finite dimensional G-CW-complex.

By repeatedly applying the $m = 0$ case of Theorem 3.3 we get:

$$I_G(X,0)^{\dim(X)} K_G(X^{(0)}) \subset K_G(X^{(\dim(X))}).$$

Using that $K_G(X^{(0)}) = I_G(X) = I_G(X,0)$, it then follows that:

$$I_G(X)^{\dim(X)+1} \subset K_G(X).$$

Since $O_G(X) = I_G(X)/K_G(X)$, the result then follows immediately. □

The next corollary of Theorem 3.3 is useful when modifying a G-space X to one of lower dimension with the same K-ideal, as we will see later in this paper. We will need the following somewhat technical definition:

Definition 3.5. *If X is a G-space, an element α is a nonzero divisor modulo $K_G(X)$, if whenever we have $\beta \in H^*(G)$ such that $\alpha\beta \in K_G(X)$, then $\beta \in K_G(X)$.*

Corollary 3.6. *If X is a finite dimensional, G-CW-complex such that for some $m \leq \dim(X)$, $I_G(X, m)$ contains a nonzero divisor modulo $K_G(X)$, then:*

$$K_G(X) = K_G(X^{(m-1)})$$

and hence

$$O_G(X) = O_G(X^{(m-1)}).$$

Proof. Since $I_G(X) = I_G(X^{(m-1)})$ for any m, and $K_G(X) \subseteq K_G(X^{(m-1)})$ in general, it is enough to show $K_G(X^{(m-1)}) \subseteq K_G(X)$.

Suppose not, then there is $\beta \in K_G(X^{(m-1)}) - K_G(X)$.

Let α be a nonzero divisor modulo $K_G(X)$ in $I_G(X, m)$. By Theorem 3.3, $\alpha^{\dim(X)-m+1}\beta \in K_G(X)$. However $\beta \notin K_G(X)$ so this contradicts the fact that α is a nonzero divisor modulo $K_G(X)$, and the proof is complete. $\square$

In the next section we construct some G-spaces with suitable properties, and use them to prove some classical theorems in the cohomology of groups.

4 Applications to Classical Theorems

Let us look at some examples to get some consequences of the machinery that has been developed in the previous section.

Fix a prime p. In this section, all coefficients for cohomology will be $\mathbb{F}_p$, the field of p elements.

4.1 Quillen-Venkov lemma

Now, we will construct a suitable action on a circle which leads to the Quillen-Venkov lemma [QuVe].

Let C_p denote the cyclic group of order p. It is easy to see that the circle S^1 has a free, smooth C_p-action. If we view the circle as the unit norm elements in the complex numbers, then multiplication with $e^{\frac{2\pi i}{p}}$ will be such an action.

Of course since this is a free action, $I_{C_p}(X) = H^*(C_p)^+$. It is easy to check that the K-ideal is the principal ideal $(\beta(e))$ of $H^*(C_p)$ where e is a generator of $H^1(C_p)$. By Corollary 3.4, it follows that $n(H^*(C_p)^+/(\beta(e))) \leq 2$ which is of course well-known since one knows $H^*(C_p)$ explicitly. However we will now modify this example in a minor way to get less trivial facts.

Let G be a group, and $u : G \to C_p$ a surjective homomorphism. Since $H^1(G; \mathbb{F}_p) \cong Hom(G; \mathbb{F}_p)$, u corresponds to a one dimensional class which we also call u. Notice that u is the image of a nonzero class in $H^1(C_p)$ under the induced map $u^* : H^*(C_p) \to H^*(G)$.

Now we can make S^1 a G-space by precomposing the original C_p-action with the map u.

We conclude $n(O_G(S^1)) \leq 2$ by Corollary 3.4. Since $\ker(u)$ is the only isotropy group for the G-space S^1 we have $I_G(S^1) = \ker\{H^*(G) \to H^*(Ker(u))\}$.

Considering the spectral sequence with $E_2^{p,q} = H^p(G; H^q(S^1))$ converging to $H_G^{p+q}(S^1)$, one sees easily that the K-ideal for the G-space S^1 is a principal ideal generated by a two dimensional class which is called the k-invariant. Since the action is obtained through $u : G \to C_p$, the k-invariant of the G action must be the image of the k-invariant of C_p action under $u^* : H^*(C_p) \to H^*(G)$. So, the K-ideal of the G action is $(\beta(u))$ where β is the Bockstein operation and $(\beta(u))$ means the principal ideal generated by $\beta(u)$ in $H^*(G)$. Thus we recover the Quillen-Venkov Lemma [QuVe]:

Corollary 4.1 (Quillen-Venkov Lemma). *If $u : G \to C_p$ is a surjective homomorphism, and α_1, α_2 are in the kernel of the restriction*

$$H^*(G) \to H^*(\ker(u)),$$

then $\alpha_1\alpha_2 \in (\beta(u))$.

We will also state an immediate consequence for essential cohomology:

Corollary 4.2. *Let G be a p-group and our field of coefficients be $\mathbb{F}_p$. If $[G, G] \neq \Phi(G)$, i.e., the commutator subgroup and the Frattini subgroup do not coincide, then $ess(G)^2 = 0$, i.e., the product of any two elements in essential cohomology is zero.*

Proof. Since $[G, G] \neq \Phi(G)$, G_{ab} is not an elementary abelian p-group. This means that we can find a surjective homomorphism $u : G \to C_p$ which factors through the quotient map $C_{p^2} \to C_p$.

It follows easily that if we view $u \in H^1(G)$, then $\beta(u) = 0$. Now if $\alpha_1, \alpha_2 \in ess(G)$, then certainly they restrict to zero in $H^*(\ker(u))$ and so by Corollary 4.1, we have $\alpha_1\alpha_2 \in (\beta(u)) = 0$ giving us the result. □

4.2 Quillen's F-injectivity theorem

An important theorem in group cohomology is the F-isomorphism theorem of Quillen [Qu1]. Part of this theorem says in particular that the kernel of restrictions to elementary abelian subgroups is a nilpotent ideal. We prove this statement using a G-space with zero K-ideal and proper isotropy subgroups. Such a G-space exists for every nonabelian finite group G, and the example we use was first discovered by Peter Symonds in [S].

Lemma 4.3. *Let G be a nonabelian finite group. Then there is a finite G-space X with $K_G(X) = 0$ and $X^G = \emptyset$ and with dimension equal to $2n - 2$ where n is the complex dimension of the smallest irreducible complex representation of G of dimension greater than one. Thus $ess(G)^{2n-1} = 0$.*

Proof. Since G is nonabelian, it has an irreducible complex representation V of dimension $n \geq 2$. Let X be the projective space $\mathbb{P}(V)$. The action of G on $H^*(\mathbb{P}(V))$ is trivial as it factors through the connected group $GL_n(\mathbb{C})$. Consider the Borel fibration:

$$X \xrightarrow{i} EG \times_G X \xrightarrow{\pi} BG$$

Since the canonical line bundle l over X extends to a line bundle l' over $EG \times_G X$, we have $c_1(l) = i^* c_1(l')$ where c_1 denotes the first Chern class. This implies that the map $i^* : H^*_G(X) \to H^*(X)$ is surjective, hence the spectral sequence for equivariant cohomology collapses at the E_2 term. Therefore, $K_G(X) = 0$. The fact that $X^G = \emptyset$ follows since if some line were left invariant under G, this would provide a 1-dimensional subrepresentation of V which contradicts irreducibility of V.

The final comment about $ess(G)$ follows from noting that $ess(G) \subseteq I_G(X)$ and using Corollary 3.4. □

Theorem 4.4 (Quillen [Qu1]). *Let G be a finite group, and let $\mathcal{E}_p$ denote the set of all elementary abelian p-subgroups in G. Then,*

$$\ker\{\text{res} : H^*(G) \to \prod_{E \in \mathcal{E}_p} H^*(E)\}$$

is a nilpotent ideal. (Recall that a nilpotent ideal is an ideal which has finite nilpotency degree as a graded ring).

Proof. Without loss of generality, assume G is a p-group. First let us also assume G is not abelian. Then, by lemma 4.3, there is a finite G-space X with K-ideal zero, and with proper isotropy subgroups. Using corollary 3.4, we can conclude that the ideal

$$\ker\{\text{res} : H^*(G) \to \prod_{H < G} H^*(H)\}$$

is nilpotent. So, by induction the result will follow once we show abelian groups also satisfy the theorem. If G is abelian but not elementary abelian, then we can find a homomorphism $u : G \to C_p$ which is surjective and factors through C_{p^2}. Note $\beta(u) = 0$, thus by corollary 4.1, we conclude the restriction $H^*(G) \to H^*(\ker(u))$ has kernel with finite nilpotency degree, thus proving the theorem by a simple induction. □

In the literature there are some other elementary proofs of Quillen's F-injectivity theorem. One of them is again due to Quillen and Venkov which uses the Quillen-Venkov lemma and Serre's theorem on the vanishing of products of Bocksteins (see [Se] or [B] for the statement). Another one is a recent proof given by Jon F. Carlson [Ca], but this proof also uses Serre's theorem. We remark here that we do not assume Serre's theorem in any stage of the proof. Since this theorem easily follows from Quillen's F-injectivity theorem, we obtain another proof of Serre's theorem [Se]:

Theorem 4.5 (Serre). *If G is a p-group which is not elementary abelian, then there exist non-zero elements $u_1, u_2, \ldots, u_m \in H^1(G)$ such that*

$$u_1 u_2 \cdots u_m = 0 \quad \textit{if} \quad p = 2,$$
$$\beta(u_1)\beta(u_2)\cdots\beta(u_m) = 0 \quad \textit{if} \quad p > 2.$$

Proof. Let α be the product of $\beta(u)$ for all nonzero elements in $u \in H^1(G)$. It is clear that α will restrict to zero on all proper subgroups. Since G is not elementary abelian, α will lie in the kernel of the map

$$\ker\{\mathrm{res} : H^*(G) \to \prod_{E \in \mathcal{E}_p} H^*(E)\}.$$

By Theorem 4.4, $(\alpha)^n$ will be zero for some n. Hence the theorem follows. (For $p = 2$ notice that $\beta(u) = u^2$.) □

4.3 Localization Theorem

In the introduction we introduced our main result as a generalization of a well known localization result. To justify this point of view, we prove here that this result follows from the main theorem:

Theorem 4.6. *Let G be an elementary abelian p-group and X a finite dimensional G-CW-complex. Then, the G action on X has a fixed point ($X^G \neq \emptyset$) if and only if the map $H^*_G(pt) \to H^*_G(X)$ is injective.*

Proof. It is clear that if X has fixed point then $K_G(X) = \mathrm{Ker}\{H^*(pt) \to H^*_G(X)\}$ is zero. Conversely, assume that $K_G(X) = 0$. Then by corollary 3.4, we have $[I_G(X)]^{\dim(X)+1} = 0$. If X has no fixed points, then $ess(G) \subseteq I_G(X)$ and so $I_G(X)$ will include the product of Bocksteins of all nonzero one dimensional classes. But, Bocksteins of one dimensional classes generate a polynomial subalgebra in the cohomology of G. This contradicts the fact that $[I_G(X)]^{\dim(X)+1} = 0$. Hence G must fix a point in X. □

5 Further Applications

5.1 Generalizations of the Quillen-Venkov lemma

In the proof of the Quillen-Venkov lemma, we used a linear action on a circle with isotropy subgroup equal to a maximal subgroup H. We can easily generalize that argument to any linear action on a sphere. In particular, if we take the representations which are induced from a one dimensional linear representation of a subgroup K, then we get the following generalization of the Quillen-Venkov lemma:

Theorem 5.1. *Let K, L be subgroups of G such that $|K : L| = p$ and let $x_L \in H^1(K)$ be a one dimensional class such that* $\ker(x_L) = L$. *If $u_1, \ldots, u_{2|G:K|} \in$*

$H^*(G)$ *such that* $\operatorname{res}^G_H u_i = 0$ *for every* $H \subseteq G$ *which satisfies* $H^g \cap K \subseteq L$ *for some* $g \in G$*, then* $u_1 \cdots u_{2|G:K|} \in \left(\mathcal{N}^G_K(\beta(x_L))\right)$ *where* $\mathcal{N}$ *is the Evens norm map.*

Proof. Let V_L be a one dimensional, complex representation of K with kernel L. Let $V = \operatorname{ind}^G_K V_L$, and $X = S(V)$, the unit sphere of the underlying $2|G : K|$ dimensional Euclidean space. The Euler class $e(X)$ of the spherical fibration $X \to EG \times_G X \to BG$ is exactly $\mathcal{N}^G_K(\beta(x_L))$, and $K_G(X)$ is the ideal generated by the euler class. Also observe that the isotropy subgroups of the G-space X are exactly the subgroups $H \subseteq G$ such that $H^g \cap K \subseteq L$ for some $g \in G$. So, an element $u \in H^*(G)$ that satisfies the conditions of the theorem will lie in $I_G(X)$. By Corollary 3.4, the obstruction group $O_G(X) = I_G(X)/K_G(X)$ has nilpotency degree less than or equal $\dim(X) + 1 = 2|G : K|$. The theorem follows. □

We can also generalize Quillen-Venkov lemma in another way: Notice for a group G, if V is a subspace of $H^1(G; \mathbb{F}_p)$, then V corresponds to a normal subgroup of G containing $[G, G]G^p$ once we identify $H^1(G; \mathbb{F}_p) = Hom(G, \mathbb{F}_p)$. Let us call this corresponding normal subgroup G_V, it is defined as

$$G_V = \cap_{f \in V} \ker(f).$$

Notice if G is a p-group and we use $V = H^1(G; \mathbb{F}_p)$ then $G_V = Frat(G)$, the Frattini subgroup.

Proposition 5.2. *If* G *is a group, and we use* $\mathbb{F}_p$ *coefficients, then if* V *is a subspace of* $H^1(G)$ *contained in the kernel of the Bockstein, and*

$$I_V = \ker\{i^* : H^*(G) \to H^*(G_V)\}$$

then $n(I_V) \leq \dim(V) + 1$*.*

Proof. Let $d = \dim(V)$ and consider the free action of an elementary abelian p-group E of rank d on a d-torus T formed by taking a product action using the actions of cyclic groups of order p on the circle discussed before. Notice of course this action induces a trivial action on $H^*(T)$. Recall

$$H^*(E) = \wedge^*(x_1, \ldots, x_d) \otimes \mathbb{F}_p[\beta(x_1), \ldots, \beta(x_d)]$$

for p odd and for $p = 2$ is a polynomial algebra on the x_i.
In the spectral sequence for $H^*_E(T)$, with appropriate choice of basis $\{x_1, \ldots, x_n\}$ for $H^1(E)$, the generators of $H^1(T) = E_2^{0,1}$ transgress to the $\beta(x_i)$.

Now consider the G-space T where we make G act via $E = G/G_V$. Notice in fact that E indeed has rank $\dim(V)$. Comparing the spectral sequence for $H^*_G(T)$ with that for $H^*_E(T)$ discussed above, we see that $d_2(E_2^{0,1}) = 0$ by our assumption that V lied inside the kernel of the Bockstein. Since $H^*(T)$ is generated by its 1-dimensional classes, we conclude that $d_2 = 0$ and in fact $E_2 = E_\infty$. Thus $K_G(T) = 0$. On the other hand, it is obvious that the unique isotropy group for the G-space T is G_V and so $I_G(T) = I_V$. Since $\dim(T) = d = \dim(V)$, the result follows from Corollary 3.4. □

5.2 Applications to Infinite Groups

We now give some examples for the cohomology of infinite groups. The first proposition follows easily by a relative cup product argument but we provide a proof from the perspective of this paper for variety:

Proposition 5.3. *Let $G = G_1 *_H G_2$ be an amalgamated product, where the amalgamation maps $H \to G_1$ and $H \to G_2$ are injective. Then if*

$$B = \ker\{i^* : H^*(G) \to H^*(G_1) \oplus H^*(G_2)\}$$

then $B^2 = 0$. (Note B consists of boundary classes coming from $H^*(H)$ in the associated Mayer-Vietoris sequence for the amalgamated product.)

Proof. It is known that G acts on a tree X such that the maximal isotropy groups are exactly the conjugates of G_1 and G_2 (see [Bn], page 52-54.)

Thus $I_G(X) = B$ and $K_G(X) = 0$ as X is contractible. Since X has dimension one, Corollary 3.4 now gives the result. □

Theorem 5.4 (Quillen). *Let Γ be a group of finite virtual cohomological dimension (vcd.) Then if $\mathfrak{F}$ is the collection of finite subgroups of Γ and $I_{\mathfrak{F}}$ is the kernel of the restriction*

$$i^* : H^*(\Gamma) \to \prod_{G \in \mathfrak{F}} H^*(G)$$

then $I_{\mathfrak{F}}^{n+1} = 0$ where if Γ' is a torsion-free subgroup of Γ of finite index, we can take $n = |\Gamma : \Gamma'| vcd(\Gamma)$.

Proof. For a group Γ of finite vcd, one can construct an acyclic Γ-CW complex X of dimension n as stated above, on which Γ acts with finite isotropy groups (see [Bn], Pg 190-191 and Exercise 3 on pg 209.)

Then $I_{\mathfrak{F}} \subseteq I_G(X)$ and $K_G(X) = 0$ (as X is acyclic) so the result follows from Corollary 3.4.

Incidentally, the word acyclic can be replaced with contractible in the above proof as long as $vcd(\Gamma) \neq 2$ Whether this can be done in general is open and would follow from the Eilenberg-Ganea conjecture which states that if $cd(H) = 2$ then there is a 2-dimensional $K(H, 1)$. (see [Bn], Pg 205-206.) □

5.3 Detection for extraspecial 2-groups

In this section, coefficients for cohomology will be the field of order 2.

Let G be an extraspecial 2-group. For simplicity let us assume that G is of real type with nondegenerate form. In other words, let G be the n-fold central product of dihedral groups of order 8.

We will now look at a suitable real representation V of dimension 2^n for G, in order to show that the cohomology of G is detected on its elementary abelian subgroups, which is a result due to Quillen [Qu2].

Let $\{a_1, .., a_n, b_1, .., b_n\}$ be generators for $G = D_8 * \cdots * D_8$ such that $\{a(i), b(i)\}$ are generators of the ith term in the central product satisfying the relations $a_i^2 = b_i^2 = 1$ and $[a(i), b(i)] = c$, where c is the central element. Let A be the elementary abelian subgroup generated by the a_i's, similarly let B be the elementary abelian subgroup generated by the b_i's. Also let us denote the central subgroup by C.

Consider the one dimensional representation $C \to GL_1(\mathbb{R}) = \mathbb{R} - 0$ which takes the element c to -1. Through the projection $A \times C \to C$, this gives us a one dimensional real $A \times C$ module, say W. When we induce W to G, we obtain a $|G : A \times C| = |B| = 2^n$ dimensional real G module V.

Notice that

$$V = W \otimes_{\mathbb{R}[A \times C]} \mathbb{R}G.$$

If w is a nonzero element of W, then we can choose a basis of the form $\{w \otimes b | b \in B\}$ for V. (Let us denote $\hat{b} = w \otimes b$ for convenience.)

In other words, we choose as a set of coset representatives for $A \times C$ in G, the elements of B.

The action on V is defined by right multiplication, so B acts via a permutation of our chosen basis. And A acts as:

$$(\hat{b})a = (w \otimes b)a = [b, a]w \otimes b = \pm \hat{b}$$

where the sign is $+$ if a commutes with b and $-$ if a does not commute with b.

Finally, C acts as

$$(\hat{b})c = (w \otimes b)c = -(w \otimes b) = -\hat{b}$$

Since G is generated by A and B, this explains how G acts on V. Notice that V is a signed permutation module for G.

We will now look at the associated projective space $\mathbb{P}(V)$. This is a G-space with $K_G(\mathbb{P}(V)) = 0$. (This follows from the same argument as used in Lemma 4.3 where Stiefel-Whitney classes are used instead of Chern classes.)

We will now describe a G-CW structure on $\mathbb{P}(V)$ and analyze its isotropy groups.

Lemma 5.5. *There is a G-CW structure on $\mathbb{P}(V)$ such that all isotropy subgroups are elementary abelian, and isotropy subgroups of cells with positive dimension have rank strictly less than the rank of the group.*

Proof. Since V is a 2^n-dimensional Euclidean space, the unit sphere in V, denoted by $S(V)$, is a $2^n - 1$ dimensional sphere with G action.

Notice that the central subgroup C acts as the antipodal map, and hence freely. Thus, the isotropy subgroups of the G action on $S(V)$ cannot include C. This implies that all the isotropy subgroups are elementary abelian and of rank strictly less than the rank of G.

Once we projectivize $S(V)$, we get $\mathbb{P}(V)$ and C acts trivially on $\mathbb{P}(V)$. Now if $x \in S(V)$, $g \in G$ with $g \cdot x = -x$, then $cg \cdot x = x$ and cg is in the isotropy subgroup of the point x. Using this observation, it is easy to see that the isotropy subgroups of the G action on $\mathbb{P}(V)$ will be of the form $C \times K$ where K is an isotropy subgroup of the G action on $S(V)$.

This shows that the isotropy subgroups of the G action on $\mathbb{P}(V)$ are elementary abelian subgroups of G.

Now, let us describe an admissible simplicial decomposition for $S(V)$. After doing this, we will take orbit representatives, and get a suitable admissible cellular decomposition for $\mathbb{P}(V)$ which gives a G-CW structure for $\mathbb{P}(V)$.

An obvious way of getting a simplicial decomposition for S^{m-1} is as follows: Let U be a set of basis vectors in $\mathbb{R}^m$. Define the vertex set as $U \cup -U$. Simplices can be taken as subsets of the vertex set such that no two elements in the set are antipodal to each other. The realization of this simplicial complex in $\mathbb{R}^m$ is homeomorphic to S^{m-1}. For example, in $\mathbb{R}^3$ the realization gives us an octahedron.

In our case, the vector space V is indexed by elements of B, and thus the simplicial complex described above has vertex set $V = B \cup -B$. Thus the simplices can be viewed as signed subsets of B with $+$'s or $-$'s used as coefficients. Since V was a signed permutation module for G, it is easy to see that this simplicial G-complex is G-homeomorphic to $S(V)$.

Notice though since B acts by permutation on the basis indexed by B, this simplicial complex will not be admissible. So, we take the barycentric subdivision to fix this. Now our new vertex sets are simplices of the original complex and the typical simplex will be a flag $s_0 < s_1 < \cdots < s_m$ where the s_i are simplices in the original complex.

Now, we take the quotient with the antipodal map to get $\mathbb{P}(V)$. So, we identify a vertex set with one where all the elements have been negated. This gives us a G-CW-complex structure on $\mathbb{P}(V)$.

To complete the proof of this lemma we just need to prove that the isotropy subgroups of positive dimensional cells have proper rank. (i.e., rank strictly less than the rank of G.)

Since our complex is admissible, the isotropy subgroup of a cell is included in the isotropy subgroups of its faces. Thus, it is enough to prove the proposition for one dimensional cells of the cellular structure of $\mathbb{P}(V)$.

Taking one such 1-cell $s' < s$, we need to show that $C_G(s' < s) = C_G(s') \cap C_G(s)$ cannot have maximal rank. Assume to the contrary that $C_G(s') = C_G(s) = E$ is a maximal elementary abelian subgroup of G.

Let $E(A)$ and $E(B)$ be its projections to A and B under quotient maps $G \to A$ and $G \to B$. It is easy to see that $E = C \times E(A) \times E(B)$, and that $E(B) = C_B(E(A))$ and $E(A) = C_A(E(B))$.

Let $B(s)$ denote the set of elements in B which appear in s. (This is just the set of elements in s with the signs dropped.)

Observe that $E(B)$ will permute the elements in $B(s)$, so $B(s) = \cup b_i E(B)$ for a set of elements $\{b_i\}$ in B. Since $E(A)$ stabilizes s, elements in $E(A)$ either centralize $B(s)$ or act by multiplication with -1 on all elements in $B(s)$. This is because elements of a act via signs depending on whether they commute or not with the given element b and because s does not contain any element and its antipode. Multiplication with -1 on all elements is possible since s is a simplex for $P(V)$.

Now, observe that if two elements in $E(A)$ act by multiplication with -1 on all elements in $B(s)$, then their product will act trivially. So, there is an index at most 2 subgroup $E(A)'$ in $E(A)$ which centralizes $B(s)$. Therefore, it centralizes $E(B)$ and all the elements in the set $\{b_i\}$.

However, by maximality of E, we have $E(B) = C_B(E(A))$, so $E(B)$ is an index at most 2 subgroup of $C_B(E(A)')$. Since the set $\{b_i\}$ is included in $C_B(E(A)')$, for every i, j we have $b_i b_j \in E(B)$. This implies that $B(s) = bE(B)$ for some $b \in B$.

Repeating the same argument for s', we find that $B(s')$ also is equal to $b'E(B)$ for some $b' \in B$. This gives a contradiction since $s' < s$. So, the isotropy subgroups of positive dimensional simplices have proper rank. This completes the proof of the lemma. □

Lemma 5.6. *Let G be an extraspecial 2-group. Then, there is a nonzero divisor element u in $H^*(G)$ such that $res^G_H u = 0$ for every maximal subgroup H with $rk(H) < rk(G)$.*

Proof. Using Poincare series calculations for these groups, we observe that $P_G(t) = P_H(t)/(1-t)$ for every maximal subgroup H with $rk(H) = rk(G) - 1$. By a Gysin sequence argument, it is easy to see that for a maximal subgroup H satisfying this Poincare series equality, the transfer map $tr^G_H : H^*(H) \to H^*(G)$ must be zero, and hence the one dimensional element $x_H \in H^1(G)$ with $\ker(x_H) = H$ will be a nonzero divisor. (See Lemma 7.9 ahead.) Taking the product of all such elements we obtain a nonzero divisor element u such that $res^G_H(u) = 0$ for every maximal H with $rk(H) < rk(G)$. □

This lemma, in particular, tells us that there is a nonzero divisor element u in $H^*(G)$ such that $res^G_E u = 0$ for every elementary abelian subgroup of proper rank containing the center C. (This is because such an elementary abelian subgroup lies inside a maximal subgroup of proper rank. Here we use that this subgroup contains C which is the Frattini subgroup of G.)

By lemma 5.5, this means that there is a nonzero divisor element in $I_G(X, 1)$. Using corollary 3.6, it follows that $0 = K_G(\mathbb{P}(V)) = K_G(\mathbb{P}(V)^{(0)}) = I_G(\mathbb{P}(V))$.

Thus we conclude the following:

Proposition 5.7 (Quillen [Qu2]). *$H^*(G)$ is detected on the cohomology of maximal elementary abelian subgroups.*

5.4 A conjecture of Quillen

Now, we consider a classical conjecture due to Quillen [Qu3] about the complex associated to the poset of nontrivial elementary abelian p-subgroups in a finite group G. This complex is usually refered to as the Quillen complex and denoted by $\mathcal{A}_p(G)$.

Conjecture 5.8 (Quillen [Qu3]). *Let G be a finite group. Then, $\mathcal{A}_p(G)$ is contractible if and only if G has a nontrivial normal elementary abelian p-subgroup.*

We prove the following:

Proposition 5.9. *Let G be a finite group. If $\mathcal{A}_p(G)$ is contractible, then for every prime q that divides $|G|$, there exists a nontrivial elementary abelian p-subgroup E such that* $\mathrm{rk}_q(N_G(E)) = \mathrm{rk}_q(G)$.

Proof. Consider $X = \mathcal{A}_p(G)$ as a G-space with conjugation action on the elementary abelian subgroups. It is clear that X is a finite G-complex and $K_G(X) = 0$ because X is assumed to be contractible. So, by corollary 3.4,

$$\ker\{res : H^*(G, \mathbb{F}_q) \to \prod_{E \in \mathcal{E}_p} H^*(N_G(E), \mathbb{F}_q)\}$$

is a nilpotent ideal. Since $\mathrm{rk}_q(G)$ is equal to the Krull dimension of $H^*(G, \mathbb{F}_q)$, and the Krull dimension of a direct sum of rings is just the maximum of the Krull dimensions of the summands, the result follows. □

The condition above gives rise to an interesting question:

Question 5.10. *Suppose G is a finite group such that for every prime q that divides $|G|$, there exists a nontrivial elementary abelian p-subgroup E such that* $\mathrm{rk}_q(N_G(E)) = \mathrm{rk}_q(G)$*. Then, is it true that G has a nontrivial, normal elementary abelian p-subgroup?*

5.5 The Universal G-sphere

We now discuss the example of the "universal G-sphere" for a p-group G. This is constructed as follows:

Let $\{M_i : i \in I\}$ be a complete set of maximal subgroups of G. Notice G/M_i is a cyclic group of order p and hence acts on S^1 via the roots of unity as before. Let X_i be the G-space which is S^1 acted on by G via G/M_i in this way. Now let $\mathfrak{U} = *_{i \in I} X_i$ be the join of these G-spaces given the natural G-action.

Thus $\mathfrak{U}$ is a sphere and G acts on it without fixed points, a set of maximal isotropy groups being exactly the set of maximal subgroups of G.

$K_G(\mathfrak{U})$ is a principal ideal generated by the Euler class of the associated sphere bundle $EG \times_G X \to BG$.

Using $\mathbb{F}_p$-coefficients, it is easy to check that this Euler class is the product $\prod_{i \in I} \beta(w(i))$ where β is the Bockstein and $w(i) \in H^1(G; \mathbb{F}_p) = Hom(G; \mathbb{Z}/p\mathbb{Z})$ is a homomorphism with kernel M_i.

Thus if G is not elementary abelian, by Serre's theorem (In his second proof of this result, it was shown that there is a product of Bocksteins of degree 1 elements corresponding to *distinct* maximals which is zero, and we use this.), the G-sphere we have constructed will have $K_G(\mathfrak{U}) = 0$ and will be fixed point free, i.e., $\mathfrak{U}^G = \emptyset$.

Now it is easy to give $\mathfrak{U}$ the structure of a G-CW-complex by giving each X_i a triangulation where there are p 0-cells (the pth roots of unity in S^1) and p 1-cells connecting adjacent 0-cells, and then using the induced cell structure on the join. Notice in particular that any cell of dimension two or higher must be supported by at least two X_i's.

Thus $Iso(2)$ is contained in the collection $\{M_i \cap M_j | i \neq j, i, j \in I\}$.

Thus

$$I_G(\mathfrak{U}, 2) = Ker\{H^*(G) \to \prod_{i \neq j} H^*(M_i \cap M_j)\}.$$

We are now ready for the application:

Proposition 5.11. *Let G be a p-group which is not elementary abelian, and $\mathfrak{U}$ the universal G-sphere.*

Then if $I_G(\mathfrak{U}, 2)$ has a nonzero divisor element, $ess(G)^2 = 0$.

Proof. By Corollary 3.6, $K_G(\mathfrak{U}) = K_G(\mathfrak{U}^{(1)}) = 0$. Thus $Y = \mathfrak{U}^{(1)}$ is a fixed point free G-graph, with $K_G(Y) = 0$. Thus $I_G(Y)^2 = 0$ by Corollary 3.4 and hence $ess(G)^2 = 0$ since $ess(G) \subseteq I_G(Y)$.

□

6 Essential Cohomology Conjecture

As defined earlier, the essential cohomology of G, denoted by $ess(G)$, is the ideal of classes which restrict to zero on every proper subgroup. When G is not a p-group, essential cohomology is zero, so we can assume G is a p-group. We are interested in the following conjecture:

Conjecture 6.1. *Let G be a p-group which is not elementary abelian. Then, $(ess(G))^2 = 0$.*

The conjecture has been proved for some 2-groups which have small cohomology length, or small number of generators (see [Mar], [Mi1]). The most remarkable positive result for the conjecture is due to Minh [Mi2] which shows that for a p-group G, the nilpotency degree of an essential class is bounded by p.

Here we will describe a possible application of our main theorem to this conjecture. First observe that if X is a G-space with no fixed points then all isotropy subgroups are proper, so the kernel of restrictions to isotropy subgroups, $I_G(X)$, will include $ess(G)$. Recall that our main theorem says $(I_G(X))^n \subseteq K_G(X)$, where $n = dim(X) + 1$ and $K_G(X)$ is the kernel of the map $H^*_G(*) \to H^*_G(X)$. So, if X is a connected one dimensional G-complex, i.e., a connected graph, with no fixed points then

$$(ess(G))^2 \subseteq (I_G(X))^2 \subseteq K_G(X).$$

Let $\mathcal{G}$ be the collection of all connected, fixed point free G-graphs. Then, we conclude that

$$(ess(G))^2 \subseteq \bigcap_{X \in \mathcal{G}} K_G(X).$$

This, in particular, implies the following:

Theorem 6.2. *If the essential cohomology conjecture is not true for a p-group G, then for any connected G-graph X the following is true: X has a fixed point if and only if the map $H^*_G(pt) \to H^*_G(X)$ is injective.*

Proof. We just need to show the "if" part. Since $(ess(G))^2$ is not zero, by above inclusion, $K_G(X)$ is nonzero for every fixed point free G action on a connected graph X. So, if the map $H^*_G(pt) \to H^*_G(X)$ is injective for a connected G-graph, then X must have a fixed point. $\square$

Now, we will investigate group actions on connected graphs on the chain level to understand the nature of the ideal $\bigcap_{X \in \mathcal{G}} K_G(X)$ better. Let $X \in \mathcal{G}$, and $C_*(X)$ be the chain complex of X in k (a field of characteristic p) coefficients. We have an exact sequence

$$0 \to H_1(X) \to C_1(X) \to C_0(X) \to k \to 0$$

and a corresponding extension class γ_X in $Ext^2(k, H_1(X))$ [all the Ext groups in this paper are over the ring kG.]

Lemma 6.3.

$$K_G(X)^n = Im\{Ext^{n-2}(H_1(X), k) \xrightarrow{\gamma_X \cdot} Ext^n(k, k)\}$$

for all $n \geq 2$.

Proof. Let M be a kG-module such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(X) & \longrightarrow & C_1(X) & \xrightarrow{\pi} & C_0(X) & \xrightarrow{i} & k & \longrightarrow & 0 \\
 & & \| & & \downarrow f_1 & & \downarrow f_0 & & \| & & \\
0 & \longrightarrow & H_1(X) & \longrightarrow & P & \longrightarrow & M & \xrightarrow{j} & k & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \Omega^{-1}(C_1(X)) & = & \Omega^{-1}(C_1(X)) & & & &
\end{array}$$

where P is the injective cover of $C_1(X)$. (M is the pushout of the maps π and f_1.)

Let D_* denote the chain complex $0 \to P \to M \to 0$. Then,

$$Ext^n(D_*, k) \cong Ext^n(M, k).$$

Also note that the chain map $f : C_*(X) \to D_*$ induces an isomorphism on homology, so it induces an isomorphism

$$f^* : Ext^n(D_*, k) \xrightarrow{\cong} Ext^n(C_*(X), k).$$

By the commutativity of the above diagram, we obtain

$$\begin{aligned} K_G(X)^n &= ker\{\ i^* : H^n_G(pt) \to H^n_G(X)\} \\ &= ker\{\ i^* : Ext^n(k,k) \to Ext^n(k, C^*(X))\} \\ &= ker\{\ i^* : Ext^n(k,k) \to Ext^n(C_*(X), k)\} \\ &= ker\{\ j^* : Ext^n(k,k) \to Ext^n(M,k)\}. \end{aligned}$$

since $i^* = f^* j^*$, and f^* is an isomorphism.

Now consider the long exact sequence

$$\cdots \to Ext^{n-1}(kerj, k) \xrightarrow{\gamma'_X \cdot} Ext^n(k,k) \xrightarrow{j^*} Ext^n(M,k)$$

$$\to Ext^n(kerj, k) \to \cdots$$

which comes from the short exact sequence $0 \to kerj \longrightarrow M \xrightarrow{j} k \to 0$ with extension class $\gamma'_X \in Ext^1(k, kerj)$. Notice that γ'_X maps to γ_X in $Ext^2(k, H_1(X))$ under the isomorphism $Ext^1(k, kerj) \cong Ext^2(k, H_1(X))$.

Thus we conclude

$$K_G(X) = Im\{Ext^{n-1}(\Omega^{-1} H_1(X), k) \xrightarrow{\gamma'_X \cdot} Ext^n(k,k)\}$$

$$= Im\{Ext^{n-2}(H_1(X), k) \xrightarrow{\gamma_X \cdot} Ext^n(k,k)\}$$

for all $n \geq 2$. □

We see from this lemma that constructing a fixed point free G-graph X such that the associated extension class is zero will be enough to conclude that for this extension $K_G(X) = 0$. This will imply $(ess(G))^2 = 0$ as we discussed before.

Thus the question of which groups G have a fixed point free G-graph with zero extension class comes up.

In the following sections, we try to answer this question by first reducing the search to a single obstruction, namely the extension class of a universal G-graph. Then, in later sections, we give necessary conditions for the obstruction to vanish and also show that it does not vanish in general.

However, even in the case where it does not vanish, it still is possible that a fixed point free G-graph with zero K-ideal exists since the required multiplication with the extension class whose image is this K-ideal, might still be zero.

7 The Universal Essential Class

Let G be a p-group, and k a field of characteristic p. If X is the set of all cosets of maximal subgroups of G, then G acts on X in the obvious way, and let kX denote the permutation module associated to G-set X.

This is just the direct sum of permutation modules of type $k[G/H]$ for all maximal subgroups H.

Let N denote the kernel of the augmentation map $\epsilon : kX \to k$. By taking duals, we see that N^* will be the cokernel of the norm map $\sigma : k \to kX$. This gives us two extensions

$$\alpha : 0 \to N \to kX \xrightarrow{\epsilon} k \to 0,$$

and

$$\alpha' : 0 \to k \xrightarrow{\sigma} kX \to N^* \to 0$$

with extension classes $\alpha \in Ext^1(k, N)$ and $\alpha' \in Ext^1(N^*, k)$. Notice that α maps to α' under the obvious isomorphism $Ext^*(k, N) \cong Ext^*(N^*, k)$.

Since the above extensions split when restricted to any proper subgroups of G, these classes are essential. We will call these classes the universal essential classes. The following proposition explains why we call them universal.

Proposition 7.1. *If $u \in Ext^n(k, k)(= H^n(G, k))$, then u is an essential class if and only if $u = \alpha \cdot v$ for some v in $Ext^{n-1}(N, k)$.*

Similarly u is an essential class if and only if $u = w \cdot \alpha'$ for some w in $Ext^{n-1}(k, N^*)$.

Proof. Consider the long exact sequence associated to the first extension above:

$$\cdots \to Ext^{n-1}(N, k) \xrightarrow{\alpha\cdot} Ext^n(k, k) \to Ext^n(kX, k) \to Ext^n(N, k) \to \cdots$$

By the theory of extensions (see [Mac]) the connecting homomorphism is just multiplication by the extension class $\alpha \in Ext^1(k, N)$, i.e., the Yoneda splice of corresponding extensions. The second map combined with Shapiro's isomorphism applied to each direct summand $k[G/H]$ gives the map

$$\oplus res^G_H : Ext^n_{kG}(k, k) \to \oplus Ext^n_{kH}(k, k)$$

where the sum is over all maximal subgroups.

It is clear now that u is an essential class, if and only if it is in the image of multiplication with α.

To see the statement for α' from the one for α, we can argue as follows:

If u is essential, its dual u' is also essential and so we can write $u' = \alpha \cdot v$. Taking the dual of the extension for the Yoneda splice of extensions v and α, one sees that u can also be written as $v' \cdot \alpha'$ where v' is just the dual of v. □

Let x and y be two essential classes. Then by above proposition, there exists two classes v and w such that $x = w \cdot \alpha'$ and $y = \alpha \cdot v$.

From this we get $xy = w \cdot (\alpha' \cdot \alpha) \cdot v$. Notice that if the middle part $\alpha' \cdot \alpha$ is zero then xy will be zero. Hence we have the following:

Proposition 7.2. *If the extension $0 \to N \to kX \to kX \to N^* \to 0$ with extension class $\alpha' \cdot \alpha$ is split, then $(ess(G))^2 = 0$.*

Actually, the relation between essential classes and the universal classes can be dualized to find a similar relationship between transfer classes and the universal classes.

Definition 7.3. *Fix a ring of coefficients k. Then we define $tr(G)$ to be the ideal of $H^*(G,k)$ generated by transfers from proper subgroups of G. We will refer to this ideal as "the ideal of proper transfers".*

Proposition 7.4. *If $u \in Ext^n(k,k) (= H^n(G,k))$, then $\alpha' \cdot u = 0$ if and only if $u \in tr(G)$.*

Similarly, $u \cdot \alpha = 0$ if and only if $u \in tr(G)$.

Proof. Applying $Ext^*(k,\cdot)$ to the extension $0 \to N \to kX \to k \to 0$, we get the following long exact sequence:

$$\cdots \to Ext^n(k,N) \to Ext^n(k,kX) \to Ext^n(k,k) \xrightarrow{\cdot\alpha} Ext^{n+1}(k,N) \to \cdots$$

Again by well known results in extension theory, the connecting homomorphism is multiplication with the extension class α and the second map combined with Shapiro's isomorphism gives the map

$$\oplus tr_H^G : \oplus Ext^n_{kH}(k,k) \to Ext^n_{kG}(k,k).$$

So, the kernel of multiplication with α' is the module generated by the image of transfers from maximal (proper) subgroups of G.

The dual statement can also be proved similarly by applying $Ext^*(\cdot,k)$ to the short exact sequence for N^*. □

Combining propositions 7.1 and 7.4, we obtain a necessary and sufficient condition for $ess(G)$ to lie inside $tr(G)$.

Remark 7.5. *This is a nice property for a group G to have since it says that either a class can be detected by restriction to some proper subgroup or it is a transfer from some proper subgroup. In either case, it "originates" from a proper subgroup. It is unknown at this time whether this property holds for all finite groups G.*

Theorem 7.6. *$ess(G) \subseteq tr(G)$ if and only if the map*

$$Ext^{n-1}(N,k) \to Ext^{n+1}(N^*,k)$$

defined by multiplication with $\alpha' \cdot \alpha$ is the zero map for all $n \geq 1$.

Proof. By proposition 7.1, we have

$$ess^n(G) = Im\{Ext^{n-1}(N,k) \xrightarrow{\alpha\cdot} Ext^n(k,k)\}.$$

By proposition 7.4, we can write

$$tr^n(G) = Ker\{Ext^n(k,k) \xrightarrow{\alpha'\cdot} Ext^{n+1}(N^*,k)\}.$$

It is clear from here that $ess(G) \subseteq tr(G)$ if and only if the stated composition homomorphism is zero for all $n \geq 1$. □

Now, we will show that $\alpha' \cdot \alpha$ is the only obstruction for the existence of a fixed point free, connected G-graph with zero extension class:

Before that we should recall the definition of a bipartite graph as one whose vertex set V can be split as a disjoint union of two nonempty sets A and B such that no two vertices in A are connected by an edge and no two vertices in B are connected by an edge. We will say that the graph is bipartite on the sets A, B. The graph will be called a complete bipartite graph on the sets A, B if each vertex of A is joined by a single edge to each vertex of B.

We are now ready for the proof:

Theorem 7.7. *There exists a fixed point free G-action on a connected graph with zero extension class if and only if $\alpha' \cdot \alpha = 0$.*

Proof. First notice that $\alpha' \cdot \alpha$ maps to α^2 under the isomorphism $Ext^*(N^*, N) \cong Ext^*(k, N \otimes N)$ where α^2 is the extension class for the extension

$$0 \to N \otimes N \to N \otimes kX \to kX \to k \to 0$$

So, it is enough to prove the statement for α^2 instead of $\alpha' \cdot \alpha$. Recall also that X is the G-set of all cosets of all maximal subgroups of G.

First we will show that α^2 is also an extension that comes from a fixed point free action on a graph. This will prove the "if" part of the proposition.

Let Y be the graph with vertex set $V = X_1 \cup X_2$ and edge set $E = X_1 \times X_2$ where both of the sets X_1 and X_2 are equal to the set X. This is the (complete) bipartite graph on the sets X_1, X_2.

It is clear that Y is connected, and admits a fixed point free action which is induced from the G action on the X_i's.

The chain complex of Y is the following complex:

$$0 \to H_1(Y) \to kX_1 \otimes kX_2 \to kX_1 \oplus kX_2 \to k \to 0$$

where the chain map $d_1 = (-id \otimes \epsilon, \epsilon \otimes id)$ because $d(x_1, x_2) = x_2 - x_1$ for every edge (x_1, x_2) of the graph Y.

From this it is easy to see that $H_1(Y) = N \otimes N$. Now consider the following commuting diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N \otimes N & \longrightarrow & N \otimes kX & \longrightarrow & kX & \xrightarrow{i} & k & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H_1(Y) & \longrightarrow & kX_1 \otimes kX_2 & \longrightarrow & kX_1 \oplus kX_2 & \xrightarrow{j} & k & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & kX & = & kX & & & &
\end{array}$$

where the first vertical sequence is the sequence $0 \to N \to kX \to k \to 0$ tensored with kX, and the second vertical sequence have maps $(id, 0)$ and projection on the second coordinate. It follows from this that horizontal sequences are extensions with the same extension class, hence the extension associated to the graph Y has extension class α^2. This completes the"if" part of the theorem.

For the converse, observe that if Z is any graph with fixed point free G action then by passing to the barycentric subdivision Z', we can assume that the G-action is admissible, that Z' has no loops or multiple edges between two vertices and that Z' is bipartite on the sets A, B where A is the vertex set of the original graph Z and B consists of the new (barycentric) vertices. Also note that the extension class for Z is equivalent to the extension class for Z' since the G-chain complexes of the two are G-chain homotopy equivalent.

We can then find a G-equivariant simplicial map $f : Z' \to Y$ as follows:

Recall $Vertex(Z') = A \sqcup B$ and $Vertex(Y) = X_1 \sqcup X_2$.

Here for the G-orbit of a vertex $y \in A$, we have $Gy \cong G/G_y$ where G_y is a proper subgroup of G since Z' is fixed point free. We can therefore choose a maximal subgroup H of G containing G_y and define a G-equivariant map from the G-orbit of $y \in A$ to the G-orbit of H in X_1. In this way we define a G-equivariant map from A to X_1. Similarly we can define a G-equivariant map from B to X_2 and hence we have defined a G-equivariant map from the 0-skeleton of Z' to the 0-skeleton of Y.

Since Z' is bipartite and Y is complete bipartite, and since both graphs have admissible actions, we can then extend this map to get a G-equivariant map f from Z' to Y. Now consider the chain map induced from $f : Z' \to Y$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(Z') & \longrightarrow & C_1(Z') & \longrightarrow & C_0(Z') & \xrightarrow{i} & k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\ 0 & \longrightarrow & H_1(Y) & \longrightarrow & C_1(Y) & \longrightarrow & C_0(Y) & \xrightarrow{j} & k & \longrightarrow & 0 \end{array}$$

From this it is easy to see that the extension class of the second row is the image of the extension class of the first row under the map

$$f_* : Ext^2(k, H_1(Z')) \to Ext^2(k, H_1(Y)).$$

So, the extension class of Y is the image of the extension class of Z' under the above map. Therefore, if Z' is a graph with zero extension class then α^2 will be zero. □

Remark 7.8. *Notice that for any G-graph one gets a group extension*

$$0 \to \pi_1(X) \to \pi_1(X_G) \to G \to 0$$

which induces

$$0 \to H_1(X) \to \Gamma \to G \to 0$$

after taking quotients with the commutator of $\pi_1(X)$*. Here* Γ *is isomorphic to* $\pi_1(X_G)/[\pi_1(X), \pi_1(X)]$.

So, if X *is the graph* Y *described above, then* $H_1(X) = N \otimes N$ *and the extension class in* $H^2(G, H_1(X))$ *is nothing other than* α^2 *in integer coefficient.*

If we take k to be the integers mod p, then the mod p reduction of this extension gives a p-group Γ_p which fits into an extension of type $0 \to V \to \Gamma_p \to G \to 0$ where V is an elementary abelian p-group. Notice that we could also start with the extension

$$0 \to F \to *H_i \to G \to 0$$

where the middle term is the free product of maximal proper subgroups, and then after abelinization and mod p reduction, we will reach an essential extension as above. It is interesting to ask if the extension class for this group extension is the same as α^2.

7.1 Vanishing of the square of the Universal Class

Let α and α' be as in the previous section. Fix k to be the field with 2 elements in this section.

In the previous section we proved that $\alpha' \cdot \alpha$ is an obstruction for the existence of fixed point free G-graphs with zero extension class. In this section, we consider 2-groups, and show that this obstruction vanishes under some conditions, but it does not vanish in general. Throughout this section we assume G is a 2-group. We start with a lemma:

Lemma 7.9. *Let x be a nonzero one dimensional class in $H^1(G,k)$ with $\ker(x) = H$ and let M be a kG-module. Then we have:*
(a) $\ker(tr_H^G : H^*(H,M) \to H^*(G,M)) = im(res_H^G : H^*(G,M) \to H^*(H,M))$.
(b) $y \in \ker(res_H^G : H^*(G,M) \to H^*(H,M))$ if and only if y is a multiple of x.
(c) $y \in im(tr_H^G : H^*(H,M) \to H^*(G,M))$ if and only if $xy = 0$.

Thus, $\alpha = x \cdot t$ for some $t \in Ext^0(k,N) = H^0(G,N)$. Similarly, $\alpha' = t' \cdot x$ for some $t' \in Ext^0(N^*,k)$.

Proof. Notice that x is the extension class for the extension $0 \to k \to k[G/H] \to k \to 0$, where the first map is the norm map, and the second one is the augmentation map. Consider the long exact sequence for this extension with coefficients in M:

$$\cdots \to Ext^{*-1}(k,M) \xrightarrow{x\cdot} Ext^*(k,M) \xrightarrow{i^*} Ext^*(k[G/H],M) \xrightarrow{j^*} Ext^*(k,M) \to \cdots$$

Using Shapiro's lemma to identify $Ext^*_{kG}(k[G/H],M)$ with $Ext^*_{kH}(k,M)$ we see easily that $i^* = res_H^G$ and $j^* = tr_H^G$.

Thus (a),(b) and (c) follow easily from the exactness of this sequence.

Also since α is an essential class, $\alpha = x \cdot t$ for some $t \in Ext^0_{kG}(k,N)$. Finally, the statement for α' can be obtained by taking duals. □

Notice that using this lemma for some nonzero x and y in $H^*(G)$, we get $\alpha' = t' \cdot x$, and $\alpha = y \cdot t$. So, if $xy = 0$, then the obstruction $\alpha' \cdot \alpha$ will vanish. Hence we can conclude the following:

Proposition 7.10. *If there exist nonzero x, y in $H^1(G,k)$ such that $xy = 0$, then $\alpha' \cdot \alpha = 0$.*

Now, we will prove that for some 2-groups this obstruction does not vanish. First, we make some observations:

Lemma 7.11 (Minh [Mi1]). *The following are equivalent:*
(i) There exist nonzero x, y in $H^1(G,k)$ such that $x \cdot y = 0$.
(ii) The restriction map $res^G_H : H^1(G,k) \to H^1(H,k)$ is not surjective for some maximal subgroup $H \subseteq G$.
(iii) The transfer map $tr^G_H : H^1(H,k) \to H^1(G,k)$ is not a zero map for some maximal subgroup $H \subseteq G$.
(iv) $\Phi(H) < \Phi(G)$ for some maximal subgroup $H \subseteq G$. Here $\Phi(H)$ and $\Phi(G)$ denote the Frattini subgroups of H and G respectively.

Proof. The equivalence of (ii) and (iii) follows from Lemma 7.9 part (a). The equivalence of (iii) and (i) follows from Lemma 7.9 part (c). Finally, that (ii) and (iv) are equivalent can be seen easily from the identity $H^1(G,k) = Hom(G/\Phi(G), \mathbb{Z}/2\mathbb{Z})$ for 2-groups.

□

Notice that the above lemma in particular tells that if $\Phi(H) = \Phi(G)$ for every maximal subgroup $H \subseteq G$, then the transfer map $tr^G_H : H^1(H,k) \to H^1(G,k)$ is a zero map for every maximal subgroup $H \subseteq G$. We prove a refinement of this result:

Lemma 7.12. *Let G be a 2-group such that $\Phi(G) = \Phi(K)$ for every subgroup K of index 4. Then, $tr^G_H : H^1(H,N) \to H^1(G,N)$ is a zero map for every maximal subgroup $H \subseteq G$.*

Proof. Recall that N is the kG module defined as the kernel of augmentation map $kX \to k$ where X is the disjoint union of cosets of maximal subgroups. Using the exact sequence $0 \to N \xrightarrow{i} kX \xrightarrow{\epsilon} k \to 0$ we get the following commuting diagram:

$$\begin{array}{ccccc}
H^1(H,N) & \xrightarrow{i^*_H} & H^1(H,kX) & \xrightarrow{\epsilon^*_H} & H^1(H,k) \\
\big\downarrow{\scriptstyle tr^G_H} & & \big\downarrow{\scriptstyle tr^G_H} & & \big\downarrow{\scriptstyle tr^G_H} \\
H^1(G,N) & \xrightarrow{i^*_G} & H^1(G,kX) & \xrightarrow{\epsilon^*_G} & H^1(G,k)
\end{array}$$

Here H is a fixed maximal subgroup of G. First we will show that $i^*_G[tr^G_H(u)]$ is zero for every element $u \in H^1(H,N)$. Notice that, by the commutativity of the above diagram, we just need to show $tr^G_H[i^*_H(u)] = 0$.

Let $\{H, H_1, \ldots, H_m\}$ be the set of maximal subgroups of G. Choosing a coset for each maximal subgroup, we obtain a set $\{x, x_1, \ldots, x_m\}$ which generates X as a G-set, hence generates kX as a kG-module. Using this basis, we can write $kX = k[G/H] \oplus \bigoplus_{i=1}^{m} k[G/H_i]$, and using Shapiro's lemma, we can replace $H^1(H, kX)$ with

$$H^1(H, k[G/H]) \oplus \bigoplus_{i=1}^{m} H^1(H \cap H_i),$$

and $H^1(G, kX)$ with

$$H^1(H) \oplus \bigoplus_{i=1}^{m} H^1(H_i).$$

By naturality of Shapiro's lemma, $tr_H^G : H^1(H, kX) \to H^1(G, kX)$ becomes $s_H \oplus [\oplus_{i=1}^m tr_{H\cap H_i}^{H_i}]$ where $s_H : H^1(H, k[G/H]) \to H^1(H, k)$ is the map induced from the augmentation map $k[G/H] \to k$.

Similarly, ϵ_H^* is replaced by $s_H + \sum_{i=1}^{m} tr_{H\cap H_i}^{H}$.

Notice that the condition in the lemma implies that $\Phi(K) = \Phi(H) = \Phi(G)$ for every maximal subgroup $H \subseteq G$, and an index 2 subgroup K of H. In particular, it says $\Phi(H \cap H_i) = \Phi(H) = \Phi(H_i)$ for all $i = 1, \ldots, m$. By Lemma 7.11, this implies that $tr_{H\cap H_i}^{H_i}$ and $tr_{H\cap H_i}^{H}$ are zero maps.

Let us write $i_H^*(u) = (v, v_1, \ldots, v_m)$ using the decomposition for $H^1(H, kX)$ above. From $\epsilon_H^*(i_H^*(u)) = 0$, we get

$$s_H(v) + \sum_{i=1}^{m} tr_{H\cap H_i}^{H}(v_i) = 0.$$

Since $tr_{H\cap H_i}^{H}$ is a zero map, we obtain $s_H(v) = 0$. Now we apply tr_H^G to $i_H^*(u)$:

$$tr_H^G[i_H^*(u)] = (s_H(v), tr_{H\cap H_1}^{H_1}(v_1), \ldots, tr_{H\cap H_m}^{H_m}(v_m)) = 0$$

because $s_H(v) = 0$ and $tr_{H\cap H_i}^{H_i}$ are zero maps. So, we see that $i_G^*[tr_H^G(u)] = 0$ for every u in $H^1(H, N)$.

Now, consider the following commuting diagram:

$$\begin{array}{ccccc}
Ext^0(k[G/H], k) & \longrightarrow & Ext^1(k[G/H], N) & \xrightarrow{i_H^*} & Ext^1(k[G/H], kX) \\
\downarrow & & \downarrow & & \\
Ext^0(k, k) & \xrightarrow{\cdot\alpha} & Ext^1(k, N) & \xrightarrow{i_G^*} & Ext^1(k, kX) \\
\downarrow{\scriptstyle x_H} & & \downarrow{\scriptstyle x_H} & & \\
Ext^1(k, k) & \xrightarrow{\cdot\alpha} & Ext^2(k, N) & &
\end{array}$$

where the vertical sequences comes from the exact sequence $0 \to k \to k[G/H] \to k \to 0$ with extension class $x_H \in Ext^1(k, k)$. With the obvious identification $Ext^1_{kG}(k[G/H], N) = H^1(H, N)$, one sees that our earlier conclusion $i_G^*[tr_H^G(u)] = 0$ implies that $tr_H^G(u)$ is a scalar multiple of α, say $tr_H^G(u) = c\alpha$.

Using, $x_H \cdot tr_H^G(u) = 0$, we get $x_H \cdot c\alpha$=0. By lemma 7.4, this implies cx_H is sum of transfers from proper subgroups. But, $\Phi(G) = \Phi(M)$ for every maximal subgroup $M \subseteq G$, so transfers from proper subgroups are zero maps. Therefore c must be zero, and hence $tr_H^G(u) = 0$. □

Lemma 7.13. *If $\alpha' \cdot \alpha = 0$, then α is a sum of transfers. Hence, for some maximal subgroup H the transfer map $tr_H^G : Ext^1_{kH}(k, N) \to Ext^1_{kG}(k, N)$ is not a zero map.*

Proof. The first part follows from proposition 7.4. The second part is obvious since α is a nonzero class when $G \neq 1$. (To see this, notice that in the long exact sequence obtained by applying $Ext_{kG}(k, -)$ to the sequence $0 \to N \to kX \to k \to 0$, α is the image of the identity in $Ext^0_{kG}(k, k)$ under the boundary map. It is easy to see that this boundary map $\delta : Ext^0(k, k) \to Ext^1(k, N)$ is injective, since the (transfer) map to the left of it is zero.)

□

Combining the last two lemmas, we conclude the following:

Proposition 7.14. *Let G be a 2-group such that $\Phi(G) = \Phi(K)$ for every subgroup K of index 4. Then, $\alpha' \cdot \alpha \neq 0$.*

There are many families of groups that satisfy the condition of this proposition. In particular, if G is an extraspecial 2-group which has no maximal elementary abelian subgroup of index 4, then G satisfies the condition.

Note that this does not necessarily mean that we cannot find a fixed point free G-graph X with $K_G(X) = 0$ for these groups. Because it is still possible that multiplication by $\alpha' \cdot \alpha$ is zero even though $\alpha' \cdot \alpha$ is not. In fact, when G is an extraspecial 2-group different than Q_8, we know that $ess(G) = 0$, so taking X as the universal G-graph, $K_G(X)$, being included in $ess(G)$, will be zero.

So, it is reasonable to ask the following question:

Question 7.15. *For which p-groups G (not elementary abelian), does there exist a fixed point free G-graph with $K_G(X) = 0$?*

Notice by the above comments that for p-groups where the answer is affirmative, the essential cohomology conjecture will hold true.

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