LINEAR COLORINGS OF SIMPLICIAL COMPLEXES AND COLLAPSING

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Abstract. A vertex coloring of a simplicial complex Δ is called a *linear coloring* if it satisfies the property that for every pair of facets (F_1, F_2) of Δ, there exists no pair of vertices (v_1, v_2) with the same color such that $v_1 \in F_1 \backslash F_2$ and $v_2 \in F_2 \backslash F_1$. The *linear chromatic number* $\mathrm{lchr}(\Delta)$ of Δ is defined as the minimum integer k such that Δ has a linear coloring with k-colors. We show that if Δ is a simplicial complex with $\mathrm{lchr}(\Delta) = k$, then it has a subcomplex Δ' with k vertices such that Δ is simple homotopy equivalent to Δ'. As a corollary, we obtain that $\mathrm{lchr}(\Delta) \geq \mathrm{Homdim}(\Delta) + 2$. We also show in the case of linearly colored simplicial complexes, the usual assignment of a simplicial complex to a multicomplex has an inverse. Finally, we show that the chromatic number of a simple graph is bounded from above by the linear chromatic number of its neighborhood complex.

1. Introduction

In this paper, we introduce a notion of linear coloring of a simplicial complex as a special type of vertex coloring. Recall that a vertex coloring of an abstract simplicial complex Δ with vertex set V is a surjective map $\kappa : V \to [k]$ where k is a positive integer and $[k] = \{1, \ldots, k\}$. We say a vertex coloring is linear if it satisfies the condition given in the abstract. Alternatively, a coloring is linear if for every two vertices u, v of Δ having the same color, we have either $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ or $\mathcal{F}(v) \subseteq \mathcal{F}(u)$ where $\mathcal{F}(u)$ and $\mathcal{F}(v)$ denote the set of facets including u and v respectively. This is actually equivalent to requiring that the set $\mathcal{F}_i = \{\mathcal{F}(u) \mid \kappa(u) = i\}$ is linearly ordered for every $i \in [k]$, which explains the rationale for our terminology.

The condition for linear coloring appears naturally when the multicomplex associated to a colored simplicial complex is studied closely. For example, in Theorem 4.4 we show that if a simplicial complex is linearly colored then we can recover it by using the multicomplex associated to it. The multicomplex associated to a simplicial complex Δ is the multicomplex whose simplices are the color combinations of the simplices on Δ. We believe that this association between simplicial complexes and multicomplexes could be very useful to study the combinatorial properties of multicomplexes although we do not investigate this direction in the present work.

Another consequence of requiring a coloring to be a linear coloring is that it gives us a natural deformation of the colored complex to a subcomplex of itself where

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the subcomplex has as many vertices as the number of colors used. In fact, we can obtain such a deformation on any subcomplex which satisfies the following condition: Given a simplicial complex Δ and a linear coloring κ of Δ with k colors, we call a subcomplex $\Delta_\kappa \subseteq \Delta$ a *representative subcomplex* if for each $i \in [k]$ there is one and only one vertex v in Δ_κ with $\kappa(v) = i$, and if it has the property that for every pair of vertices u, v with the same color, we have $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ whenever $u \in \Delta$ and $v \in \Delta_\kappa$. The main result of the paper is the following:

Theorem 1.1. *Let Δ be a simplicial complex on V, and let $\kappa : V \to [k]$ be a k-linear coloring map. If Δ_κ is a representative subcomplex of Δ, then Δ_κ is a strong deformation retract of Δ.*

This allows us to gain information on the homotopy type of a simplicial complex by coloring it linearly. For example it is clear that if a simplicial complex can be linearly colored using k colors then its homology dimension will be less than or equal to $k - 2$.

We also introduce the notion of LC-reduction by saying that a simplical complex Δ LC-reduces to its subcomplex Δ', denoted by $\Delta \searrow_{\mathrm{LC}} \Delta'$, if there exist a sequence of subcomplexes $\Delta = \Delta_0 \supseteq \Delta_1 \supseteq \ldots \supseteq \Delta_t = \Delta'$ such that for all $0 \leq r \leq t-1$, the subcomplex Δ_{r+1} is a representative subcomplex of Δ_r with respect to some linear coloring κ_r of Δ_r. We study various questions arising from this definition. For example, we show that if $X_1 \searrow_{\mathrm{LC}} X_2$ and Y is any simplical complex, then $X_1 * Y \searrow_{\mathrm{LC}} X_2 * Y$. The main result about LC-reduction is the following:

Theorem 1.2. *Let Δ be a simplicial complex and Δ' be a subcomplex in Δ. If Δ LC-reduces to Δ', then Δ NE-reduces to Δ' (also called strong collapsing), in particular Δ and Δ' have the same simple homotopy type.*

In fact, Theorem 1.2 implies Theorem 1.1, but we still give a separate proof for Theorem 1.1 using the basic techniques of poset homotopy due to Quillen [8]. The reason for this is that we believe that Theorem 1.1 is interesting in its own right for understanding the topology of simplicial complexes and should have an independent proof accessible to a topologist. We view Theorem 1.2 as a combinatorial version of Theorem 1.1.

In the rest of the paper, we give some applications of LC-reduction. The first application we give is closely related to a theorem by Kozlov [4] about monotone maps and NE-reduction. We prove that if $\varphi\colon P \to P$ is a closure operator on a finite poset P, then $\Delta(P) \searrow_{\mathrm{LC}} \Delta(\varphi(P))$, and we conclude that, in this case, $\Delta(P)$ collapses to $\Delta(\varphi(P))$. Our second application is related to graph coloring. We show that a linear coloring of the neighborhood complex of a simple graph gives a (vertex) coloring for the graph.

We organize the paper as follows: In Section 2, we give the definition of a linear coloring and its equivalent formulations to ease the computations. Then, we prove our main result Theorem 1.1 in Section 3. In Section 4, we describe an association between linearly colored simplicial complexes and multicomplexes. In the following two sections, we introduce *LC*-reduction and prove Theorem 1.2. The last two sections are devoted to applications of LC-reduction. In Section 7, we consider linear colorings of order complexes of posets and prove the reduction theorem for closure operators. Finally, in the last section, we consider the linear colorings of neighborhood complexes associated to simple graphs.

2. Linear coloring of a simplicial complex

We start with some basic definitions related to multisets.

Definition 2.1. A *multiset* M on a set A is a function $M\colon A \to \mathbb{N} := \{0, 1, 2, \dots\}$, where $M(a)$ is regarded as the number of repetitions of $a \in A$. We say that $a \in A$ is an *element* of M, and write $a \in M$, if $M(a) > 0$. The *cardinality* (or *size*) of a multiset M is defined by $\|M\| := \sum_{a \in A} M(a)$.

Note that every multiset M on A can be regarded as a monomial on the set A where the degree of $a \in A$ is equal to $M(a)$. The elements of $M(a)$ will be the elements of a with nonzero degree, and the cardinality will be equal to the total degree of the monomial. The usual division relation on monomials gives rise to the definition of submultisets, and the union and the intersection of multisets can be defined with the following formulas:

$$(M_1 \cup M_2)(a) = M_1(a) + M_2(a);$$
$$(M_1 \cap M_2)(a) = \min(M_1(a), M_2(a)).$$

Now we recall the definition of vertex coloring of a simplicial complex.

Definition 2.2. Let Δ be a finite (abstract) simplicial complex on V. Let $[k]$ denote the set $\{1, \dots, k\}$. A surjective map $\kappa : V \to [k]$ is called a (vertex) coloring of Δ using k colors.

Given a coloring κ of a simplicial complex Δ, we can associate a multiset to each of its faces as follows: If S is a face of Δ, then we define the multiset S_κ on $[k]$ by setting $S_\kappa(t)$ equal to the order of the set $\{v \in S\colon \kappa(v) = t\}$ for each $t \in [k]$. We define the linear coloring in its most technical form as follows:

Definition 2.3. Let Δ be a finite abstract simplicial complex on V and let $\mathcal{F}$ denote the set of all facets of Δ. A surjective map $\kappa\colon V \to [k]$ is called a *k-linear coloring* of Δ if and only if $\|F_\kappa \cap F'_\kappa\| = |F \cap F'|$ for any two facets $F, F' \in \mathcal{F}$.

Note that if Δ is linearly colored with κ, then for distinct facets F, F' of Δ, the multisets F_κ and F'_κ must be also different. Otherwise, we would have $|F \cap F'| = |F| = |F'|$ which cannot happen since F and F' are distinct. We can rephrase this by saying that the color combinations (with multiplicities) used in different facets must be different.

Every complex with n vertices can be linearly colored using n colors by giving a different color to each vertex. We call a linear coloring *trivial* if it is such a coloring.

Definition 2.4. The *linear chromatic number* of a simplicial complex Δ, denoted by $\mathrm{lchr}(\Delta)$, is defined to be the minimum integer k such that Δ has a k-linear coloring.

Since there is always the trivial linear coloring, the linear chromatic number of a simplicial complex is well defined and it is less than or equal to the number of vertices of the complex.

Definition 2.5. Let Δ be a simplicial complex and let κ be a k-linear coloring map. Define $V_i := \{v \in V \mid \kappa(v) = i\}$ and set $c_i^\kappa := \mathrm{card}(V_i)$ for each $i \in [k]$. Then, κ is said to be a linear coloring of *type* $c_\kappa(\Delta) = (c_1^\kappa, \dots, c_k^\kappa)$.

Example 2.6. In Figure 1(a), we illustrate a 2-dimensional simplicial complex admitting a 2-linear coloring of type $(3,1)$, whereas Figure 1(b) shows a linear coloring of type $(1,1,1,1)$. Note that the complex in Figure 1(b) is a 1-dimensional complex with $\mathrm{lchr}(\Delta) = 4$. For the simplicial complex depicted in Figures 1(c) and 1(d), the map given at Figure 1(c) is a 4-linear coloring of type $(2,1,1,2)$, while the coloring given in Figure 1(d) is not a linear coloring.

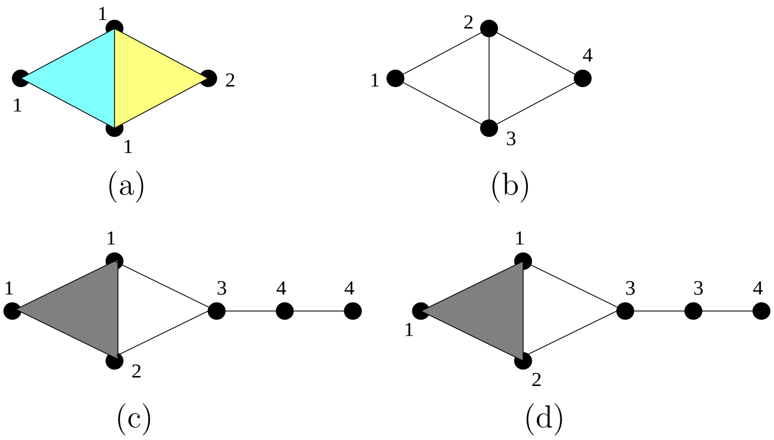

FIGURE 1. Linear colorable complexes and a non-linear coloring

To understand the definition of linear coloring better, we now give an equivalent condition for linear coloring. This is the same as the condition given in the abstract of the paper.

Proposition 2.7. *Let Δ be a finite abstract simplicial complex on V. A coloring $\kappa\colon V \to [k]$ of its vertices is a k-linear coloring of Δ if and only if for every pair of facets (F_1, F_2) of Δ, there exists no pair of vertices (v_1, v_2) with the same color such that $v_1 \in F_1 \backslash F_2$ and $v_2 \in F_2 \backslash F_1$.*

Proof. In general $\|(F_1)_\kappa \cap (F_2)_\kappa\| \geq |F_1 \cap F_2|$ for every pair of facets (F_1, F_2) of Δ. So, the equality does not hold if and only if there is a pair of vertices (v_1, v_2) with the same color such that $v_1 \in F_1 \backslash F_2$ and $v_2 \in F_2 \backslash F_1$. $\square$

There is even a nicer description of the condition which makes a coloring linear. To describe this we first introduce the following definition.

Definition 2.8. Let Δ be a simplicial complex and v be a vertex of Δ. The set of facets of Δ containing v is called *the facet set of* v and denoted by $\mathcal{F}(v)$.

We have the following:

Proposition 2.9. *Let Δ be a simplicial complex with vertex set V, and let $\kappa\colon V \to [k]$ be a coloring of Δ. The coloring κ is linear if and only if for every $i \in [k]$, the set $\mathcal{F}_i = \{\mathcal{F}(v) : \kappa(v) = i\}$ is linearly ordered by inclusion.*

Proof. Assume that κ is a linear coloring. Let $v_1, v_2 \in V$ such that $\kappa(v_1) = \kappa(v_2)$. Suppose that there exist facets $F_1 \in \mathcal{F}(v_1)\backslash\mathcal{F}(v_2)$ and $F_2 \in \mathcal{F}(v_2)\backslash\mathcal{F}(v_1)$. Then, it is clear that $v_1 \in F_1\backslash F_2$ and $v_2 \in F_2\backslash F_1$. This contradicts with the fact that κ is a linear coloring. So, either $\mathcal{F}(v_1) \subseteq \mathcal{F}(v_2)$ or $\mathcal{F}(v_2) \subseteq \mathcal{F}(v_1)$ holds. This shows that for each i, the set $\mathcal{F}_i$ is linearly ordered by inclusion. It is clear that the converse also holds. □

3. Deformation to a representative subcomplex

In this section we prove Theorem 1.1 stated in the introduction. Throughout the section, let Δ be a simplicial complex with vertex set V and let $\kappa : V \to [k]$ be a linear coloring of Δ. By Proposition 2.9, the set $\mathcal{F}_i = \{\mathcal{F}(v) : \kappa(v) = i\}$ is linearly ordered by inclusion for each $i \in [k]$. So, for each color i, there is a vertex (possibly more than one) such that $\kappa(v) = i$ and $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ for every $u \in V$ with $\kappa(u) = i$. This leads to the following definition:

Definition 3.1. A subcomplex Δ_κ of Δ is said to be a *representative subcomplex* with respect to κ if for each $i \in [k]$ there is one and only one vertex in $v \in \Delta_\kappa$ with $\kappa(v) = i$ and that $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ for every $u \in V$ with $\kappa(u) = i$.

Although a linearly colored complex may have many different representing subcomplexes, they are unique up to an isomorphism of simplicial complexes.

Proposition 3.2. *Let Δ be a simplicial complex with linear coloring κ. Suppose that Δ_κ and Δ'_κ are two subcomplexes of Δ which are representative with respect to κ. Then, Δ_κ and Δ'_κ are isomorphic as simplicial complexes.*

Proof. Let x, y be two vertices with $\mathcal{F}(x) = \mathcal{F}(y)$. Consider the map $f : V \to V$ such that $f(x) = y, f(y) = x$ and $f(z) = z$ for all the other vertices. We claim that f extends to an isomorphism of simplicial complexes. For this it is enough to show that if $S \in \Delta$, then $f(S) \in \Delta$. This is clear if x, y are both in S or if neither of them are in S. Suppose S is such that $x \in S$ and $y \notin S$. Let F be a facet that includes S. Since $x \in F$, we must have $y \in F$ by the assumption that $\mathcal{F}(x) = \mathcal{F}(y)$. This gives that $f(F) = F$. From this we can conclude that $f(S) \subseteq F$ and hence $f(S)$ is a simplex in Δ. Similarly, if S is a simplex with $y \in S$ and $x \notin S$, we can prove again $f(S)$ is in Δ using the equality $\mathcal{F}(x) = \mathcal{F}(y)$.

Let Δ_κ and Δ'_κ be two different choices of representative subcomplexes. Composing isomorphisms of the above type, we can find an isomorphism $f : \Delta \to \Delta$ such that f takes the image of Δ_κ to the image of Δ'_κ. □

We are now ready to prove Theorem 1.1:

Proof of Theorem 1.1. Let Δ_κ be a representing set for Δ with respect to κ, and let V_κ denote the vertex set of Δ_κ. Let $r : \Delta \to \Delta_\kappa$ be the map defined by

$$r(S) = \{v \in V_\kappa \mid \kappa(v) = \kappa(u) \text{ for some } u \in S\}.$$

It is easy to see that r is a retraction, i.e., for each simplex S of Δ_κ, $r(S) = S$. To see that r is a simplicial map, take a simplex S in Δ and let F be a facet including S. The facet F belongs to the set $\mathcal{F}(u)$ for every $u \in S$. By the definition of representing sets, F belongs to the set $\mathcal{F}(v)$ for every $v \in r(S)$ as well. This shows that $r(S) \subseteq F$, and hence $r(S)$ is a simplex of Δ.

Now, we need to show that the composition

$$f : \Delta \xrightarrow{r} \Delta_\kappa \xrightarrow{inc} \Delta$$

is homotopic to identity with a homotopy relative to Δ_κ. Consider f as a poset map between corresponding face posets. If there exists another poset map $g : \Delta \to \Delta$ such that $S \leq g(S) \geq f(S)$ for all $S \in \Delta$, then by Quillen's criteria for homotopy equivalence of poset maps (see, for example, [8]), we can conclude that $id \simeq g \simeq f$. In the above argument we showed that for every simplex S in Δ, a facet F including S also includes $f(S)$ as a subcomplex. Thus, the set $S \cup f(S)$ is a simplex of Δ. Thus we can define $g : \Delta \to \Delta$ as the map $g(S) = S \cup f(S)$ and conclude that f is homotopic to identity. Since both f and g are equal to identity on Δ_κ, the required relativeness condition for the homotopy also holds. This completes the proof. $\square$

Recall that the *homology dimension* $\mathrm{Homdim}(\Delta)$ of a finite simplicial complex Δ is defined to be the integer

$$\mathrm{Homdim}(\Delta) := \min\{i \mid \widetilde{H}_j(\Delta; \mathbb{Z}) = 0 \text{ for all } j > i\}$$

with the convention that $\widetilde{H}_{-1}(\Delta; \mathbb{Z}) = \mathbb{Z}$. The following is an immediate corollary of Theorem 1.1.

Corollary 3.3. *$lchr(\Delta) \geq Homdim(\Delta) + 2$.*

Proof. This is because a simplex with n vertices can have non-trivial homology only at dimensions $i \leq n - 2$. $\square$

We can also obtain a linear coloring analogue of a well-known result of Lovász on graph colorability (see [7]). To state this, we first introduce some terminology about connectedness. Let $\widetilde{H}_i(\Delta)$ denote the *reduced simplicial homology groups* of a simplicial complex Δ over $\mathbb{Z}$. A simplicial complex Δ is said to be *k-acyclic* if $\widetilde{H}_r(\Delta) = 0$ for all $r \leq k$, and it is called *acyclic* if it is k-acyclic for all $k \in \mathbb{Z}$. Further, Δ is called *k-connected* if it is k-acyclic and simply connected, $k \geq 1$.

Corollary 3.4. *If Δ is non-acyclic and k-connected ($k \geq 1$), then $lchr(\Delta) \geq k + 3$.*

Proof. Assume that Δ admits a $(k+2)$-linear coloring κ and let Δ_κ be a representative subcomplex of Δ with respect to κ. Then, Δ is homotopy equivalent to Δ_κ by Theorem 1.1, where Δ_κ is a simplicial complex with $k + 2$ vertices. Such a complex is at most $(k + 1)$-dimensional. Since Δ is non-acyclic, the dimension of Δ_κ cannot be less than $k + 1$ by k-connectivity. On the other hand, if $\dim(\Delta_\kappa) = k + 1$, then it is a $(k + 1)$-simplex which is contractible; hence, it is acyclic, a contradiction. $\square$

4. Multicomplexes associated to linear colorings

In this section, we will discuss an association between multicomplexes and linearly colored simplicial complexes. The results in this section are not used anywhere else in the paper, but we believe that this connection is interesting from enumerative point of view. The main result of this section is that associated to each linearly colored simplicial complex there is a multicomplex such that the simplicial complex can be recovered from this associated multicomplex. Note that if the coloring is an arbitrary coloring, not a linear one, there is still a multicomplex associated to this coloring but we can not recover the simplicial complex from the associated multicomplex. So, the existence of an invertible association is a special property of linear coloring.

We start with the definition of a multicomplex. More details on this material can be found in [2] and [10].

Definition 4.1. A *multicomplex* Γ is a collection of multisets over a set A such that if $M \in \Gamma$ and $M' \subseteq M$, then $M' \in \Gamma$. The elements of Γ are usually called the *faces* of Γ.

Note that the faces of Γ are ordered by inclusion, giving a lattice after adjoining a maximal element. We call the resulting lattice the *face lattice* of Γ and denote it by $L(\Gamma)$. Every multiset M includes a submultiset which is formed by all its elements with no repetitions. We denote this submultiset by $u(M)$ and call it the *underlying set* of M.

If M is a face of a multicomplex Γ, the underlying set $u(M)$ of M is called the *underlying face* of Γ with respect to M. It is easy to see that the collection of all underlying faces of a multicomplex Γ is a simplicial complex. We call this simplicial complex the *underlying simplicial complex* of Γ and denote it by $u(\Gamma)$.

Now, we consider complexes with a linear coloring.

Proposition 4.2. *If Δ is a k-linearly colored complex with coloring map κ, then the collection $\{S_\kappa : S \in \Delta\}$ of multisets is a multicomplex. We call this multicomplex the* associated multicomplex *of the couple (Δ, κ) and denote it by $\Gamma(\Delta, \kappa)$.*

Proof. Let M' be a submultiset of an S_κ where S is a simplex in Δ. Then, it is clear that S has a subset S' such that S'_κ is equal to M'. □

This gives us an assignment $(\Delta, \kappa) \to \Gamma(\Delta, \kappa)$ from the set of linearly colored simplicial complexes to multicomplexes. The following shows that this assignment is surjective.

Proposition 4.3. *Given a multicomplex Γ over $[k]$, there exists a simplicial complex Δ and a k-linear coloring map $\kappa : \Delta \to [k]$ such that $\Gamma = \Gamma(\Delta, \kappa)$.*

Proof. Let Γ be an arbitrary multicomplex over $[k]$. For each $i \in [k]$, let $n_i := \max\{M(i) \colon M \in \Gamma\}$ and let $V_i := \{a^i_r \colon 1 \le r \le n_i\}$. We next define a simplicial complex $\Delta(\Gamma)$ on $V := \cup_{i=1}^k V_i$ as follows: We first associate a subset S_M of V to every multiset $M \in \Gamma$ by taking $a^i_1, a^i_2, \ldots, a^i_j \in S_M$ whenever $M(i) = j$ for any $i \in [k]$. Now, $\Delta(\Gamma)$ is the k-linear colorable simplicial complex generated by the subsets $F_M \subseteq V$ for which M is a facet of Γ, and the linear coloring map $\kappa \colon V \to [k]$ of $\Delta(\Gamma)$ is given by $\kappa(a^i_r) = i$ for all $i \in [k]$. □

The construction given above gives us a unique simplicial complex associated to a multicomplex Γ. Let us denote this simplicial complex $\Delta(\Gamma)$. The following shows that the assignment $\Gamma \to \Delta(\Gamma)$ is, in fact, inverse to the assignment $(\Delta, \kappa) \to \Gamma(\Delta, \kappa)$.

Theorem 4.4. *Let Δ be a simplicial complex on V, and let $\kappa : V \to [k]$ be a k-linear coloring of Δ. Suppose $\Gamma = \Gamma(\Delta, \kappa)$ is the multicomplex associated to the linear coloring κ and let $\Delta(\Gamma)$ be the simplicial complex as in Proposition 4.3. Then, $\Delta(\Gamma)$ is isomorphic to Δ.*

Proof. One can show this using a labeling technique. Note that the coloring $\kappa : V \to [k]$ gives a partitioning of $V = \cup_{i=1}^{k} V_i$ such that V_i is the set of vertices colored by i. Let n_i denote the number of elements in V_i for each $i \in [k]$. As before let $\mathcal{F}(v)$ denote the set of facets in Δ including v as a vertex. Recall that by Proposition 2.9, for each $i \in [k]$, the set $\mathcal{F}_i = \{\mathcal{F}(v) : v \in V_i\}$ is linearly ordered by inclusion. We can label the vertices of Δ in the following way: Let $V = \{v_r^i : i \in [k], r \in [n_i]\}$ where for all i, the vertex v_r^i belongs to V_i and $\mathcal{F}(v_t^i) \subseteq \mathcal{F}(v_r^i)$ whenever $1 \leq r \leq t \leq n_i$.

Recall that the simplicial complex $\Delta(\Gamma)$ on $V := \cup_{i=1}^{k} V_i$ is defined as follows. The subset S_M of V to every multiset $M \in \Gamma$ is defined by taking $a_1^i, a_2^i, \ldots, a_j^i \in S_M$ whenever $M(i) = j$ for any $i \in [k]$. Now, $\Delta(\Gamma)$ is the simplicial complex generated by the subsets $F_M \subseteq V$ for which M is a facet of Γ.

We claim that the assignment $f : \Delta \to \Delta(\Gamma)$ defined by $f(v_r^i) = a_r^i$ for every $i \in [k]$ and $r \in [n_i]$ is an isomorphism of simplicial complexes. To prove this claim, it is enough to show that S is a simplex in Δ if and only if $f(S)$ is a simplex in $\Delta(\Gamma)$. Note that we can prove each direction starting with a facet. Let F be a facet in Δ. To show that $f(F)$ is a simplex in $\Delta(\Gamma)$, we need to show that F satisfies the property that if $v_t^i \in F$, then v_r^i is in F for every $1 \leq r \leq t$. This follows from the fact that $\mathcal{F}(v_t^i) \subseteq \mathcal{F}(v_r^i)$ for every $1 \leq r \leq t \leq n_i$. So, $f(F) \in \Delta(\Gamma)$ as desired. For the other direction, let F be a facet in $\Delta(\Gamma)$, and let M be the corresponding face in Γ. Then, there is a facet F' in Δ such that for each $i \in [k]$, a vertex from V_i appears exactly $M(i)$ times. Recall that the facets of Δ satisfy the property that if v_t^i is in a facet, then v_r^i is also in that facet for every $1 \leq r \leq t$. So, we can conclude that $F' = f^{-1}(F)$, and hence $f^{-1}(F)$ is in Δ. This completes the proof. □

The labeling technique given in the above proof will be used later in the paper. Note that if Δ is labeled as above then we can use it to give a specific representing set and a retraction onto it by taking $\Delta_\kappa = \{v_1^i : i \in [k]\}$ and $r : \Delta \to \Delta_\kappa$ as the map defined by $r(S) = \{v_1^i : i \in u(S_\kappa)\}$ for every simplex S in Δ.

5. LC-reduction of a simplicial complex

In this section we introduce the concept of LC-reduction and study its basic properties. We start with the definition of LC-reduction.

Definition 5.1. Let Δ be a simplicial complex and Δ' be a subcomplex of Δ. If there exist a sequence of subcomplexes $\Delta = \Delta_0 \supseteq \Delta_1 \supseteq \ldots \supseteq \Delta_t = \Delta'$ such that Δ_{r+1} is a representative subcomplex in Δ_r with respect to some linear coloring κ_r of Δ_r for all $0 \leq r \leq t-1$, then we say Δ LC-*reduces* to Δ', and write $\Delta \searrow_{\mathrm{LC}} \Delta'$.

By Theorem 1.1, if Δ LC-reduces to a subcomplex Δ', then Δ' is a strong deformation retract of Δ.

For our purposes it is desirable to be able to express an LC-reduction as a composition of LC-reductions which are primitive in some sense. In this context, the appropriate definition of primitiveness can be given as follows:

Definition 5.2. A linear coloring of a simplicial complex Δ with n vertices is called a *primitive linear coloring* if there is a pair of vertices u, v in Δ such that $\kappa(u) = \kappa(v)$ and the remaining vertices of Δ are colored using distinct colors. An LC-reduction is called *primitive* if it involves only one linear coloring and that coloring is primitive.

Given a primitive coloring κ involving vertices u and v, we have either $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ or $\mathcal{F}(v) \subseteq \mathcal{F}(u)$. In the first case, the subcomplex $\mathrm{del}_\Delta(u) = \{S \in \Delta \mid u \notin S\}$ will be a representative subcomplex, and in the second case $\mathrm{del}_\Delta(v)$ will be representative. In the case of equality either of these sets can be taken as a representative subcomplex. Note that an LC-reduction $\Delta \searrow_{\mathrm{LC}} \Delta'$ is primitive if and only if the number of vertices in Δ' is exactly one less than the number of vertices in Δ.

Proposition 5.3. *Any LC-reduction $\Delta \searrow_{LC} \Delta'$ can be expressed as a sequence of primitive LC-reductions.*

Proof. It is enough to prove the proposition for a LC-reduction involving only one coloring. So, we can assume $\Delta' = \Delta_\kappa$ for some coloring κ of Δ. Suppose that the vertices Δ are labeled as in the proof of Theorem 4.4. So, if V is the set of vertices of Δ, then we can write $V = \{v_r^i \colon i \in [k], r \in [n_i]\}$ where $\mathcal{F}(v_t^i) \subseteq \mathcal{F}(v_r^i)$ whenever $1 \le r \le t \le n_i$. We can assume that Δ_κ is the subcomplex generated by the vertices $\{v_1^i \mid i = 1, \ldots, k\}$.

Let $\kappa(i,j)$ denote the primitive linear coloring involving vertices v_j^i and v_{j+1}^i for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i - 1$. It is easy to see that if we apply LC-reductions associated to primitive linear colorings $\kappa(i, n_i-1), \kappa(i, n_i-2), \ldots, \kappa(i, 1)$ in this order for each $i = 1, \ldots, k$, then we obtain an LC-reduction to Δ_κ. □

Some complexes cannot be LC-reduced further to any proper subcomplex.

Definition 5.4. A simplicial complex Δ is called LC-*irreducible* if it admits only a trivial linear coloring.

The following is clear from the definition.

Proposition 5.5. *A simplicial complex Δ is LC-irreducible if and only if for every pair of vertices u, v, the facet sets $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are not comparable by inclusion.*

A typical example of an LC-irreducible complex is the boundary of a simplex. Another example would be a complex whose realization is an n-gon.

It is easy to see that every simplicial complex Δ LC-reduces to an LC-irreducible subcomplex, although the resulting LC-irreducible subcomplex can be quite different depending on the choices we make. Let us call a subcomplex Δ' of Δ an LC-*core* of Δ if it is irreducible and if Δ LC-reduces to it. The homotopy type of an LC-core is uniquely determined by the homotopy type of Δ, but it is not easy to see what other properties of LC-cores of Δ are invariants of Δ. One would expect that at least the number of vertices of a core is an invariant of the simplicial complex, but since we do not know this at this point, we define a concept of linear dimension in the following way.

Definition 5.6. Let Δ be a simplicial complex. The *linear dimension* of Δ, denoted by $\mathrm{lindim}(\Delta)$, is defined to be the smallest integer n such that Δ has a core with n vertices.

Note that $\mathrm{lindim}(\Delta)$ is also the smallest integer n such that Δ LC-reduces to a simplicial complex with n vertices. It is easy to see that linear dimension is related to the homological dimension of the complex. We can easily adopt the proof of Corollary 3.3 to obtain the following.

Proposition 5.7. *For any finite simplicial complex Δ, we have*

$$lchr(\Delta) \geq lindim(\Delta) \geq Homdim(\Delta) + 2.$$

An interesting family of simplicial complexes are the ones with linear dimension equal to one. These are the complexes which can be LC-reduced to a point. We say a simplicial complex Δ is *LC-contractible* if $\Delta \searrow_{\mathrm{LC}} \{x\}$ for some vertex x of Δ.

Now, we investigate the behavior of LC-reduction under the join operator. Recall that the join of two simplicial complexes X and Y, denoted by $X * Y$, is defined as the simplicial complex which includes both X and Y as subcomplexes and includes also the sets of the form $S \cup T$ where $S \in X$ and $T \in Y$.

Proposition 5.8. *Let $X_1 \searrow_{LC} X_2$ and let Y be an arbitrary simplicial complex. Then, $X_1 * Y \searrow_{LC} X_2 * Y$.*

Proof. It is enough to prove the result for a primitive LC-reduction. Let $X_1 \searrow_{\mathrm{LC}} X_2$ be a primitive reduction involving vertices $u, v \in X_1$. Without loss of generality we can assume $v \in X_2$. Recall that in this case X_2 is the subcomplex $\mathrm{del}_{X_1}(u) = \{S \in X_1 \mid u \notin S\}$. Since $\mathrm{del}_{X_1 * Y}(u) = \mathrm{del}_{X_1}(u) * Y$, we just need to show that the primitive coloring involving u and v is still a linear coloring in $X_1 * Y$. We know that $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ in X_1. Let F be a facet of $X_1 * Y$ including the vertex u. Then either F is a facet of X_1 or F is of the form $S \cup T$ where S and T are facets of X_1 and Y respectively. In the first case, $F \in \mathcal{F}(u)$, so $v \in F$ can be seen easily. In the second case, the facet S belongs to the set $\mathcal{F}(u)$, and again we can conclude $v \in S$. This gives $v \in F$ since $F = S \cup T$. This shows that the inclusion $\mathcal{F}(u) \subseteq \mathcal{F}(v)$ still holds for facet sets in $X_1 * Y$. This completes the proof. $\square$

6. LC-reduction, nonevasive reduction, and collapsing

The aim of this section is to describe an equivalent way to define LC-reduction and as a consequence prove Theorem 1.2 stated in the introduction.

Let Δ be a simplicial complex and v be a vertex in Δ. The link of v in Δ is defined as the subcomplex $\mathrm{lk}_\Delta(v) = \{S \in \Delta \mid v \notin S, S \cup \{v\} \in \Delta\}$ and the deletion of v is defined as the subcomplex $\mathrm{del}_\Delta(v) = \{S \in \Delta \mid v \notin S\}$. The Nonevasiveness of a simplicial complex is defined inductively by declaring that a point is nonevasive and a simplicial complex Δ is nonevasive if it has a vertex v such that both its deletion

$\mathrm{del}_\Delta(v)$ and its link $\mathrm{lk}_\Delta(v)$ are nonevasive. One also defines the concept of nonevasive reduction as a generalization of nonevasiveness.

Definition 6.1. (Kozlov [4], Welker [11]). Let Δ be a simplicial complex and Δ' be a subcomplex of Δ. We say that Δ NE-*reduces* to Δ', denoted by $\Delta \searrow_{\mathrm{NE}} \Delta'$, if there exist a sequence $\Delta = \Delta^1, \Delta^2, \ldots, \Delta^{t+1} = \Delta'$ of subcomplexes and a sequence of vertices $v_1, \ldots, v_t$ such that $V(\Delta^r) = V(\Delta^{r+1}) \cup \{v_r\}$ and $\mathrm{lk}_{\Delta_r}(v_r)$ is nonevasive for every $1 \leq r \leq t$.

It is well known that if a simplicial complex is a cone then it is nonevasive. So, if there exist a sequence $\Delta = \Delta^1, \Delta^2, \ldots, \Delta^{t+1} = \Delta'$ of subcomplexes and a sequence of vertices $v_1, \ldots, v_t$ such that $V(\Delta^r) = V(\Delta^{r+1}) \cup \{v_r\}$ and $\mathrm{lk}_{\Delta_r}(v_r)$ is a cone for every $1 \leq r \leq t$, then this would imply that Δ NE-reduces to Δ'. The following shows that LC-reduction is equivalent to the existence of such sequences.

Theorem 6.2. *Let Δ be a simplicial complex and Δ' be a subcomplex of Δ. Then, Δ LC-*reduces *to Δ' if and only if there exist a sequence $\Delta = \Delta^1, \Delta^2, \ldots, \Delta^{t+1} = \Delta'$ of subcomplexes and a sequence of vertices $v_1, \ldots, v_t$ such that $V(\Delta^r) = V(\Delta^{r+1}) \cup \{v_r\}$ and $lk_{\Delta_r}(v_r)$ is a cone for every $1 \leq r \leq t$*

Proof. Suppose Δ and Δ' are simplicial complexes such that Δ LC-reduces to Δ'. Without loss of generality, we can assume that $\Delta' = \Delta_\kappa$ where κ is a primitive coloring. Let u and v be in Δ such that they are both colored with the same color. Suppose $u \in \Delta'$ and hence $\mathcal{F}(v) \subseteq \mathcal{F}(u)$. We claim that $\mathrm{lk}_\Delta(v)$ is a cone with apex u. Let S be a simplex in $\mathrm{lk}_\Delta(v)$. Let F be a facet of Δ which includes $S \cup \{v\}$. Since $F \in \mathcal{F}(v)$, we have $F \in \mathcal{F}(u)$. This implies that $S \cup \{u\}$ is a simplex in $\mathrm{lk}_\Delta(v)$. We have shown that for every simplex S in $\mathrm{lk}_\Delta(v)$, $S \cup \{u\}$ is also a simplex in $\mathrm{lk}_\Delta(v)$. This means $\mathrm{lk}_\Delta(v)$ is a cone with apex u.

Conversely, suppose that Δ and Δ' are two simplicial complexes such that Δ has one more vertex than Δ', say v, and $\mathrm{lk}_\Delta(v)$ is a cone in Δ' with apex u. We claim that $\mathcal{F}(v) \subseteq \mathcal{F}(u)$. Assume otherwise that $\mathcal{F}(v)$ is not contained in $\mathcal{F}(u)$, and let F be a facet containing v that does not contain u. Then, $F \backslash \{v\}$ is a simplex in $\mathrm{lk}_\Delta(v)$ and since $\mathrm{lk}_\Delta(v)$ is a cone with apex u, we have $(F\backslash\{v\}) \cup \{u\}$ is a simplex in $\mathrm{lk}_\Delta(v)$. Thus, $F \cup \{u\}$ is a simplex in Δ, which contradicts to the fact that F is a facet which does not include u. $\square$

As an immediate corollary, we obtain the following

Corollary 6.3. *Let Δ be a simplicial complex and Δ' be a subcomplex of Δ. If $\Delta \searrow_{LC} \Delta'$, then $\Delta \searrow_{NE} \Delta'$.*

Now, we recall the definition of collapsing.

Definition 6.4. A face S of a simplicial complex Δ is called *free* if S is not maximal and there is a unique maximal face in Δ that contains S. If S is a free face of Δ then the simplicial complex $\Delta[S] := \Delta \backslash \{T \in \Delta \mid S \subseteq T\}$ is called an *elementary collapse* of Δ. If Δ can be reduced to a subcomplex Δ' by a sequence of elementary collapses, then we say Δ *collapses* to Δ' and denote it by $\Delta \searrow \Delta'$.

It is well known that nonevasive reduction is a collapsing by a result of Kahn, Saks, and Sturtevant (see Proposition 1 in [3]). So, we conclude the following

Corollary 6.5. *Let Δ be a simplicial complex and Δ' be a subcomplex in Δ. If $\Delta \searrow_{LC} \Delta'$, then $\Delta \searrow \Delta'$.*

Recall that the relation $\Delta \searrow \Delta'$ can be completed to an equivalence relation. The resulting equivalence class of a simplicial complex is called the simple homotopy type of Δ. So, Corollary 6.5 says, in particular, that if Δ LC-reduces to Δ', then Δ and Δ' have the same simple homotopy type. Note that the proof of Theorem 1.2 is now complete.

We conclude this section with an example which shows that the converse of Proposition 6.5 does not hold in general.

Example 6.6. Let Δ be the 2-dimensional simplicial complex on $V = \{a, b, c, d, e, f\}$ with the set of facets

$$\mathcal{F}(\Delta) = \{\{a, b, c\}, \{a, b, e\}, \{a, d, e\}, \{b, e, f\}, \{d, e, f\}, \{b, c, f\}, \{c, d, f\}\}.$$

The resulting simplicial complex can be thought of as the boundary of a 3-crosspolytope with one facet removed (see Figure 2). It is clear that Δ is collapsible and NE-reduces to a point, but it does not LC-reduce to a point (in fact it is LC-irreducible).

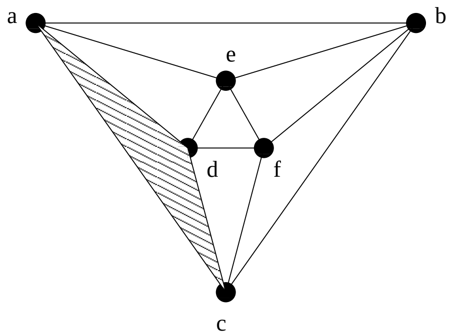

FIGURE 2. A 3-crosspolytope without the facet $\{a, c, d\}$

7. Linear coloring of posets

Let P be a finite partially ordered set. We denote by $\Delta(P)$ its order complex, i.e., the set of all chains in P. When P has maximal and minimal elements, we denote them by $\hat{0}$ and $\hat{1}$, respectively. The elements of P that cover $\hat{0}$ are called *atoms*, and the elements that are covered by $\hat{1}$ are called *coatoms*. We denote the set of atoms and coatoms of a bounded poset P by $at(P)$ and $co(P)$ respectively. We write $\overline{P}$ for the poset $P \backslash \{\hat{0}, \hat{1}\}$, and call it the proper part of P. The set of maximal chains of P is denoted by $\mathcal{M}$, and in particular $\mathcal{M}_x$ denotes the maximal chains containing the element $x \in P$. For a given subset $S \subseteq P$, we denote by $\bigwedge S$ and $\bigvee S$, the greatest lower bound and the least upper bound (when exist) of S respectively.

Throughout, by a linear coloring of P, we mean a linear coloring of $\Delta(P)$. We may rephrase the definition of a linear coloring for posets as follows.

Lemma 7.1. *A surjective mapping $\kappa\colon P \to [k]$ is a k-linear coloring of P if and only if $\kappa(x) = \kappa(y)$ implies either $\mathcal{M}_x \subseteq \mathcal{M}_y$ or $\mathcal{M}_y \subseteq \mathcal{M}_x$ for any two elements $x, y \in P$.*

This implies, in particular, that in a linearly colored poset P any two elements $x, y \in P$ having the same color must be comparable. In fact, more is true. Let P be a poset linearly colored with κ, and let $x, y \in P$ be such that $\kappa(x) = \kappa(y)$. Suppose $\mathcal{M}_x \subseteq \mathcal{M}_y$. Let z be an element in P such that x is comparable with z, i.e, either $x < z$ or $z < x$. Then, there is a maximal chain M including x and z. Since $\mathcal{M}_x$ is included in $\mathcal{M}_y$, the chain M must also include y. Thus, z and y are also comparable. Similarly, we can show that if $\mathcal{M}_x \subseteq \mathcal{M}_y$, then every element of P which is comparable with y is also comparable with x. We define the following:

Definition 7.2. Let P be a poset and $x, y \in P$. We say y *dominates* x, denoted by $x \prec y$, if every element z which is comparable with x is also comparable with y.

We have seen above that in a linearly colored poset P any two elements $x, y \in P$ having the same color must be comparable by domination. The converse of this statement also holds:

Proposition 7.3. *Let P be a poset and $\kappa : P \to [k]$ be a coloring of P. Then, κ is a linear coloring if and only if for every pair $x, y \in P$ with $\kappa(x) = \kappa(y)$, either $x \prec y$ or $y \prec x$.*

Proof. We only need to prove one direction. Let $x, y \in P$ be such that $\kappa(x) = \kappa(y)$ and $x \prec y$. Then every element $z \in P$ which is comparable with x is also comparable with y. We claim that in this case the inclusion $\mathcal{M}_x \subseteq \mathcal{M}_y$ holds. Let M be a maximal chain in $\mathcal{M}_x$. Note that all the elements in M are comparable with x, so they must be also comparable with y. If y is not in M, then by adding y to M we would get a longer chain which will contradict with the maximality of M. So, y must lie already in M. Thus, $M \in \mathcal{M}_y$. $\square$

We have the following:

Proposition 7.4. *Let P be a poset and let $x, y \in P$ such that $x \prec y$. Then, $\Delta(P) \searrow_{LC} \Delta(P \backslash \{x\})$.*

Proof. Consider the primitive linear coloring κ that involves only x and y. The proposition follows from the fact that $\Delta(P)_k = \mathrm{del}_{\Delta(P)}(x) = \Delta(P\backslash\{x\})$. $\square$

It is easy to see that if an element is minimal or maximal, then it dominates all other elements. So, if a poset has a minimal or maximal element, then it is LC-contractible.

Now, we consider monotone poset maps and prove a reduction theorem for them.

Definition 7.5. Let P be a poset. An order-preserving map $\varphi\colon P \to P$ is called a *monotone map* if either $x \leq \varphi(x)$ or $x \geq \varphi(x)$ for any $x \in P$. If φ is a monotone map which also satisfies $\varphi^2 = \varphi$, then it is called a *closure operator* on P.

Note that when $\varphi : P \to P$ is a closure operator then $\mathrm{Fix}(\varphi) = \varphi(P)$, and the equality $P = \varphi(P)$ holds only when φ is the identity map.

Lemma 7.6. *Let P be a finite poset, and let $\psi : P \to P$ be a monotone map on P which is different than the identity map. Then there exists an $x \in P \backslash Fix(\psi)$ such that $x \prec \psi(x)$.*

Proof. Assume to the contrary that for all $x \in P \backslash \mathrm{Fix}(\psi)$, we have $x \not\prec \psi(x)$. Start with $y_0 \in P \backslash \mathrm{Fix}(\psi)$ such that $y_0 \not\prec \psi(y_0)$. This means that there exists an element $y_1 \in P$ such that y_1 is comparable with y_0 but not with $\psi(y_0)$.

Note that since ψ is a monotone map either $y_0 < \psi(y_0)$ or $\psi(y_0) < y_0$ holds. We look at each case separately.

Case 1: Assume $y_0 < \psi(y_0)$ holds. Then, we must have $y_0 < y_1$, because otherwise we have $y_1 < y_0 < \psi(y_0)$ which contradicts the assumption that y_1 and $\psi(y_0)$ are not comparable. Also note that y_1 cannot be an element of $\mathrm{Fix}(\psi)$, because otherwise $y_1 = \psi(y_1) < \psi(y_0)$ implies that y_1 and $\psi(y_0)$ are comparable, which is again a contradiction. So, we have $y_1 \in P \backslash \mathrm{Fix}(\psi)$.

Now, let's apply the same arguments for y_1. First we have $y_1 \not\prec \psi(y_1)$ by our starting assumption, so there exists a y_2 such that y_2 is comparable with y_1 but not with $\psi(y_1)$. Since ψ is a monotone map, we again have either $y_1 < \psi(y_1)$ or $\psi(y_1) < y_1$. Now we claim that actually the second inequality cannot hold. Suppose it holds, i.e., $\psi(y_1) < y_1$. Then we get $\psi(y_0) < \psi(y_1) < y_1$ which gives $\psi(y_0)$ and y_1 are comparable and hence a contradiction. So, we have $y_1 < \psi(y_1)$. This allows us to continue in the same way and obtain an infinite ascending sequence $y_0 < y_1 < y_2 < \cdots$ of distinct elements in P. But, this is in contradiction with the fact that P is a finite poset.

Case 2: Assume $y_0 > \psi(y_0)$ holds. Then, arguing as above we find a descending infinite sequence $y_0 > y_1 > y_2 > \cdots$ of distinct elements in P and again reach a contradiction. □

The main result of this section is the following:

Theorem 7.7. *Let $\varphi \colon P \to P$ be a closure operator on a finite poset P. Then, $\Delta(P) \searrow_{LC} \Delta(\varphi(P))$.*

Proof. We will prove the result by induction on $n = |P \backslash \varphi(P)|$. If $n = 0$, then there is nothing to prove. So assume $n \geq 1$, i.e., φ is not the identity. Then, by Lemma 7.6 there exists an $x \in P \backslash \varphi(P)$ such that $x \prec \varphi(x)$. By Proposition 7.4, we have $\Delta(P) \searrow_{\mathrm{LC}} \Delta(P \backslash \{x\})$. Since $x \notin \varphi(P)$, the restriction of φ to $P \backslash \{x\}$ induces a closure operator $\overline{\varphi} : P \backslash \{x\} \to P \backslash \{x\}$. Applying the induction assumption, we obtain $\Delta(P \backslash \{x\}) \searrow_{\mathrm{LC}} \Delta(\overline{\varphi}(P \backslash \{x\}))$ which gives $\Delta(P \backslash \{x\}) \searrow_{\mathrm{LC}} \Delta(\varphi(P))$ since $\overline{\varphi}(P \backslash \{x\}) = \varphi(P)$. Combining this with the above reduction, we conclude that $\Delta(P) \searrow_{\mathrm{LC}} \Delta(\varphi(P))$. □

Remark 7.8. It has been pointed to us that the Kozlov's argument in [4] works in this generality, so a proof for Theorem 7.7 can also be given using the arguments in [4].

Corollary 7.9. *For a finite poset P, if $\bar{x} = \bigwedge \{c \in co(P) \colon x \leq c\}$ exists for all $x \in P$ then $P \searrow_{LC} R$, where $R = \{\bar{x} \mid x \in P\}$. If in addition, $\bigwedge co(P)$ exists then $\Delta(P)$ is LC-contractible.*

Proof. The map $\varphi\colon P \to P$ defined by $\varphi(x) = \bar{x}$ is a closure operator. Hence, by Theorem 7.7, $\Delta(P) \searrow_{\mathrm{LC}} \Delta(R)$, since $\mathrm{Fix}(\varphi) = \varphi(P) = R$. On the other hand, when it exists, $\bigwedge co(L)$ is the minimal element of R, therefore $\Delta(R)$ is LC-contractible, so is $\Delta(P)$. □

In particular, the above corollary says that the proper part of a lattice is LC-reducible to the proper part of the sublattice of elements that are the meet of coatoms. This result is well-known when the LC-reduction is replaced by homotopy equivalence (see Theorem 10.8 in [1]).

Another interesting invariant in poset theory is the order dimension of a poset which is defined as follows:

Definition 7.10. The *order dimension* of a finite poset P, denoted by $\mathrm{ordim}(P)$, is defined to be the smallest integer n such that P can be embedded in $\mathbb{N}^n$ as an induced subposet (an induced subposet is a subposet which inherits all the relations of the poset.)

There is a very nice paper by Reiner and Welker [9] which proves that the order dimension of a lattice L is greater than $\mathrm{Homdim}(\overline{L}) + 2$, where $\overline{L}$ denotes the proper part of the lattice L. Recall that there is a similar inequality for the linear dimension of a poset (see Proposition 5.7). The obvious question is whether there is any connection between the order dimension of a lattice and the linear dimension of its proper part. Unfortunately these invariants are not comparable by inclusion as the following examples show.

Example 7.11. Consider the poset P which is an antichain with three elements. Let L be the lattice obtained form P by adding minimal and maximal elements. It is clear that $\overline{L} = P$ has linear dimension exactly 3. But, the order dimension of L is equal to 2 since we can embed L in $\mathbb{N}^2$ by taking the minimal element to $(0, 0)$, the maximal element to $(2, 2)$ and the 3 middle points to the points $(0, 2), (1, 1), (2, 0)$. This shows that there is a lattice L where $\mathrm{ordim}(L) < \mathrm{lindim}(\overline{L})$.

For the other direction, consider the poset $P = \{a, b, c\}$ where $a \leq b$, $a \leq c$, and b and c are not comparable. It is easy to see that P is LC-reducible to a point, so $\mathrm{lindim}(P) = 1$. Let L be the lattice obtained from P by adding $\hat{0}$ and $\hat{1}$. It is clear that L is not linear, so $\mathrm{ordim}(L) > 1 = \mathrm{lindim}(\overline{L})$.

We end the section with an application of Corollary 7.9 to subgroup lattices.

Corollary 7.12. *Let G be a finite p-group (p a prime). Then, $\overline{\mathcal{L}(G)}$ is LC-contractible if and only if G is not elementary abelian, where $\mathcal{L}(G)$ is the subgroup lattice of G.*

Proof. It is known that if G is elementary abelian, then the Euler characteristic of $\overline{\mathcal{L}(G)}$ is bigger than 1 (see for example [8]). Thus, $\overline{\mathcal{L}(G)}$ cannot be LC-contractible. Conversely, if G is not elementary abelian, then the intersection of the maximal subgroups of G is non-trivial. Therefore, by Corollary 7.9, $\overline{\mathcal{L}(G)}$ is LC-contractible. □

8. Linear graph colorings

In this final section, we consider linear colorings of neighborhood complexes associated to simple graphs.

Let $G = (V, E)$ be a simple graph. We recall that a (vertex) coloring of G is a surjective mapping $\nu \colon V \to [n]$ such that $\nu(x) \neq \nu(y)$ whenever $(x, y) \in E$. The *neighborhood* of a vertex $v \in V$ is defined to be $\mathcal{N}(v) := \{u \in V : (u, v) \in E\}$, and the *neighborhood complex* of G, denoted by $\mathcal{N}(G)$, is the simplicial complex whose simplices are those subsets of V which have a common neighbor. Note that facets of $\mathcal{N}(G)$ are those subsets of V which are maximal with respect to inclusion and have a single common neighbor, i.e., the set of neighbors of a vertex. We start with the following observation.

Proposition 8.1. *Let $G = (V, E)$ be a simple graph and let $\mathcal{N}(G)$ denote its neighborhood complex. If $\kappa \colon V \to [k]$ is a k-linear coloring of $\mathcal{N}(G)$, then κ is a coloring of the underlying graph G.*

Proof. Assume that κ is not a coloring of the underlying graph G. Therefore, there exist $x, y \in V$ such that $(x, y) \in E$ and $\kappa(x) = \kappa(y)$. By the definition of a linear coloring, either $\mathcal{F}(x) \subseteq \mathcal{F}(y)$ or $\mathcal{F}(y) \subseteq \mathcal{F}(x)$. So, without loss of generality, assume $\mathcal{F}(x) \subseteq \mathcal{F}(y)$. Let $\mathcal{N}(z)$ be a facet of $\mathcal{N}(G)$ such that $\mathcal{N}(y) \subseteq \mathcal{N}(z)$. Since there is an edge between x and y, we have $x \in \mathcal{N}(y)$, and hence $x \in \mathcal{N}(z)$. This implies that $\mathcal{N}(z) \in \mathcal{F}(x)$, and gives $\mathcal{N}(z) \in \mathcal{F}(y)$. Therefore, $y \in \mathcal{N}(z)$ and hence $z \in \mathcal{N}(y)$. However, together with $\mathcal{N}(y) \subseteq \mathcal{N}(z)$, this implies $z \in \mathcal{N}(z)$ which is a contradiction since G is a simple graph and has no loops. $\square$

The following is immediate:

Corollary 8.2. *For any graph G, we have lchr$(\mathcal{N}(G)) \geq \chi(G)$, where $\chi(G)$ denotes the (vertex) chromatic number of G.*

It is easy to see that a coloring of G may not give rise to a linear coloring of its neighborhood complex $\mathcal{N}(G)$. So, in general the equality does not hold.

Example 8.3. Consider the graph which is a hexagon, i.e., $G = (V, E)$ with $V = \{v_1, \dots, v_6\}$ and $E = \{(v_i, v_{i+1}) \mid 1 \leq i \leq 5\} \cup \{(6, 1)\}$. Note that $\chi(G) = 2$, but $\mathrm{lchr}(\mathcal{N}(G)) = 6$ since $\mathcal{N}(G)$ is a disjoint union of two (empty) triangles.

We now give a sufficient condition for a coloring of a graph to be a linear coloring of its neighborhood complex.

Proposition 8.4. *A coloring $\nu \colon V \to [k]$ of $G = (V, E)$ is a k-linear coloring of $\mathcal{N}(G)$ if either $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds for every $x, y \in V$ with $\nu(x) = \nu(y)$.*

Proof. Assume that whenever $\nu(u) = \nu(v)$ for any two vertices $u, v \in V(G)$, then one of the inclusions $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds. Let $u, v \in V(G)$ be two such vertices and let $\mathcal{N}(u) \subseteq \mathcal{N}(v)$. To verify that $\mathcal{F}(u) \subseteq \mathcal{F}(v)$, let $\mathcal{N}(y)$ be a facet of $\mathcal{N}(G)$ containing u. Then we must have $y \in \mathcal{N}(v)$, since $y \in \mathcal{N}(u) \subseteq \mathcal{N}(v)$. Hence, $v \in \mathcal{N}(y)$. $\square$

The converse of Proposition 8.4 does not hold in general as illustrated in Figure 3. It is easy to see that the given vertex coloring of G is indeed a linear coloring of $\mathcal{N}(G)$ with $\nu(u) = \nu(v) = 1$; however, there is no inclusion relation between the neighborhoods of u and v.

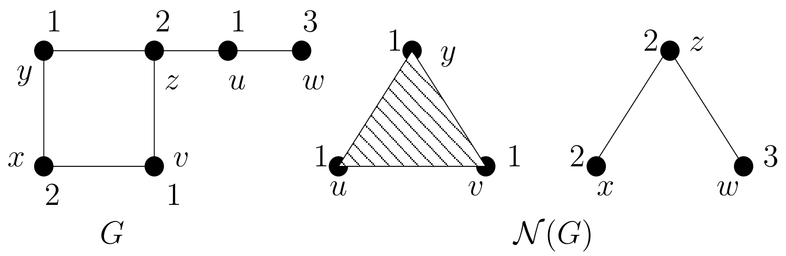

FIGURE 3. A linear coloring of the neighborhood complex of a simple graph.

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