QUASILINEAR ACTIONS ON PRODUCTS OF SPHERES

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Abstract. For some small values of f, we prove that if G is a group having a complex (real) representation with fixity f, then it acts freely and smoothly on a product of $f+1$ spheres with trivial action on homology.

1. introduction

Some of the most interesting questions about group actions on topological spaces are about group actions on products of spheres. Historically, these questions are generalizations of questions about group actions on a sphere. In general, one is interested in whether a given finite group G can act freely on a product of k spheres for a given integer $k \geq 1$. One of the long lasting conjectures in this subject states that if a group G acts freely on a product of k spheres then $\mathrm{rk}(G) \leq k$, where $\mathrm{rk}(G)$ is the rank of the group G defined as the largest integer s such that $(\mathbb{Z}/p)^s \subset G$ for some prime number p. In terms of construction of actions, it is known that every finite group can act freely on a product of spheres. But the following problem is still open.

Problem. Show that every finite group can act freely on a product of spheres with trivial action on homology.

The type of action demanded in the above problem is much harder to construct since we are not allowed to permute the spheres in the product. In this paper, we are interested in the following question, which is closely related to the above problem.

Question. Suppose that G is a finite group which has a complex representation or an oriented real representation with fixity f. Then, does it act freely and smoothly on a product of $f+1$ spheres with trivial action on homology?

Let G be a finite group and F be a field. The *fixity* of an F-representation V of G is defined as the maximum value of $\dim_F V^g$ among all $1 \neq g \in G$. Note that if G has a complex representation with fixity f, then it acts freely on the coset space $U(n)/U(n-f-1)$ which has the integral homology of a product of $f+1$ spheres. So, the above problem is asking whether one can replace a free action on a space which has the homology of a product of spheres with a free smooth action on a product of spheres.

It is clear that if G has a representation with fixity zero, then it acts freely on a sphere. Adem, Davis, and Ünlü [1] showed that if G has a complex representation V of dimension

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n with fixity one, then G acts freely and smoothly on $S^{2n-1} \times S^{4n-5}$. They construct this action by taking the unit sphere of the Whitney sum double of the tangent bundle of $S(V)$, where $S(V)$ denotes the unit sphere of V. We observe that their argument can be extended to some higher values of fixity. In particular, for complex representations we prove the following:

Theorem 1.1. *If G is a finite group which has a faithful n-dimensional complex representation with fixity 2, then G acts freely and smoothly on $X = S^{2n-1} \times S^{4n-5} \times S^{q(4n-8)-1}$ for some $q \geq 2$ with trivial action on homology.*

The value of q in the above product depends on the value of n. For example, if $n \geq 4$, then the last sphere can be taken as low as S^{8n-17}. This shows in particular that finite subgroups of $SU(4)$ act freely and smoothly on $X = S^7 \times S^{11} \times S^{15}$. We prove Theorem 1.1 using pullbacks of bundles over quaternionic Stiefel manifolds.

For real representations, we prove a similar result.

Theorem 1.2. *Let G be a finite group which has a faithful representation $\rho : G \to SO(n)$ with fixity $f \leq 4$. Assume that $n \geq 12$ when $f = 4$. Then G acts freely and smoothly on a product of $f + 1$ spheres with trivial action on homology.*

For fixity $f \leq 3$, we can prove the above statement using a similar argument to the complex case, but for $f = 4$ this method seems to fail. So we use a slightly different argument which involves a sequence of $\mathbb{R}$-algebras satisfying certain properties. The argument gives a recipe for constructing free actions for all values of f, but because of the well known dimension restriction on division algebras over $\mathbb{R}$, at this point we can only make it work for $f \leq 4$.

Finally, we would like to remark that, as was explained in [1], to obtain a free action of a finite group on a finite complex homotopy equivalent to a product of k spheres, it is enough to construct an action on a product of $k - 1$ spheres with rank 1 isotropy. Then, using a technique given in [2] one gets a free action on a finite complex having homotopy type of a product of k spheres. This allows us to state the following as a corollary of Theorems 1.1 and 1.2.

Corollary 1.3. *Let G be a finite group. If G has a complex representation with fixity 3, then G acts freely on a finite complex X homotopy equivalent to a product of 4 spheres with trivial action on homology. If G has a real representation with fixity 5 having dimension $n \geq 12$, then G acts freely on a finite complex X homotopy equivalent to a product of 6 spheres with trivial action on homology.*

We organize the paper as follows: In Section 2, some basic lemmas and Theorem 1.1 are proved. In this section, we also discuss how Theorem 1.2 can be proved for $f \leq 3$ using an argument similar to the complex case. In Section 3, we prove Theorem 1.2 for $f \leq 4$ using a more general argument.

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2. Proof of Theorem 1.1

Throughout this section, we will use the notation and terminology used in [6]. Let F denote the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$. For a real number the conjugation is defined by $\overline{x} = x$, for a complex number $x = a+ib$ by $\overline{x} = a-ib$, and for a quaternion $x = a + ib + jc + kd$ by $\overline{x} = a - ib - jc - kd$. On the vector space F^n, we can define an inner product (v, w) by taking

$$(v, w) = v_1\overline{w}_1 + v_2\overline{w}_2 + \cdots + v_n\overline{w}_n.$$

The classical group $U_F(n)$ is defined as the subgroup of $GL(n, F)$ that preserves the inner product. For $F = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, we have different notations for the group $U_F(n)$. The orthogonal group $O(n)$ is the subgroup of $GL(n, \mathbb{R})$ formed by $n \times n$ real matrices A satisfying the property $(Av, Aw) = (v, w)$ for all $v, w \in \mathbb{R}^n$. Similarly, the unitary group $U(n)$ is the subgroup of $GL(n, \mathbb{C})$ which preserves the inner product, and $Sp(n)$ is the subgroup of $GL(n, \mathbb{H})$ which preserves the inner product for vector space $\mathbb{H}^n$. We define classical groups $SO(n)$ and $SU(n)$ as subgroups of $O(n)$ and $U(n)$ formed by matrices with determinant equal to 1.

In our proofs, we will be using fibre bundles arising from Stiefel manifolds. Let $V_k(F^n)$ denote the subspace of F^{nk} formed by the k-tuples of vectors $(v_1, v_2, \ldots, v_k)$ such that $v_i \in F^n$ for all $i = 1, \ldots, k$, and for every pair (i, j), we have $(v_i, v_j) = 1$ if $i = j$ and zero otherwise. There is a homeomorphism between $V_k(F^n)$ and the coset space $U_F(n)/U_F(n-k)$ (see Theorem 1.3 in Chapter 8 of [6]).

There is a sequence of fibre bundles

$$V_n(F^n) \to \cdots \to V_{k+1}(F^n) \to V_k(F^n) \to \cdots \to V_2(F^n) \to V_1(F^n)$$

where the map $q_k : V_{k+1}(F^n) \to V_k(F^n)$ is defined by $q_k(v_1, \ldots, v_{k+1}) = (v_1, \ldots, v_k)$ and the fibre of q_k is $V_1(F^{n-k}) = S^{c(n-k)-1}$ where $c = \dim_{\mathbb{R}} F$ (see Theorem 3.8 and Corollary 3.9 in Chapter 8 of [6]). Note that the sphere bundle $q_k : V_{k+1}(F^n) \to V_k(F^n)$ is the sphere bundle of the vector bundle $\overline{q}_k : \overline{V}_{k+1}(F^n) \to V_k(F^n)$ where $\overline{V}_{k+1}(F^n)$ is the space formed by $(k+1)$-tuples $(v_1, \ldots, v_{k+1})$ where $(v_1, \ldots, v_k) \in V_k(F^n)$ and $(v_i, v_{k+1}) = 0$ for all $i = 1, \ldots, k$.

Lemma 2.1. *The vector bundle $\overline{q}_k : \overline{V}_{k+1}(F^n) \to V_k(F^n)$ is stably trivial.*

Proof. Note that a vector bundle ξ is stably trivial if there is a bundle isomorphism $\xi \oplus \tau^i \cong \tau^j$ for some trivial bundles τ^i and τ^j of dimensions i and j. Note that in our case we can consider the bundle $\xi = (\overline{q}_k : \overline{V}_{k+1}(F^n) \to V_k(F^n))$ as a subbundle of the trivial bundle $V_k(F^n) \times F^n$ which is orthogonal to the bundle θ defined as follows: Let $\theta : E \to V_k(F^n)$ denote the bundle with total space

$$E = \{((v_1, v_2, \ldots, v_k), w) \mid w \in \langle v_1, \ldots, v_k \rangle,\ (v_1, \ldots, v_k) \in V_k(F^n),\ w \in F^n\}$$

and with obvious projection map. It is easy to see that $\xi \oplus \theta$ is isomorphic to the trivial bundle $V_k(F^n) \times F^n$ and that θ is a trivial bundle. □

Another way to see this result is the following: Note that a vector bundle ξ over a paracompact base space B is stably trivial if its classifying map $\hat{\xi} : B \to BU_F(n)$ becomes null-homotopic when it is composed with $BU_F(n) \to BU_F(n+i)$ for some i. In the above lemma, the classifying map for $\xi = (\overline{q}_k : \overline{V}_{k+1}(F^n) \to V_k(F^n))$ is a map of the form $\phi : V_k(F^n) \to BU_F(n-k)$. Note that $BU_F(n-k)$ can be described as the space of $(n-k)$-dimensional subspaces in F^∞. Under this description, the classifying map ϕ takes a k-tuple $(v_1, \dots, v_k)$ of vectors in F^n to the subspace orthogonal to the subspace generated by v_i's.

We also have a fibration of the form

$$V_k(F^n) \to BU_F(n-k) \to BU_F(n)$$

which comes from the homeomorphism $V_k(F^n) \cong U_F(n)/U_F(n-k)$. The projection map $BU_F(n-k) \to BU_F(n)$ is given by inclusion of an $(n-k)$-dimensional subspace in F^∞ to an n-dimensional subspace in F^∞. The inclusion of a fibre over a subspace of dimension n is exactly the map described above. So, the map $V_k(F^n) \to BU_F(n-k)$ in the above fibration coincides with the classifying map ϕ. So, composing the classifying map ϕ with $BU_F(n-k) \to BU_F(n)$ gives a null-homotopic map. This shows that the bundle $\xi = (\overline{q}_k : \overline{V}_{k+1}(F^n) \to V_k(F^n))$ is stably trivial

Another important ingredient in our proofs is a theorem about stably trivial bundles. Let ξ be a vector bundle $E \to B$ over an n-dimensional CW-complex B. The following is a special case of a theorem given in [6].

Lemma 2.2 ([6], Theorem 1.5, Chapter 9)**.** *Let ξ be a k-dimensional F vector bundle with $\dim_{\mathbb{R}} F = c$. Suppose that the base space B is an n-dimensional CW-complex. If ξ is stably trivial and $n \leq c(k+1) - 2$, then ξ is trivial.*

Proof. Let $f : B \to BU_F(k)$ denote the classifying map for ξ. Assume that the composition of f with $BU_F(k) \to BU_F(k+j)$ is null-homotopic for some j. Then f lifts to a map $f : B \to V_j(F^{k+j})$. Using the fibration sequence for Stiefel manifolds, one can easily show that $\pi_i(V_j(F^{k+j})) = 0$ for $i \leq c(k+1) - 2$. So, the result follows from the so called compression lemma in homotopy theory (see Lemma 4.6 in [4] or Corollary 7.13 in [3]). □

Lemma 2.2, in particular, shows that given a stably trivial vector bundle over a finite dimensional complex, we can obtain a trivial vector bundle by taking Whitney sum multiples of it sufficiently many times.

Remark 2.3. Note that in vector bundle theory if the vector bundle in question is a smooth vector bundle $E \to B$ over a smooth manifold B, then a continuous trivialization can be replaced by a smooth trivialization leading to a diffeomorphism $S(E) \approx B \times S(V)$ where $S(E)$ is the total space of the corresponding sphere bundle. This is explained in detail in Chapter 4 of [5] (see also Proposition 6.20 in [7]). Throughout the paper, we use these diffeomorphisms without further explanation. Note that the differential structure on the product $B \times S(V)$ is the product differential structure and $S(V)$ always denotes the standard sphere, not an exotic one.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since the theorem obviously holds for the trivial group, we can assume $G \neq 1$. Let V be a faithful complex representation of G with $\dim V = n$ and fixity 2. Since V is faithful, we must have $n \geq 3$. Consider the pull-back diagram

$$\begin{array}{ccc} E_1 & \longrightarrow & V_2(\mathbb{H}^n) \\ \downarrow & & \downarrow{\scriptstyle q_1} \\ V_1(\mathbb{C}^n) & \xrightarrow{f} & V_1(\mathbb{H}^n) \end{array}$$

where f is the map induced from the inclusion of $\mathbb{C}$ into $\mathbb{H}$ defined by $a+ib \to a+ib+j0+k0$. Since the map $f : V_1(\mathbb{C}^n) = S^{2n-1} \to V_1(\mathbb{H}^n) = S^{4n-1}$ is null-homotopic, the bundle $E_1 \to V_1(\mathbb{C}^n)$ is a trivial bundle with fibre S^{4n-5}. So $E_1 \approx S^{2n-1} \times S^{4n-5}$.

Now, consider the pull back bundle

$$\begin{array}{ccc} E_2 & \longrightarrow & \overline{V}_3(\mathbb{H}^n) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle q_2} \\ E_1 & \xrightarrow{\tilde{f}} & V_2(\mathbb{H}^n) \end{array} .$$

The bundle ξ is stably trivial since it is a pullback of a stably trivial bundle. Taking a q-fold Whitney sum $\xi \oplus \cdots \oplus \xi$ of ξ, we obtain a bundle $\xi^{\oplus q} : E_2(q) \to E_1$. By Lemma 2.2, the bundle $\xi^{\oplus q}$ is a trivial bundle when

$$\dim E_1 \leq 4(q(n-2)+1) - 2 = 4q(n-2) + 2 .$$

Note that since $n \geq 3$, we can always find a q which makes this inequality hold. In fact, since $\dim E_1 = 6n-6$, the inequality holds even for $q = 2$ when $n \geq 4$.

Note that the total space $E_2(q)$ can be considered as the subspace of $\mathbb{C}^{n(3+2q)}$ formed by $(3+2q)$-tuples of complex vectors $(v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, \ldots, v_{3,2q})$ satisfying the property

$$((v_{1,1}, 0), (v_{2,1}, v_{2,2}), (v_{3,2i-1}, v_{3,2i})) \in \overline{V}_3(\mathbb{H}^n)$$

for all $i = 1, \ldots, q$. Here we consider a pair of complex numbers $(a+ib, c+id)$ in $\mathbb{H}$ as the quaternion $y = a+ib+jc+kd$.

Let V be an n-dimensional faithful complex representation of G, and let $W = \mathbb{H} \otimes_{\mathbb{C}} V$. By taking the average, we can assume the inner product on W is G-invariant. Note that if we consider elements of W as pairs of complex vectors (v_1, v_2), then the G-action on W can be written as a diagonal action $g(v_1, v_2) = (gv_1, gv_2)$. From this it is easy to see that the G-action on $\overline{V}_3(W)$ gives a G-action on $E_2(q)$ for any q.

Let X denote the total space of the sphere bundle of $E_2(q)$. Then $X \approx S^{2n-1} \times S^{4n-5} \times S^{q(4n-8)-1}$ and G acts smoothly on X. Since V has (complex) fixity 2, the quaternionic representation W has (quaternionic) fixity 2. This implies that if $g \in G$ fixes a point

$$(v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, \ldots, v_{3,2q}) \in X,$$

then we must have $v_{3,i} = 0$ for all i, which is not possible. Thus G acts freely on X. □

For real representations the argument given in the above proof can be extended to prove that if a finite group has a faithful real representation $\rho : G \to SO(n)$ with fixity

$f \leq 3$, then G acts freely and smoothly on a product of $f+1$ spheres with trivial action on homology. For this, we consider the diagram

$$\begin{array}{ccccc} E_2 & \longrightarrow & E_2' & \longrightarrow & V_3(\mathbb{H}^n) \\ \downarrow & & \downarrow & & \downarrow q_2 \\ E_1 & \xrightarrow{f'} & V_2(\mathbb{C}^n) & \xrightarrow{h} & V_2(\mathbb{H}^n) \\ \downarrow & & \downarrow q_1 & & \\ V_1(\mathbb{R}^n) & \xrightarrow{f} & V_1(\mathbb{C}^n) & & \end{array}$$

where each square in the diagram is a pullback square and the maps f and h are maps induced by inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. Since the map f is null-homotopic, $E_1 \approx S^{n-1} \times S^{2n-3}$. The space $V_2(\mathbb{H})$ is $(4n-6)$-connected. For $n \geq 2$, the inequality $\dim E_1 = 3n-4 \leq 4n-6$ holds, so by the compression lemma in homotopy theory, the composition $h \circ f'$ is also null-homotopic. This implies that $E_2 \approx S^{n-1} \times S^{2n-3} \times S^{4n-9}$.

Note that if G has a real representation with fixity $f = 1$ or 2, then G acts freely and smoothly on E_f in the diagram. The G-actions are induced from the G-actions on Stiefel manifolds as in the complex case. For $f = 3$, we first consider the pullback square

$$\begin{array}{ccc} E_3 & \longrightarrow & \overline{V}_4(\mathbb{H}^n) \\ \downarrow \xi & & \downarrow q_3 \\ E_2 & \xrightarrow{h \circ f'} & V_3(\mathbb{H}^n) \end{array}$$

and then we take a q-fold Whitney sum of ξ to make the dimension condition in Lemma 2.2 hold. For this we need q to satisfy $\dim E_2 = 7n - 13 \leq q(4n-12)+2$. Note that since we have $n \geq 4$, there always exists a q which makes this inequality hold. If $n \geq 9$, the inequality holds even for $q = 2$. Let X denote the total space of the sphere bundle of $E_3(q)$, then X is a product of 4 spheres. If G has a faithful real representation $\rho : G \to SO(n)$ with fixity $f = 3$, then G acts freely and smoothly on X.

Note that in the above construction the space E_1 can be considered as the space of tuples of n-dimensional real vectors $(v_{1,1}, v_{2,1}, v_{2,2})$ such that $v_{1,1} + i0$ is orthogonal to $v_{2,1} + iv_{2,2}$ as complex vectors. This is equivalent to saying that $v_{1,1} \perp v_{2,1}$ and $v_{1,1} \perp v_{2,2}$ as real vectors. It is clear from this that if G acts orthogonally on $\mathbb{R}^n$, then it will act on E_1 with diagonal action on coordinates. In a similar way, the space E_2 can be considered as the space of tuples of the form

$$(v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4})$$

where $v_{1,1} \perp v_{k,l}$ for all k, l with $k \geq 2$ and $v_{2,1} + iv_{2,2}$ is perpendicular to $v_{3,1} + iv_{3,2}$ and $v_{3,3} + iv_{3,4}$ as complex vectors. This is equivalent to saying that as pairs of real vectors, $(v_{2,1}, v_{2,2})$ is perpendicular to $(v_{3,1}, v_{3,2})$ and $(v_{3,3}, v_{3,4})$, and that $(-v_{2,2}, v_{2,1})$ is perpendicular to $(v_{3,1}, v_{3,2})$ and $(v_{3,3}, v_{3,4})$. Note that this second relation makes it possible to conclude that the G-action on E_2 is free for a representation with fixity $f = 2$. If we had just taken all the tuples

$$(v_{1,1}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{3,3}, v_{3,4})$$

with $v_{1,1} \perp v_{k,l}$ for all k, l with $k \geq 2$ and $(v_{2,1}, v_{2,2})$ is perpendicular to $(v_{3,1}, v_{3,2})$ and $(v_{3,3}, v_{3,4})$ we would still have a product of spheres with a G-action but the action would no longer be free for a real representation with fixity 2. In the next section, we elaborate on this idea and give a more general construction of "quasilinear" actions.

Remark 2.4. In Theorem 1.2, if we start with an $O(n)$ representation instead of an $SO(n)$ representation, we still get a free action on a product of spheres, but the resulting action on homology will no longer be trivial since G may act on some homology classes via the sign representation. However, the action on $\mathbb{F}_2$ homology will be trivial.

3. Proof of Theorem 1.2

Let V be a real representation of G. Suppose that we are given a sequence of $\mathbb{R}$-algebras $A_1 \subset A_2 \subset \cdots \subset A_s$ which are possibly non-commutative and non-associative. For each i, define $V_i = V \otimes_{\mathbb{R}} A_i$. Then, V_i is a real representation with the G-action given by left multiplication $g(v \otimes a) = gv \otimes a$. Note that V_i is also a module over A_i where the action of $a' \in A$ on $v \otimes a$ is given by $(v \otimes a)a' = v \otimes (aa')$. In the non-associative case, this does not give a module structure in the usual sense by it still gives a $\mathbb{R}$-bilinear map $V_i \times A_i \to V_i$ which is what we need.

If there is an inner product on V, we can define an inner product on V_i by $(v \otimes a, v' \otimes a') = (v, v')(a, a')$ where the inner products on A_i's can be taken in such a way that $A_i \subset A_{i+1}$ is a subspace as an inner product space. This gives us a sequence of real representations of G

$$V_1 \subset V_2 \subset \cdots \subset V_s$$

such that each V_i is a module over A_i.

From now on we assume that $V_1 \subset V_2 \subset \cdots \subset V_s$ is a sequence of real representations such that each V_i is a module over A_i, but not necessarily of the form $V_i = V \otimes_{\mathbb{R}} A_i$ for a fixed real representation V. We choose an inner product on each V_i such that $V_i \subset V_{i+1}$ is an inclusion of inner product spaces. For each $u \in V_i$, let $\langle u \rangle_{A_i}$ denote the vector space $\{ua \mid a \in A_i\}$. We denote the norm of a vector by $|u|$. We define

$$W_k(\{V_i\}_{i=1}^s) = \{(v_1, v_2, \ldots, v_k) \mid v_i \in V_i \text{ with } |v_i| = 1, \text{and } v_j \perp \langle v_i \rangle_{A_j} \text{ for all } i < j\}.$$

To simplify notation, we write $W_k(V)$ for $W_k(\{V_i\}_{i=1}^s)$. The subspace $\overline{W}_k(V)$ is defined as the space of k-tuples as above but we no longer require the last vector v_k to be a unit vector.

We will prove that under certain conditions, the map $q_k : \overline{W}_{k+1}(V) \to W_k(V)$ defined by $q_k(v_1, \ldots, v_{k+1}) = (v_1, \ldots, v_k)$ is a projection map of a stably trivial vector bundle. To find these conditions, we consider some subbundles of the trivial bundle $W_k(V) \times V_{k+1} \to W_k(V)$ which are defined in the following way: For each $i \in \{1, \ldots, k\}$, let

$$E_i = \{((v_1, \ldots, v_k), w) \mid w \in \langle v_i \rangle_{A_{k+1}},\ (v_1, \ldots, v_k) \in W_k(V)\},$$

and let $q_{k,i} : E_i \to W_k(V)$ be the obvious projection map $((v_1, \ldots, v_k), w) \to (v_1, \ldots, v_k)$.

Note that for each nonzero $u \in V_i$, the vector space $\langle u\rangle_{A_{k+1}}$ is isomorphic to $A_{k+1}/\operatorname{Ann}_{A_{k+1}}(u)$ where $\operatorname{Ann}_{A_{k+1}}(u) = \{a \in A_{k+1} \mid ua = 0\}$. Suppose that $\operatorname{Ann}_{A_{k+1}}(u)$ is equal to a fixed subspace $B_{k+1,i} \subset A_{k+1}$ for every $u \in V_i$. Then, we can choose a basis $\{a_1, \dots, a_m\}$ for A_{k+1} such that $\{a_1, \dots, a_l\}$ is a basis for $B_{k+1,i}$. Using this basis, we can easily express $w \in \langle v_i\rangle_{A_{k+1}}$ uniquely as $\sum_{t=l+1}^{m} \gamma_t(v_i a_t)$ where $\gamma_t \in \mathbb{R}$. We conclude the following:

Lemma 3.1. *Suppose that* $\operatorname{Ann}_{A_{k+1}}(u)$ *is equal to a fixed subspace* $B_{k+1,i} \subset A_{k+1}$ *for every nonzero* $u \in V_i$. *Then, the map* $q_{k,i} : E_i \to W_k(V)$ *defined above is the projection map of a fibre bundle* $\theta_{k,i}$ *which is a trivial bundle with dimension* $\dim_{\mathbb{R}}(A_{k+1}/B_{k+1,i})$.

If a filtration $\{V_i\}_{i=1}^s$ satisfies the condition given in the above lemma for all i, k with $1 \le i \le k \le s-1$, then we say it is a *uniform* filtration. In our applications, this condition holds often with $B_{k+1,i} = 0$ for all i, k. Another condition that we can impose on our filtration is the following:

Definition 3.2. We say a sequence $\{V_i\}_{i=1}^s$ is *separable* at k if for every $(v_1, \dots, v_k) \in W_k(V)$, we have

$$\langle v_j\rangle_{A_{k+1}} \cap \langle v_i\rangle_{A_{k+1}} = 0$$

for every i, j with $1 \le i < j \le k$. Then, we say the sequence $\{V_i\}_{i=1}^s$ is a *separable* sequence if it is separable at every k with $1 \le k \le s-1$.

For uniform, separable sequence of real representations, the following is true.

Lemma 3.3. *Let* $\{V_i\}_{i=1}^s$ *be a uniform, separable sequence. Then, the space*

$$E = \{((v_1, \dots, v_k), w) \mid w \in \sum_{i=1}^{k} \langle v_i\rangle_{A_{k+1}},\ (v_1, \dots, v_k) \in W_k(V)\},$$

together with the obvious projection map $E \to W_k(V)$ *defines a vector bundle isomorphic to* $\theta_{k,1} \oplus \theta_{k,2} \oplus \cdots \oplus \theta_{k,k}$.

Let ξ_k denote the orthogonal complement of the bundle $\theta_{k,1} \oplus \cdots \oplus \theta_{k,k}$ in the trivial bundle $\tau_k : W_k(V) \times V_{k+1} \to W_k(V)$. Then, it is not very difficult to see that ξ_k is the bundle with the projection map $q_k : \overline{W}_{k+1}(V) \to W_k(V)$ described above. So, we conclude the following:

Proposition 3.4. *Let* $\{V_i\}_{i=1}^s$ *be a uniform, separable sequence of real representations. Then, the vector bundle* $\xi_k : \overline{W}_{k+1}(V) \to W_k(V)$ *is a stably trivial vector bundle of dimension* $\dim_{\mathbb{R}} V_{k+1} - \sum_{i=1}^{k} \dim_{\mathbb{R}}(A_{k+1}/B_{k+1,i})$.

Let $m_1 = \dim_{\mathbb{R}} V_1$, and for each $1 \le k \le s-1$, define

$$m_{k+1} = \dim_{\mathbb{R}} V_{k+1} - \sum_{i=1}^{k} \dim_{\mathbb{R}}(A_{k+1}/B_{k+1,i}).$$

Note that a CW-structure can be given for the space $W_k(V)$, and as CW-complex the dimension of $W_k(V)$ is $\sum_{i=1}^{k}(m_i - 1)$. By Lemma 2.2, we have the following:

Proposition 3.5. *Let $\{V_i\}_{i=1}^s$ be a uniform, separable sequence of representations. Suppose that*

$$\sum_{j=1}^{k}(m_j-1)\leq m_{k+1}-1$$

holds for all $1\leq k\leq s-1$. Then, the vector bundles $\xi_1,\dots\xi_s$ associated to this sequence are all trivial vector bundles. In particular, for each $k\leq s$, we have $W_k(V)\approx S^{m_1-1}\times S^{m_2-1}\times\cdots\times S^{m_k-1}$.

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a finite group and V be a faithful n-dimensional oriented real representation of G with fixity 4. We assume that $n\geq 12$. Consider the sequence of $\mathbb{R}$-algebras

$$A_1=\mathbb{R}\subset A_2=\mathbb{C}\subset A_3=\mathbb{H}\subset A_4=\mathbb{O}\subset A_5=\mathbb{S}$$

where $\mathbb{O}$ denotes the octonions, and $\mathbb{S}$ denotes the sedenions. These algebras are the first five steps of the Cayley-Dickson construction, where A_i is defined as the pairs of elements (a,b) with $a,b\in A_{i-1}$ and the multiplication is given by

$$(a,b)(c,d)=(ac-d^*b,da+bc^*)$$

and the conjugation for a pair is defined by $(a,b)^*=(a^*,-b)$.

One particular property of the above sequence is that for every $a_i\in A_i$ and $a_j\in A_j$ with $1\leq i<j\leq 5$, if $a_ia_j=0$, then $a_i=0$ or $a_j=0$. This is because the algebras $\mathbb{R},\mathbb{C},\mathbb{H}$, and $\mathbb{O}$ are division algebras. The sedenions $\mathbb{S}$ is not a division algebra but we have $(a,0)(c,d)=0\Rightarrow ac=0,da=0$, so the above property still holds since $\mathbb{O}$ is a division algebra. Because of this property, we can conclude that the sequence $\{V_i\}_{i=1}^5$ with $V_i=V\otimes_{\mathbb{R}}A_i$ is a uniform sequence with $B_{k+1,i}=0$ for all $1\leq i\leq k\leq 4$.

We claim that the sequence $\{V_i\}_{i=1}^5$ is also separable. To show this, we will need to show that for every $k\leq 4$, the following holds: If $v_i\in V_i$ and $v_j\in V_j$ with $1\leq i<j\leq k$, then $\langle v_i\rangle_{A_{k+1}}\cap\langle v_j\rangle_{A_{k+1}}=0$. Note that for every u,t with $1\leq u<t\leq 5$, the algebra A_t is a free module over A_u. So, to show that $\langle v_i\rangle_{A_{k+1}}\cap\langle v_j\rangle_{A_{k+1}}=0$, it is enough to show that $\langle v_i\rangle_{A_j}\cap\langle v_j\rangle_{A_j}=0$. By choosing a free basis for A_j as an A_i-module, we can express v_i in V_j as a tuple $(v_i,0,\dots,0)$ and v_j as a tuple $(v_{j,1},\dots,v_{j,d})$ where $d=\dim_{A_i}A_j$.

If $\langle v_i\rangle_{A_j}\cap\langle v_j\rangle_{A_j}\neq 0$, then there exists an $a\in A_i$ such that $v_ia=\sum_{l=1}^{d}v_{j,l}b_l$ for some $b_l\in A_i$. Since $i<4$, the algebra A_i is an associative division algebra, so b_l is invertible. Using the fact that $v_j\perp\langle v_i\rangle_{A_j}$, we obtain that $v_ia(b_l)^{-1}\perp v_{j,l}$ for all l. This gives that v_ia is perpendicular to $v_{j,l}b_l$ for all l. But then v_ia would be perpendicular to itself which is a contradiction. So, the sequence $\{V_i\}_{i=1}^5$ is a separable sequence.

Since $B_{k+1,i}=0$ for all k,i with $1\leq i\leq k\leq 4$, we have $m_1=\dim_{\mathbb{R}}V_1=n$ and

$$m_{k+1}=\dim_{\mathbb{R}}V_{k+1}-\sum_{i=1}^{k}\dim_{\mathbb{R}}(A_{k+1}/B_{k+1,i})=n2^k-\sum_{i=1}^{k}2^k=(n-k)2^k.$$

So, to apply Proposition 3.5, we need the following inequalities to hold:

$$\begin{aligned} n-1 &\leq 2n-3 \\ 3n-4 &\leq 4n-9 \\ 7n-13 &\leq 8n-25 \\ 15n-38 &\leq 16n-65 \end{aligned} \tag{3.5}$$

Note that the first three inequalities are satisfied when $n \geq 12$. The last inequality needs a higher bound for n, but we can always use the Whitney sum on the last bundle, so we really don't need the last inequality to hold. To see this, consider the bundle $\overline{W}_5(V) \to W_4(V)$. Since the sequence $\{V_i\}_{i=1}^4$ is splitting, we get $W_4(V) \cong S^{n-1} \times S^{2n-3} \times S^{4n-9} \times S^{8n-25}$. Taking a q-fold Whitney sum of this bundle, we get a real vector bundle, say $E(q)$, with dimension $q(16n-64)-1$. For this bundle to be trivial, we need the inequality

$$15n-38 \leq q(16n-64)-1$$

to hold. Since we have already assumed $n \geq 12$, the inequality holds even for $q=2$.

Finally, we need to show that G acts freely on the total space of the sphere bundle for $E(q)$. Let us denote this space by X. As before, we consider an element in $E(q)$ as a tuple of real vectors $(v_{1,1}, v_{2,1}, v_{2,2}, \dots, v_{5,1}, \dots, v_{5,16q})$ in V which satisfies certain orthogonality conditions. Note that the inner product on V_5 respects the decomposition of it as a direct sum of smaller V_i's, so all these orthogonality conditions can be expressed in terms of these real coordinate vectors. If we choose the inner product on V as a G-invariant inner product, then G will act on $E(q)$ and hence on X. Thus G acts smoothly on X.

Finally, we will show that the action is free when the fixity f of V is 4. Suppose that we have an element $(v_{1,1}, v_{2,1}, v_{2,2}, \dots, v_{5,1}, \dots, v_{5,16q})$ fixed by $1 \neq g \in G$. Then, the vector $(v_{5,1}, \dots, v_{5,16})$ will lie in V_5^g and will be orthogonal to subspaces $\langle v_i \rangle_{A_5}$ of V_i^g for every $v_i \in V_i$. Here $v_1 = v_{1,1}$, $v_2 = (v_{2,1}, v_{2,2})$, $v_3 = (v_{3,1}, \dots, v_{3,4})$, and $v_4 = (v_{4,1}, \dots, v_{4,8})$. So, $(v_{5,1}, \dots, v_{5,16})$ lies in a vector space of dimension $16f - 16 \cdot 4$ which is zero when $f=4$. So, G acts freely on X. □

We conclude the paper with the proof of Corollary 1.3.

Proof of Corollary 1.3. Let V be a complex representation of G with fixity $f=3$. Repeating the argument in Theorem 1.1, we obtain a smooth action of G on

$$X = S^{2n-1} \times S^{4n-5} \times S^{q(4n-8)-1}.$$

Note that if H is a subgroup of G that fixes a point on X, then it also fixes a point on $V_3(W)$ where $W = \mathbb{H} \otimes_{\mathbb{C}} V$. Since V has fixity 3, it acts freely on $V_4(W)$. Thus H acts freely on the fibre of $q_3 : V_4(W) \to V_3(W)$. The fibre of this map is homeomorphic to a sphere, so H has periodic cohomology by classical Smith theory. Now, from Theorem 3.2 of [2], we can conclude that G acts freely on a finite complex Y homotopy equivalent to $X \times S^N$ for some N.

In the real case the proof is similar. Let X be the G-space constructed in the proof of Theorem 1.2. Now suppose that H is subgroup of G fixing a point

$$(v_{1,1}, v_{2,1}, v_{2,2}, \dots, v_{5,1}, \dots, v_{5,16q}) \in X.$$

Then, we can find indices i_j such that the 5-tuple $(v_{1,1}, v_{2,i_2}, v_{3,i_3}, v_{4,i_4}, v_{5,i_5})$ is a point on the Stiefel manifold $V_5(V)$ and it is fixed by H under the usual G action on the Stiefel manifold. Since G acts freely on $V_6(V)$, we obtain that H acts freely on the fibre of $V_6(V) \to V_5(V)$. The rest follows as in the complex case. □

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