Set Covering and Serre's Theorem on the Cohomology Algebra of a p-Group

Ergün Yalçın

Last Revised on April 20, 2000

Abstract

We define a group theoretical invariant, denoted by $s(G)$, as a solution of a certain set covering problem, and show that it is closely related to $chl(G)$, the cohomology length of a p-group G. By studying $s(G)$, we improve the known upper bounds for the cohomology length of a p-group, and determine $chl(G)$ completely for extra-special 2-groups of real type.

1991 *Mathematics Subject Classification.* Primary: 20J06; Secondary: 20D15, 20D60, 51E20.

Key words and phrases. Cohomology length, extra-special p-group, set covering problem, ovoid.

1 Introduction

A classical theorem by Serre [13] states that if G is a p-group which is not elementary abelian then there exist non-zero elements $u_1, u_2, \ldots, u_m \in H^1(G, \mathbf{F}_p)$ such that

$$\begin{aligned} u_1 u_2 \cdots u_m &= 0 \quad \text{if} \quad p = 2, \\ \beta(u_1)\beta(u_2)\cdots\beta(u_m) &= 0 \quad \text{if} \quad p > 2. \end{aligned}$$

Since its first appearance many different proofs have been given for this theorem, and there has been some interest in finding the minimum number of one dimensional classes required for a vanishing product. For a p-group G, which is not elementary abelian, the smallest integer m that satisfies the conclusion of Serre's Theorem is usually referred as *the cohomology length of* G and denoted by $chl(G)$. Since we will be studying this invariant, throughout the paper we assume that the groups considered are p-groups that are **not elementary abelian** unless otherwise specified.

Although Serre's original proof does not provide an estimate for $chl(G)$, later proofs, including an independent proof by Serre himself, give upper bounds for $chl(G)$ in terms of $k = \dim_{\mathbf{F}_p} H^1(G, \mathbf{F}_p)$ and the prime p. The first known upper bound, $chl(G) \leq p^k - 1$, was given by O. Kroll [6]. This was improved by Serre [14] to $chl(G) \leq \frac{p^k-1}{p-1}$, and later by T. Okuyama and H. Sasaki [9] to $chl(G) \leq (p+1)p^{k-2}$. The most recent result states that $chl(G) \leq (p+1)p^{[\frac{k}{2}]-1}$, which is due to P.A. Minh [7].

In his proof of Serre's Theorem, P.A. Minh uses a detection theorem for extra-special p-groups. We will exploit this theorem further to relate the cohomology length to a group theoretical and combinatorial invariant defined as follows: Let G be a p-group and S be a subset of G. We say S is a *representing set* if it includes at least one non-central element from each maximal elementary abelian subgroup of G. If G is not p-central, i.e., if there exists an element of order p which is not central, then one can always find a representing set for G. For a p-group G which is not p-central we define $s(G)$ as the minimum cardinality of a representing set for G. To make our statements more general, we are using Quillen's definition of extra-special p-groups which only requires that the Frattini subgroup to be a cyclic group of order p (see Section 2 for properties of such groups.) We prove the following:

Proposition 1.1. *If G is an extra-special p-group which is not p-central, then $chl(G) \leq s(G)$. Moreover, if G has self-centralizing maximal elementary abelian subgroups, then $chl(G) = s(G)$.*

As a combinatorial problem computing $s(G)$ is a *set covering problem* (See [3]) where the adjacency matrix has a very special form due to the special structure of extra-special p-groups. Although there exist algorithms to solve set covering problems, and they provide upper bounds for minimal order covering set, these results are too general and do not provide sufficiently sharp bounds in our case. Using some counting arguments for extra-special p-groups with a small number of generators, we find an upper bound for $s(G)$, and conclude:

Theorem 1.2. *If G is a p-group and $k = \dim_{\mathbf{F}_p} H^1(G, \mathbf{F}_p)$, then $chl(G) \leq p+1$ if $k \leq 3$ and*

$$chl(G) \leq (p^2 + p - 1)p^{[\frac{k}{2}]-2} \quad \text{if} \quad k \geq 4.$$

In Section 5, we calculate the invariant $s(G)$ when G is an extra-special 2-group of real type, i.e., G is isomorphic to central product of D_8's. We conclude:

Theorem 1.3. *Let G_n be an extra-special 2-group isomorphic to an n-fold central product of D_8's. Then,*

$$chl(G_n) = \begin{cases} 2^{n-1} + 1 & \text{if} \quad n \leq 4, \\ 2^{n-1} + 2^{n-4} & \text{if} \quad n \geq 5. \end{cases}$$

For an extra-special 2-group G, the invariant $s(G)$ is equal to the minimum cardinality of a set of points that meets every maximal subspace of the associated quadric. Such sets are called *sets closest to ovoids* and were first introduced by K. Metsch [8]. By Metsch's calculations for elliptic quadrics, we determine $s(G)$ for the corresponding family of extra-special 2-groups. We also do similar calculations for parabolic quadrics to complete the calculations of $s(G)$ for extra-special 2-groups.

We conclude the paper with a discussion of open problems.

Throughout the paper we use the following notations: For a group G, let $Z(G)$, $\Phi(G)$ and $[G,G]$ denote the center, the Frattini subgroup and the commutator subgroup of G respectively. We write $\mathbf{C}_G(g)$ for the centralizer of an element $g \in G$, and $\mathbf{C}_S(g)$ for the set of elements in a set $S \subseteq G$ that commutes with g. As it was mentioned before, we use Quillen's definition of extra-special p-groups since this is the form in which we have most of the cohomological information. Notice that this is different from usually accepted group theoretical definition of extra-special p-group where one must have $\Phi(G) = [G,G] = Z(G)$. Finally, all the cohomology groups in the paper are in $\mathbf{Z}/p$ coefficients and they are simply denoted by $H^*(G)$.

Acknowledgments: I would like to thank Jonathan Pakianathan for many helpful conversations and especially for pointing out the references on ovoids. I am also greatful to Jon F. Carlson for his comments on the earlier version of this paper, and would like to acknowledge that some of the results in Section 3 were already known to him.

2 Properties of Extra-special p-Groups

Let G be an extra-special p-group. By this we mean that G fits into an extension

$$1 \to \mathbf{Z}/p \to G \to V \to 1$$

where V is a vector space over $\mathbf{Z}/p$. If $G \cong G_0 \times \mathbf{Z}/p$ for some $G_0 \subseteq G$, then $chl(G) = chl(G_0)$ and $s(G) = s(G_0)$. Hence, without loss of generality we can assume that G has no proper direct factors. It follows that if G is represented by the extension class $\alpha \in H^1(V)$, then there exists a basis such that α is in one of the following forms (See [7]):

–for $p = 2$ and $k = 2n$:	(a) $x_1y_1 + x_2y_2 + \cdots + x_ny_n$
	(b) $x_1^2 + y_1^2 + x_1y_1 + x_2y_2 + \cdots + x_ny_n$
–for $p = 2$ and $k = 2n+1$:	(c) $x_0^2 + x_1y_1 + x_2y_2 + \cdots + x_ny_n$
–for $p > 2$ and $k = 2n$:	(d) $x_1y_1 + x_2y_2 + \cdots + x_ny_n$
	(e) $\beta(x_1) + x_1y_1 + x_2y_2 + \cdots + x_ny_n$
–for $p > 2$ and $k = 2n+1$:	(f) $\beta(x_0) + x_1y_1 + x_2y_2 + \cdots + x_ny_n$

where $k = \dim_{\mathbf{Z}_p} V$. From this it is easy to see that every maximal elementary abelian subgroup of G is of rank n when α is in the form (b) and $n+1$ otherwise. Observe also that if G is of the form (a) or (d), then $\mathbf{C}_G(E) = E$ for every maximal elementary abelian subgroup $E \subseteq G$. It is well known that when G is an extra-special p-group of type (e) or (f), then $chl(G) \leq p$ (see page 107 in [1].) Therefore, we can ignore these cases and assume that all the extra-special p-groups are in one of the forms $(a) - (d)$.

Let $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be a set of generators of G satisfying $x_i(a_k) = \delta_{i,k}$ and $y_i(b_k) = \delta_{i,k}$ with δ being the Kronecker symbol (when k is odd, we take $\{a_0, a_1, \ldots, a_n, b_1, \ldots, b_n\}$ as a set of generators.) Let G_m denote the subgroup generated by $\{a_i, b_i \mid i \leq m\}$, then G_m is

isomorphic to the central product $G_{m-1} * \langle a_m, b_m \rangle$. In general, when G is an extra-special group with $k = 2n$ or $2n+1$, we write $G = G_n$ for convenience.

In the following sections we use some counting arguments involving the number of maximal elementary abelian subgroups of G_n. We do this calculation here (See also [12] or [5]):

Lemma 2.1. *Let $t(G_n)$ denote the number of maximal elementary abelian subgroups of G_n. Then for each of the cases listed above, we have*

$$(a)\ t(G_n) = \prod_{i=1}^{n}(2^{i-1}+1), \qquad (b)\ t(G_1) = 1,\quad t(G_n) = \prod_{i=2}^{n}(2^i+1) \quad \textit{for} \quad n \geq 2,$$
$$(c)\ t(G_n) = \prod_{i=1}^{n}(2^i+1), \qquad (d)\ t(G_n) = \prod_{i=1}^{n}(p^i+1).$$

Proof. It is easy to verify the value of $t(G_1)$ for each case, so we can assume $n \geq 2$. Consider the set of pairs (E, g) where E is a maximal elementary abelian subgroup of G_n and g is an element in E which is not central. It is easy to see that the order of this set is $t(G_n)\cdot(|E|-p)$. On the other hand, we can count this set starting from non-central elements of order p. Observe that if g is an element of order p that is not central, then the centralizer $\mathbf{C}_G(g)$ is isomorphic to $G_{n-1} \times \mathbf{Z}/p$ where $G_{n-1} \subseteq G_n$ is the subgroup generated by $\{a_i, b_i \mid i \leq n-1\}$. So, g is included in exactly $t(G_{n-1})$ maximal elementary abelian subgroups. If $\mu(G_n)$ is the number of elements in G_n of order less than equal to p, then the order of above set is equal to $(\mu(G_n) - p)\cdot t(G_{n-1})$. Setting the results of two different ways of counting equal, we find that $t(G_n)/t(G_{n-1}) = (\mu(G_n) - p)/(|E| - p)$.

In the case of (d), $\mu(G_n) = p^{2n+1}$ and $|E| = p^{n+1}$. Hence,

$$\frac{t(G_n)}{t(G_{n-1})} = \frac{p^{2n+1}-p}{p^{n+1}-p} = p^n + 1,$$

therefore $t(G_n) = (p^n+1)(p^{n-1}+1)\cdots(p+1)$. For $p = 2$, we need to calculate $\mu(G_n)$ since its value is not obvious. We do this by using a recursive relation:

Observe that G_n is covered by a_nG_{n-1}, eG_{n-1}, b_nG_{n-1} and $a_nb_nG_{n-1}$ which are the cosets of the subgroup G_{n-1}. Therefore we have

$$\mu(G_n) = 3\mu(G_{n-1}) + (|G_{n-1}| - \mu(G_{n-1}))$$

from which we obtain

$$\mu(G_n) = 2^{n-1}\mu(G_1) + 2^{n-2}(2^{n-1}-1)|G_1|$$

for all $n \geq 2$. Substituting the values of $\mu(G_1)$ and $|G_1|$ for each case, we find that $\mu(G_n) = 2^{2n} + 2^n$ in the case of (a), $\mu(G_n) = 2^{2n} - 2^n$ for $n \geq 2$ in the case of (b), and $\mu(G_n) = 2^{2n+1}$ in the case of (c). Using these we obtain the values of $t(G_n)$ as listed in the lemma. □

Lemma 2.2. *Let $E_r \subseteq G_n$ be an elementary abelian subgroup of rank $r < rk(G_n)$. Then, the number of maximal elementary abelian subgroups in G_n that include E_r is equal to $t(G_{n-r+1})$, where $G_{n-r+1} \subseteq G_n$ is the subgroup generated by $\{a_i, b_i \mid i \leq n-r+1\}$.*

Proof. If E is a maximal elementary abelian subgroup that includes E_r, then $E \subseteq \mathbf{C}_G(E_r) \cong (E_r/\Phi(G)) \times G_{n-r+1}$. Hence, maximal elementary abelian subgroups that include E_r are in one to one correspondence with maximal elementary abelian subgroups of G_{n-r+1}. □

3 Representing Sets

In this section we discuss the notion of representing sets and prove Proposition 1.1.

Definition 3.1. Let G be a p-group, and S be a subset of G. We say S is a representing set for G if $S \cap (E - Z(G)) \neq \emptyset$ for every maximal elementary abelian subgroup $E \subseteq G$.

If G is a p-group which is not p-central, i.e. it has an element of order p which is not central, one can always find a representing set for G. For a p-group G which is not p-central we define $s(G)$ as the minimum of cardinalities of representing sets for G. This restriction does not affect our results, since we will be working with extra-special p-groups and the only p-central p-group in this class is the group of unit quaternions Q_8 which is known to have cohomology length 3. We also would like to note that when p is odd, p central groups have rather small cohomology length:

Proposition 3.2. *If p is an odd prime and G is a p-central p-group, then $chl(G) \leq p$.*

Proof. We will be using two well known facts about the cohomology length of a p-group (for further information on Serre's theorem see [1], [2], or [4]). If L is a factor group of a p-group G, i.e. $L \cong G/N$ for some normal subgroup $N \subseteq G$, then $chl(G) \leq chl(L)$. This is a direct consequence of the fact that if π is the quotient map $G \to L = G/K$, then the induced map $\pi^* : H^1(L) \to H^1(G)$ is injective. Also recall that the cohomology length of an extra-special p-group of type (e) or type (f) is known to be less than p. Therefore, to prove the result we only need to show that a p-central p-group (p is odd) which is not elementary abelian always has a factor group isomorphic to an extra-special p-group of exponent p^2.

Let G be a p-central p-group where p is an odd prime. Recall that a p-central group is a group where every element of order p is central. So, it has a unique maximal elementary abelian subgroup. Moreover, when p is odd, taking the quotient with the maximal elementary abelian subgroup gives again a p-central p-group. So, if E is the maximal elementary abelian subgroup of G and if the quotient group G/E is not elementary abelian, then G will have the desired factor group by induction. Thus we can assume G/E is elementary abelian. Furthermore, we

can assume that the Frattini subgroup $\Phi(G)$ is equal to E, because otherwise $G \cong G_0 \times \mathbf{Z}/p$ for some p-central subgroup $G_0 \subseteq G$. Again the result follows by induction.

Now, since the group G itself is not elementary abelian, the subgroup of $G^p \subseteq G$ generated by the pth powers is nontrivial and it is included in $\Phi(G) = E$. Let M be a maximal subgroup of E that does not include G^p. Then, the factor group G/M is an extra-special p-group (possibly abelian) of exponent p^2. So, the proof is complete. □

Remark 3.3. The conclusion of Proposition 3.2 is not true for 2-central groups. The simplest example is the case $G = Q_8$ where the group is 2-central but the cohomology length is equal to 3.

We now quote an important detection theorem for the cohomology of extra-special p-groups. Let $\mathcal{EA}_G$ be the set of all maximal elementary abelian subgroups of G, and let H (G) denote the subalgebra of $H^*(G)$ generated by one dimensional classes when $p = 2$ and Bocksteins of one dimensional classes when $p > 2$.

Theorem 3.4 (Quillen [12], Tezuka-Yagita [15]). *The map*

$$\prod_{E \in \mathcal{EA}_G} \mathrm{res}^G_E \ : \ \mathcal{H}(G) \to \prod_{E \in \mathcal{EA}_G} H^*(\mathbf{C}_G(E))$$

is an injection.

Now, we introduce some notation: Given an element $s \in G - Z(G)$, we can define an homomorphism $\Psi_s : G \to Z/p$ by letting $\Psi(g) = [s, g]$ for all $g \in G$. Since $H^1(G) \cong \mathrm{Hom}(G, Z/p)$, the homomorphism Ψ uniquely defines a one dimensional cohomology class which we will denote by u_s. Note that u_s satisfies the property $\mathbf{C}_G(s) = \ker u_s$. For simplicity, let v_s denote $\beta(u_s)$ when p is odd and u_s when $p = 2$.

Lemma 3.5. *If S is a representing set for an extra-special p-group G which is not p-central, then $\sigma = \prod_{s \in S} v_s = 0$ in $H^*(G)$. Hence, $chl(G) \leq s(G)$.*

Proof. Let S be a representing set for G. For every $E \in \mathcal{EA}_G$, there exists a non-central element $s \in S$ such that $s \in E$, i.e., $\mathbf{C}_G(E) \subseteq \mathbf{C}_G(s) = \ker u_s$. Therefore,

$$\mathrm{res}^G_{\mathbf{C}_G(E)} v_s = \mathrm{res}^{\mathbf{C}_G(s)}_{\mathbf{C}_G(E)} \mathrm{res}^G_{\mathbf{C}_G(s)} v_s = 0.$$

Thus $\mathrm{res}^G_{\mathbf{C}_G(E)} \sigma = 0$ for every $E \in \mathcal{EA}_G$. Applying Theorem 3.4, we obtain $\sigma = 0$. □

Remark 3.6. In fact, Lemma 3.5 holds for a larger class of groups. Let G be a p-group which is not p-central. Suppose that G satisfies the above detection theorem, for example, G is such that $H^*(G)$ is Cohen-Macaulay. If S is a representing set for G, then for every $s \in S$ we can find a one dimensional class u_s such that $\mathbf{C}_G(s) \subseteq \ker u_s$. As in the proof of Lemma 3.5, we can conclude $\prod_{s \in S} v_s = 0$. Hence, $chl(G) \leq s(G)$.

If G is an extra-special p-group of type (a) or (d), we will prove conversely that given a vanishing product of m one dimensional classes (respectively vanishing product of Bocksteins of m one dimensional classes when p is odd), one can find a representing set of order less than m. For this we first observe that if G is an extra-special p-group of type (a) or (d), and if M is an index p subgroup of G, then $|Z(M)| > |Z(G)|$, i.e. there is an element $g \in G - Z(G)$ such that $M = \mathbf{C}_G(g)$. So, for every $u \in H^1(G, \mathbf{F}_p)$ there is a non-central element $g \in G$ such that $C_G(g) = \ker u$. As above, let v_i denote $\beta(u_i)$ when p is odd and u_i when $p = 2$.

Lemma 3.7. *Let G be an extra-special p-group of the form (a) or (d). Let $\{u_1, u_2, \ldots, u_m\}$ be a set of nonzero classes in $H^1(G)$, and $S = \{g_1, ..., g_m\}$ be a set of corresponding noncentral group elements which satisfy $\mathbf{C}_G(g_i) = \ker u_i$ for $i = 1, \ldots, m$. If $\sigma = \prod_{i=1}^{m} v_i = 0$, then S is a representing set for G. Hence, $s(G) \leq chl(G)$.*

Proof. Take an element $E \in \mathcal{EA}_G$. Note that the images of v_i under the restriction map $\mathrm{res}^G_E : H^*(G) \to H^*(E)$ lie in the polynomial subalgebra of $H^*(E)$. Thus, $\mathrm{res}^G_E\ \sigma = \prod_{i=1}^{m} \mathrm{res}^G_E\ v_i = 0$ implies that $\mathrm{res}^G_E\ v_i = 0$ for some i. It follows that $E \subseteq \ker u_i = \mathbf{C}_G(g_i)$, and hence, $g_i \in \mathbf{C}_G(E) = E$. Since this is true for all $E \in \mathcal{EA}_G$, we conclude that S is a representing set. □

Proof of Proposition 1.1. Follows from Lemma 3.5 and Lemma 3.7. □

Remark 3.8. Another obvious choice for a definition of a representing set is the one that requires the set to include at least one non-central element from each maximal abelian subgroup. For any non-abelian p-group G, we can always find a set that represents maximal abelian subgroups, so we can define $sA(G)$ as the order of smallest such set in G. Using Minh's arguments in [7], one can show that $chl(G_n) \leq sA(G_n)$. However, in the case of a extra-special 2-group of type (a), this gives a weaker upper bound. For example in the case of $G = D_8$, $chl(G) = s(G) = 2$ but $sA(G) = 3$. In general the following relations are known for each type:

(a) $chl(G) = s(G) < sA(G)$,

(b) $chl(G) \leq sA(G) \leq s(G)$,

(c) $chl(G) \leq s(G) = sA(G)$.

We conclude this section with some examples of representing sets from which we obtain some of the previously known upper bounds for $chl(G)$. In all the examples below, G is an extra-special p-group which is not p-central, and $k = \dim_{\mathbf{F}_p} H^1(G)$.

Example 3.9. For every nontrivial cyclic subgroup $C \subseteq G/\Phi(G)$, choose an element $g \in G$ such that the image of g under quotient map generates C. Let S be the set of all chosen

elements. It is easy to see that every maximal elementary abelian subgroup includes a subgroup of the form $\langle \Phi(G), g\rangle$ for some noncentral element $g \in G$. Hence S is a representing set. Since the set S is in one to one correspondence with the projective space of the vector space $V = G/\Phi(G)$, we obtain Serre's upper bound: $chl(G) \leq |S| \leq |P(V)| = \frac{p^k-1}{p-1}$.

Example 3.10. Let $H \subseteq G$ be an index p^2 subgroup of G such that $\mathrm{rk}(H) = \mathrm{rk}(G) - 1$. Let S be as in the previous example, and $S' = S \cap (G - H)$. Since every maximal elementary abelian subgroup includes at least one element from $G - H$, it includes a subgroup of the form $\langle \Phi(G), g\rangle$ for some $g \in G - H$. Hence S' is a representing set. This gives Okuyama and Sasaki's bound: $chl(G) \leq |S'| = (p^{k+1} - p^{k-1})/(p^2 - p) = (p+1)p^{k-2}$.

Example 3.11. Minh's upper bound was obtained using maximal abelian subgroups. So, we will construct a set that represents maximal abelian subgroups. Let A be a maximal abelian subgroup in G_n and A' be an index p subgroup of A that includes the center. We form S by picking one non-central element from each abelian subgroup of the form $\langle g, Z(G_n)\rangle$ where $Z(G)$ is the center of G_n and g is an element in $\mathbf{C}_G(A') - A'$. It is easy to see that every maximal abelian subgroup has a nonempty intersection with $\mathbf{C}_G(A') - A'$, so S is a set that represents maximal elementary abelian subgroups. This gives Minh's upper bound:

$$chl(G) \leq |S| \leq (p^2-1)|A'|/(p-1)|Z(G)| = (p+1)p^{\lceil \frac{k}{2} \rceil - 1}.$$

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2 stated in the introduction. We continue to use the notation introduced in Section 2. In particular, G_n denotes an extra-special group with $k = 2n$ or $k = 2n+1$ with a basis $\{a_i, b_i \mid i \leq n\}$ chosen as in Section 2. For every $m \leq n$, $G_m \subseteq G_n$ denotes the subgroup generated by $\{a_i, b_i \mid i \leq m\}$.

Lemma 4.1. $s(G_2) \leq p^2 + p - 1$.

Proof. For each case listed in Section 2, we show that there is a representing set of order less than $p^2 + p - 1$ by using the calculations done for $t(G_2)$ in Lemma 2.1.

(a) Let $S = \{a_1, b_1, a_1b_1a_2b_2\}$. Every element in S is included in $t(G_1) = 2$ maximal elementary abelian subgroups. Since elements in S are pairwise non-commuting, there are no maximal elementary abelian subgroups that include two elements in S. So, elements in S represent 6 distinct maximal elementary abelian subgroups. Since $t(G_2) = 6$, we conclude that S is a representing set.

(b) In this case there are only 5 maximal elementary abelian subgroups which are $\langle a_2, c\rangle$, $\langle b_2, c\rangle$, $\langle a_1a_2b_2, c\rangle$, $\langle a_1b_1a_2b_2, c\rangle$, and $\langle b_1a_2b_2, c\rangle$. Hence, the set $S = \{a_2, b_2, a_1a_2b_2, a_1b_1a_2b_2, b_1a_2b_2\}$ is a representing set.

(*c*) Let $S = \{a_1, b_1, a_0a_1b_1a_2, a_0a_1b_1b_2, a_1b_1a_2b_2\}$. Each element $s \in S$ represents $t(G_1) = 3$ maximal elementary abelian subgroups. The elements in S are chosen in such a way that they are pairwise non-commuting. Therefore, neither of the two appears in the same maximal elementary abelian subgroup, thus S represents a total of 15 distinct maximal elementary abelian subgroups. But, this is all there is since $t(G_2) = (2^2 + 1)(2 + 1) = 15$. So, S is a representing set.

(*d*) For each nontrivial element in $G_1/\Phi(G_1)$ we choose a representative in G_1 and form the set S_1. Now let $S_2 = \{a_2^i b_2 \mid i = 0, 1, \ldots, (p-1)\}$, and $S = a_2S_1 \cup S_2$. It is clear that $|S| = |S_1| + |S_2| = p^2 + p - 1$. Let E be a maximal elementary abelian subgroup of G_2. If $E \cap S_2 = \emptyset$, then there is an element $g \in E$ that commutes with none of the elements in S_2. Since the centralizers of elements in $\{a_2\} \cup S_2$ cover G, the element g lies in $\mathbf{C}_G(a_2) - G_1 = a_2G_1$. Therefore $\langle g, \Phi(G)\rangle \cap S_1 \neq \emptyset$. Hence S is a representing set. (It is possible to reach the same conclusion through a counting argument. S_1 divides into $p + 1$ subsets where each subset represents $[1 + p(p - 1)]$ maximal elementary abelian subgroups, and each element in S_2 represents $p + 1$ maximal elementary abelian subgroups. So, we get a total of $(p+1)[1+p(p-1)]+p(p+1) = (p^2+1)(p+1)$ distinct maximal elementary abelian subgroups represented which are all there is in G_2.) □

Lemma 4.2. *$s(G_n) \leq p \cdot s(G_{n-1})$ for $n \geq 3$.*

Proof. Let S' be a representing set for G_{n-1} with $|S'| = s(G_{n-1})$. Set $S = \{a_n^i s | s \in S', i = 0, 1, \ldots, (p-1)\}$. Let E be a maximal elementary abelian subgroup of G_n. If $E \cap G_{n-1}$ is a maximal elementary abelian subgroup of G_{n-1}, then E is represented by S' and hence by S. Otherwise $E = \langle E', a_nx, b_ny\rangle$ for some $x, y \in G_{n-1}$ where $E' \subseteq G_{n-1}$ is an elementary abelian subgroup of rank $n - 1$. Observe that $\langle E', x\rangle$ is a maximal elementary abelian subgroup of G_{n-1}, so there is an $s \in S'$ which represents $\langle E', x\rangle$. Then, $a_n^i s \in E$ for some i and hence $S \cap (E - Z(G)) \neq \emptyset$. Thus, S is a representing set for G_n and $s(G_n) \leq |S| = p \cdot s(G_{n-1})$. □

Now, we prove our main theorem.

Proof of Theorem 1.2. By Lemma 4.1 and Lemma 4.2, we obtain $s(G_m) \leq (p^2 + p - 1)p^{m-2}$ for $m \geq 2$. By Proposition 1.1, it follows that $chl(G_m) \leq (p^2 + p - 1)p^{m-2}$ when $m \geq 2$. By earlier calculations we also know that $chl(G_1) \leq p + 1$. We can extend this result to arbitrary p-groups as follows: Let G be a p-group with $k = H^1(G) = 2n$ or $2n + 1$. Then, G has a factor group isomorphic to some G_m with $m \leq n$. For any factor group L of G, we have $chl(G) \leq chl(L)$, so the theorem follows. □

5 Calculations for extra-special 2-groups of type (a)

In this section we compute the invariant $s(G)$ for extra-special 2-groups of type (a) and obtain Theorem 1.3 as a corollary. Let G_n denote an extra-special 2-group of type (a) with $\dim_{\mathbf{F}_2} H^1(G) = 2n$. The number of maximal elementary abelian subgroups in G_n is calculated in Section 2 as

$$t(G_n) = \prod_{i=1}^{n} (2^{i-1} + 1)$$

and by Lemma 2.2, every noncentral element of order 2 is included in exactly $t(G_{n-1})$ maximal elementary abelian subgroups. So, if S is a representing set, then it must have at least $t(G_n)/t(G_{n-1}) = 2^{n-1} + 1$ elements. Thus,

Lemma 5.1. $s(G_n) \geq 2^{n-1} + 1$.

Observe that a set of noncentral elements of order 2 with $|S| = 2^{n-1} + 1$ is a representing set if and only if every maximal elementary abelian subgroup includes only one element from S. The last statement is true if and only if the elements in S are pairwise non-commuting (such a set is called a non-commuting set.) We conclude:

Lemma 5.2. *Let S be a set of elements of order 2 in G_n such that $|S| = 2^{n-1} + 1$. Then, S is a representing set if and only if S is a non-commuting set.*

Now, the following is a easy consequence of these Lemmas:

Lemma 5.3. $s(G_n) = 2^{n-1} + 1$ *for* $n \leq 4$.

Proof. Let $\{a_1, b_1, \ldots, a_n, b_n\}$ be a generating set as described in Section 2. By previous lemmas, it is enough to find a subset $S_n \subseteq G_n$ of order $2^{n-1} + 1$ such that S_n is a set of pairwise non-commuting elements of order 2. Let

$$S_1 = \{a_1, b_1\}, \quad S_2 = \{a_1, b_1, a_1b_1a_2b_2\}, \quad S_3 = \{a_1, b_1, a_1b_1a_2a_3b_3, a_1b_1b_2a_3b_3, a_1b_1a_2b_2\},$$

$$S_4 = \{a_1, b_1, a_1b_1a_2a_4b_4, a_1b_1b_2a_4b_4, a_1b_1a_2b_2a_3, a_1b_1a_2b_2b_3, a_1b_1a_3b_3a_4, a_1b_1a_3b_3b_4, a_1b_1a_2b_2a_3b_3a_4b_4\}.$$

It is straight forward to check that for all i, all the elements in S_i are of order 2 and pairwise non-commuting. □

Unfortunately, the equality $s(G_n) = 2^{n-1} + 1$ does not hold in general. This is because in general the orders of non-commuting sets in G_n are much smaller than $2^{n-1} + 1$. Let $nc(G)$ denote the order of largest non-commuting set in G. The following calculation was originally done by Marty Isaacs (See [11]):

Lemma 5.4. *If G is an extra-special 2-group of order 2^{2n+1}, then $nc(G) = 2n + 1$.*

Proof. It is easy to see that there is a non-commuting set of order $2n + 1$ defined inductively by letting $X_n = \{a_n\} \cup \{b_n\} \cup a_n b_n X_{n-1}$ and $X_1 = \{a_1, b_1, a_1 b_1\}$. Now we will show that one can not find a non-commuting set larger than this. Let X be a set of pairwise non-commuting elements in G_n. Take two elements x and y in X. Observe that the rest of the elements in X should lie in the coset $xyC_G(\langle x, y\rangle)$. Since $C_G(\langle x, y\rangle)$ is an extra-special group of order 2^{2n-1}, by induction $|X| - 2 \leq 2n - 1$ and hence $|X| \leq 2n + 1$. We conclude that $nc(G) = 2n + 1$. □

Lemma 5.5. $2^{n-1} + 1 < s(G_n) \leq 2^{n-1} + 2^{n-4}$ *for every* $n \geq 5$.

Proof. When $n \geq 5$, we have $2n + 1 < 2^{n-1} + 1$. So, by the above lemmas, $2^{n-1} + 1 < s(G_n)$. The second inequality follows from Lemma 5.3 and Lemma 4.2. □

As stated in Theorem 1.3, we claim that $s(G_n) = 2^{n-1} + 2^{n-4}$ when $n \geq 5$. For the proof we need the following:

Lemma 5.6. *Let S be a representing set for G_n with minimum order. Then, for every $s \in S$, there exist at least 2^{n-1} elements in S that do not commute with s, i.e.,*

$$|S - \mathbf{C}_S(s)| \geq 2^{n-1} \quad \textit{for every} \quad s \in S.$$

Proof. Take an element $s \in S$. Observe that there exists a maximal elementary abelian subgroup $E \subseteq G_n$ such that $E \cap S = \{s\}$. Because, otherwise $S - \{s\}$ is a representing set with smaller order, contradicting the assumption that S is a representing set with minimum order. Consider the set $\{E'_1, \ldots, E'_m\}$ of index 2 subgroups of E that include the center of G_n, but do not include the element s. An easy calculation shows that there are exactly $(2^n - 1) - (2^{n-1} - 1) = 2^{n-1}$ such subgroups, so $m = 2^{n-1}$. Each E'_i is included in 2 maximal elementary abelian subgroups one of which is E. For each E'_i we call the other maximal elementary abelian subgroup E_i. Since $E_i \cap S \neq \emptyset$, for each i there is an element $s_i \in S$ such that $s_i \in E_i - E'_i$. If $s_i = s_j$ for some $i \neq j$, then s_i commutes with both E'_i and E'_j, and hence commutes with $E_i E_j = E$. Then, $s_i \in E \cap E_i = E'_i$ which is in contradiction with $E'_i \cap S = \emptyset$. So, the s_i's are distinct. Finally, if for some i the element s_i commutes with s, then s_i commutes with $\langle E_i, s\rangle = E$, and hence $s_i \in E \cap E_i = E'_i$. This again leads to a contadiction, so for each i, s_i does not commute with $s \in S$. Hence the proof of the lemma is complete. □

For the proof of the claim $s(G_n) = 2^{n-1} + 2^{n-4}$ for $n \geq 5$, it remains to show that if S is a representing set, then there exists an element $s \in S$ such that s commutes with at least 2^{n-4} elements in S, i.e. $|\mathbf{C}_S(s)| \geq 2^{n-4}$. For this we need a stronger version of the above lemma:

Lemma 5.7. *Let S be a representing set for G_n with minimum order and let g be a non-central element G which is not included in S. Suppose further that there exists a maximal elementary abelian subgroup $E \subseteq G_n$ such that $g \in E$ and $S \cap E = \{s\}$. Then, there exist at least 2^{n-2} elements in S that do not commute with g.*

Proof. The proof is similar to the proof of Lemma 5.6. In this case we take the set $\{E'_1, \ldots, E'_m\}$ as the set of index 2 subgroups of E that include the center and the element gs, but do not include s. Counting such subgroups we find that $m = 2^{n-2}$. Observe that the elements $s_1, \ldots, s_m$, chosen as in the proof of Lemma 5.6, will commute with gs, but they will not commute with s, hence will not commute with g. □

Lemma 5.8. *If S is a representing set for G_n, then there is an element $s \in S$ such that*

$$|\mathbf{C}_S(s)| \geq 2^{n-4}.$$

Proof. The lemma is true for $n \leq 4$, so assume $n \geq 5$. Since $2n + 1 < 2^{n-1} + 1 < |S|$ for $n \geq 5$, there exist $a, b \in S$ such that $[a, b] = 1$. Without loss of generality we can assume S is a representing set with minimum order. Then by Lemma 5.6, we have $|S - \mathbf{C}_S(a)| \geq 2^{n-1}$. If $|\mathbf{C}_S(b)| \geq 2^{n-4}$, then we are done. So, assume $|\mathbf{C}_S(b)| < 2^{n-4}$. Since the commutator subgroup of G_n is isomorphic to $\mathbf{Z}/2$, every element in G commutes with a, b or ab. Therefore $\mathbf{C}_S(a) \cup \mathbf{C}_S(b) \cup \mathbf{C}_S(ab) = S$. Hence $|\mathbf{C}_S(ab)| > 2^{n-1} - 2^{n-4} > 2^{n-4}$. If $ab \in S$ then we are done. So, assume $ab \notin S$.

If there exists a maximal elementary abelian subgroup $E \subseteq G_n$ such that $ab \in E$ and $|E \cap S|$=1, then by Lemma 5.7, we have $|S - \mathbf{C}_S(ab)| \geq 2^{n-2}$, which implies either $|\mathbf{C}_S(a)| \geq 2^{n-3}$ or $|\mathbf{C}_S(b)| \geq 2^{n-3}$, hence the lemma will be true. So, assume contrary that for every maximal elementary abelian subgroup $E \subseteq G$ that includes the element ab, we have $|S \cap E| \geq 2$. Now we consider the following cases:

Case 1: Suppose that there exists an element $s \in S$ such that $abs \notin S$. Let E_3 be the subgroup generated by the elements ab and s, and the central element $c \in G_n$. Let $\mathcal{E}_3$ denote the set of maximal elementary abelian subgroups in G_n that include E_3. By minimality of S, we have $S \cap cS = \emptyset$, and by the above assumption $abs \notin S$. So, $E_3 \cap S = \{s\}$. By Lemma 2.2, E_3 is included in $t(G_{n-2})$ maximal elementary abelian subgroups, i.e., $|\mathcal{E}_3| = t(G_{n-2})$. Recall that for every maximal elementary abelian subgroup E, we have $|E \cap S| \geq 2$. So, for every $E \in \mathcal{E}_3$, we have $E \cap (S - \{s\}) \neq \emptyset$. Note that an element $s' \in S - \{s\}$ is included in $t(G_{n-3})$ maximal elementary abelian subgroups in $\mathcal{E}_3$. So, there are at least $t(G_{n-2})/t(G_{n-3}) = 2^{n-3}+1$ elements in $S - \{s\}$ which are included in a maximal elementary abelian subgroup $E \in \mathcal{E}_3$. Since each of these elements will commute with s, we conclude that $|\mathbf{C}_S(s)| \geq 2^{n-3}+2 \geq 2^{n-4}$.

Case 2: Suppose that for every $s \in \mathbf{C}_S(ab)$, $abs \in S$, and there exists an element $x \in S$ such that $[x, ab] \neq 1$. Then, x commutes with either s or abs for every $s \in S$. Since $\mathbf{C}_S(ab)$

can be written as disjoint union of X and abX for some $X \subseteq S$, we obtain

$$|\mathbf{C}_S(x)| \geq |\mathbf{C}_S(ab)|/2 > 2^{n-2} - 2^{n-5} > 2^{n-4}.$$

Case 3: Finally, we assume that for every $s \in \mathbf{C}_S(ab)$, $abs \in S$, and $S \subseteq \mathbf{C}_S(ab)$. Take an element $x \in G_n$ of order 2 such that $[ab, x] \neq 1$. Let $H \subseteq G_n$ be the centralizer of $\langle ab, x\rangle$. Subgroup H is isomorphic to G_{n-1} and $\mathbf{C}_G(ab)$ is a disjoint union of H and abH. So, $S \subseteq \mathbf{C}_G(ab)$ can be written as a disjoint union $(S \cap H) \sqcup (S \cap abH)$. Since $S = abS$, we have $S \cap H = abS \cap H$, and hence $ab(S \cap H) = S \cap abH$. So, $S = S' \sqcup abS'$ where $S' = S \cap H \subseteq H$.

Now, we claim that $S' \subset H$ is a representing set for H. Let E' be a maximal elementary abelian subgroup of H. Then, $E = \langle E', ab\rangle$ is a maximal elementary abelian subgroup for G_n, and hence $E \cap S \neq \emptyset$. Since $E = E' \sqcup abE'$, we have

$$E \cap S = (E' \sqcup abE') \cap (S' \sqcup abS') = (E' \cap S') \sqcup ab(E' \cap S').$$

Thus $E' \cap S' \neq \emptyset$. Therefore, S' is a representing set for H. By induction there is an element $s' \in S'$ such that $|\mathbf{C}_{S'}(s')| \geq 2^{n-5}$. Since $|\mathbf{C}_S(s')| = 2\,|\mathbf{C}_{S'}(s')|$, we obtain that $|\mathbf{C}_S(s')| \geq 2^{n-4}$ for $s' \in S$.

Since we have considered all the possible cases, the proof of the lemma is complete. □

Proof of Theorem 1.3. By Lemma 1.1, we have $s(G_n) = chl(G_n)$, so it is enough to prove the theorem for $s(G_n)$. By Lemma 5.3, we know $s(G_n) = 2^{n-1} + 1$ for all $n \leq 4$, and by Lemma 5.5, we have $2^{n-1} + 1 < s(G_n) \leq 2^{n-1} + 2^{n-4}$ for all $n \geq 5$. Finally, Lemma 5.6 and Lemma 5.8 imply that $s(G_n) \geq 2^{n-1} + 2^{n-4}$. So, the proof is complete. □

6 Calculations for other types of extra-special 2-groups

The arguments used in the previous section can easily be extended to other types of extra-special 2-groups to calculate $s(G)$ for these groups. In fact $s(G)$ corresponds to an invariant in combinatorial algebra and the case of type (b) has already been calculated by Klaus Metsch [8]. We explain here briefly the relation between representing sets and the sets which are called *sets closest to ovoids*.

Let $PG(n, q)$ denote the projective space of $(n + 1)$-dimensional vector space over $\mathbf{F}_q$, the finite field of q-elements. A *non-singular quadric* in $PG(n, q)$ is the variety $V(F)$ of a non-singular quadratic form

$$F = \sum_{i=0}^{n} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j.$$

Projective linear group $PGL(n + 1, q)$ acts on non-singular quadrics and under this action there is one orbit when n is even, and there are two orbits when n is odd. The following are the orbit representatives for these orbits:

–for $n = 2m+1$, $Q^+(2m+1,q) = \mathbb{V}(x_0x_1 + x_2x_3 + \cdots + x_{2m}x_{2m+1})$, hyperbolic;
$Q^-(2m+1,q) = \mathbb{V}(f(x_0,x_1) + x_2x_3 + \cdots + x_{2m}x_{2m+1})$, elliptic;
–for $n = 2m$, $Q(2m,q) = \mathbb{V}(x_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m})$, parabolic;

where f is irreducible over $\mathbf{F}_q$.

An *ovoid* of a quadric Q is defined as the set of points of the quadric which intersect every maximal subspace of the quadric in exactly one point. The existence of ovoids in a given quadric has been studied extensively by many combinatorial algebraists (See [5] for a survey of known results.) In conjuction with this study, Klaus Metsch [8] introduced the concept of *a set closest to ovoids* which are defined as a set of points on the quadric that intersects every maximal subspace in the quadric at at least one point. He also calculated the minimal cardinality of such sets in the elliptic quadric $Q^-(2m+1,q)$ as $q^{m+1} + q^{m-1}$.

Lets define $s(Q)$ as the minimal cardinality of a set of points in Q such that it includes at least one point from every maximal subspace of Q. Observe that given a extra-special 2-group G of order 2^{n+1} which does not split as $G \cong G_0 \times Z/2$, there is a corresponding non-singular quadric in $PG(n,2)$, and representing sets in G correspond to the sets closest to ovoids. Hence, $s(G_m) = s(Q^+(2m-1,2))$, $s(G_m) = s(Q^-(2m-1,2))$, and $s(G_m) = s(Q(2m,2))$ respectively in the case of (a), (b) and (c). So, the calculations in the previous section give the following:

Proposition 6.1. *The quadric $Q = Q^+(2m+1,2)$ has an ovoid for $m \leq 3$, i.e., $s(Q) = 2^m+1$ for $m \leq 3$. For $m \geq 3$, we have $s(Q^+(2m+1,2)) = 2^m + 2^{m-3}$.*

On the other hand from K. Metsch's calculation we obtain [8]:

Proposition 6.2. *If G_n is an extra-special 2-group of type (b) such that $|G_n| = 2^{2n+1}$, then $s(G_n) = 2^n + 2^{n-2}$ for $n \geq 2$.*

To complete the calculations for extra-special 2-groups (or quadrics in $PG(n,2)$), we include the following calculation:

Proposition 6.3. *If G_n is an extra-special 2-group of type (c) such that $|G_n| = 2^{2n+2}$, then*

$$s(G_n) = \begin{cases} 2^n + 1 & \text{if} \quad n \leq 2, \\ 2^n + 2^{n-2} & \text{if} \quad n \geq 3. \end{cases}$$

As a consequence we obtain:

Corollary 6.4. *The quadric $Q = Q(2m,2)$ has an ovoid for $m \leq 2$, i.e., $s(Q) = 2^m + 1$ for $m \leq 2$. For $m \geq 3$, we have $s(Q) = 2^m + 2^{m-2}$.*

Proof of Proposition 6.3. Let G_n be as in the proposition, and let $\{a_0, a_1, b_1, \ldots, a_n, b_n\}$ be a basis as in Section 2. It is clear that the sets

$$S_1 = \{a_1, b_1, a_0a_1b_1\},$$

$$S_2 = \{a_1, b_1, a_0a_1b_1a_2, a_0a_1b_1b_2, a_1b_1a_2b_2\}$$

are pairwise non-commuting sets formed by elements of order 2 in G_1 and G_2 respectively. Since $t(G_n)/t(G_{n-1}) = 2^n + 1$, the set S_i is a representing set for G_i for $i = 1, 2$. By Lemma 5.4, there are no pairwise non-commuting sets of order $2^n + 1$ when $n \geq 3$, so $s(G_n) > 2^n + 1$ for $n \geq 3$. By Lemma 4.2, we also know $s(G_n) \leq 2^n + 2^{n-2}$. Now, let S be a representing set of minimal order. We will show that $|S| \geq 2^n + 2^{n-2}$. This will complete the proof of the proposition.

The argument in Lemma 5.6 can be repeated easily for this case. Since index 2 subgroups of a maximal elementary abelian groups are included in 3 maximal elementary abelian subgroups, we find $|S - \mathbf{C}_S(s)| \geq 2^n$ for every $s \in S$. Now, we will show inductively that there is an $s \in S$ such that $|\mathbf{C}_S(s)| \geq 2^{n-2}$. Take $a, b \in S$ such that $[a, b] \neq 1$. We assume $|\mathbf{C}_S(a)| < 2^{n-2}$, because otherwise we are done. This gives $|\mathbf{C}_S(a_0ab)| > 2^n - 2^{n-2} = 2^{n-1} + 2^{n-2}$. So, we can also assume $a_0ab \notin S$.

If there exists a maximal elementary abelian subgroup $E \subseteq G_n$ such that $a_0ab \in E$ and $|E \cap S|$=1, then by an argument similar to the one in Lemma 5.8, we have $|S - \mathbf{C}_S(a_0ab)| \geq 2^{n-1}$, which implies $|\mathbf{C}_S(b)| \geq 2^{n-2}$. So, assume contrary that for every maximal elementary abelian subgroup $E \subseteq G$ that includes the element a_0ab, we have $|S \cap E| \geq 2$. Now we consider the following cases:

Case 1: Suppose that there exists an element $s \in S$ such that $a_0abs \notin S$. Let E_3 be the subgroup generated by the elements a_0ab and s, and the central element $c \in G_n$. Let $\mathcal{E}_3$ denote the set of maximal elementary abelian subgroups in G_n that include E_3. By Lemma 2.2, $|\mathcal{E}_3| = t(G_{n-2})$. Since $E_3 \cap S = \{s\}$, and $|E \cap S| \geq 2$ for every $E \in \mathcal{E}_3$, we have $E \cap (S - \{s\}) \neq \emptyset$. Note that an element $s' \in S - \{s\}$ is included in $t(G_{n-3})$ maximal elementary abelian subgroups in $\mathcal{E}_3$. So, there are at least $t(G_{n-2})/t(G_{n-3}) = 2^{n-2} + 1$ elements in $S - \{s\}$ which are included in a maximal elementary abelian subgroup $E \in \mathcal{E}_3$. Since each of these elements will commute with s, we conclude that $|\mathbf{C}_S(s)| \geq 2^{n-2} + 2 \geq 2^{n-2}$.

Case 2: Suppose that for every $s \in \mathbf{C}_S(a_0ab)$, $abs \in S$, and there exists an element $x \in S$ such that $[x, a_0ab] \neq 1$. Then, x commutes with either s or a_0abs for every $s \in S$. Since $\mathbf{C}_S(a_0ab)$ can be written as disjoint union of X and a_0abX for some $X \subseteq S$, we obtain

$$|\mathbf{C}_S(x)| \geq |\mathbf{C}_S(a_0ab)|/2 > (2^{n-1} + 2^{n-2})/2 > 2^{n-2}.$$

Case 3: Finally, we assume that for every $s \in \mathbf{C}_S(a_0ab)$ $a_0abs \in S$, and $S \subseteq \mathbf{C}_S(a_0ab)$. Repeating the argument in the proof of Lemma 5.8, we obtain $S = S' \sqcup a_0abS'$ for some $S' \subseteq S$ such that S' is a representing set for a subgroup isomorphic to G_{n-1}. So, by induction we obtain $|\mathbf{C}_S(s)| \geq 2^{n-2}$ for some $s \in S$.

The proof of the proposition is complete. □

7 Open Problems

We now would like to list some problems which we find interesting:

Problem 7.1. Show that the equality $chl(G) = s(G)$ holds for all extra-special 2-groups.

Motivation for this problem is clear, since it will complete the calculation of cohomology lengths of extra-special 2-groups, and it will give the best possible upper bounds for cohomology lengths of 2-groups obtained by using extra-special factor groups.

For odd primes, we do know that $chl(G) \leq p$ when G is an extra-special p-group of type (e) and (f), and we proved in this paper that if G is of type (d), then $chl(G) = s(G)$. So, for odd primes what remains is the following calculation:

Problem 7.2. Calculate $s(G_n)$ in terms of p and n, when G_n is an extra-special p-group of type (d).

As in the case of $p = 2$, for the calculation of $s(G_n)$ for odd primes we need a good understanding of non-commuting structure of G_n. In particular, one would like to know how big the invariant $nc(G_n)$ is:

Problem 7.3. Calculate $nc(G_n)$ in terms of p and n for extra-special p-groups of order p^{2n+1}.

In a recent joint work with Jon Pakianathan [10], we study simplicial complexes associated with the non-commuting structure of a group. Although this study has many different aspects, we hope that it will also provide a good understanding of the invariant $nc(G)$, and it will eventually help us to solve some of these problems.

References

[1] A. Adem and R.J. Milgram, *The cohomology of finite groups*, Grundlehran der Math. Wissenschaften 309. Springer-Verlag, Berlin/New York 1994.

[2] D. Benson, *Representations and cohomology II*, Cambridge Studies in Advanced Mathematics 31, Cambridge University Press, Cambridge.

[3] N. Christofides, *Graph theory, an algorithmic approach*, Academic Press, 1975.

[4] L. Evens, *The cohomology of groups*, Oxford Mathematical Monographs, Clarendon Press, 1991.

[5] J.W.P. Hirshfeld and J.A. Thas, *General Galois Geometries*, Oxford Science Publications, Oxford, 1991.

[6] O. Kroll, *A representation theoretical proof of a theorem of Serre*, Århus Preprint, May 1986.

[7] P.A. Minh, *Serre's theorem on the cohomology algebra of a p-group*, Bull. London Math. Soc. **30** (1998), 518-520.

[8] K. Metsch, *The sets closest to ovoids in* $Q^-(2n+1,q)$, Bull. Belg. Math. Soc. **5** (1998), 389-392.

[9] T. Okuyama and H. Sasaki, *Evens' norm maps and Serre's theorem on the cohomology algebra of a p-group*, Arch. Math **54** (1990), 331-339.

[10] J. Pakianathan and E. Yalçın, *On commuting and non-commuting complexes*, J. Algebra **236** (2001), 396-418.

[11] L. Pyber, *The number of pairwise non-commuting elements and the index of the centre in a finite group*, J. London Math. Soc. (2) **35** (1987), 287-295.

[12] D. Quillen, *The mod* 2 *cohomology rings of extra-special* 2*-groups and the spinor groups*, Math. Ann. **194** (1971), 197-212.

[13] J.P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology **3** (1965), 413-420.

[14] J.P. Serre, *Une relation dans la cohomologie des p-groupes*, C.R. Acad. Sci. Paris **304** (1987), 587-590.

[15] M. Tezuka and N. Yagita, *The varieties of the mod p cohomology rings of extra-special p-groups for an odd prime p*, Math. Proc. Camb. Phil. Soc. **94** (1983), 449-459.

Department of Mathematics
McMaster University
Hamilton, ON, Canada
L8S 4K1

Current Address:
Department of Mathematics
Bilkent University
Ankara, 06533, Turkey
E-mail: yalcine@fen.bilkent.edu.tr