A NOTE ON SERRE'S THEOREM IN GROUP COHOMOLOGY

ERGÜN YALÇIN

ABSTRACT. In [7], Serre proves that if G is a p-group which is not elementary abelian, then a product of Bocksteins of one dimensional classes is zero in the mod p cohomology algebra of G, provided that the product includes at least one nontrivial class from each line in $H^1(G, \mathbb{F}_p)$. For $p = 2$, this gives that $(\sigma_G)^2 = 0$ where σ_G is the product of all nontrivial one dimensional classes in $H^1(G, \mathbb{F}_2)$. In this note, we prove that if G is a nonabelian 2-group, then σ_G is also zero.

1. INTRODUCTION

Let G be a finite p-group and let $H^*(G, \mathbb{F}_p)$ denote the cohomology algebra of G in mod p coefficients. One of the most important theorems about the algebra structure of $H^*(G, \mathbb{F}_p)$ is Serre's theorem [6] which states that if G is not an elementary abelian group, then there exist nontrivial one dimensional classes $x_1, x_2, \ldots, x_m \in H^1(G, \mathbb{F}_p)$ such that $\beta(x_1)\beta(x_2)\cdots\beta(x_m) = 0$. The original proof by Serre is an existence proof giving no indication how such a vanishing product should be chosen. In his second proof, Serre gives an explicit set of one dimensional classes which has a vanishing product. He proves the following:

Theorem 1.1 (Serre [7]). *Let G be a p-group which is not elementary abelian. Then, the product $\prod \beta(x)$ is zero if the product includes at least one nontrivial class from each line in $H^1(G, \mathbb{F}_p)$.*

When G is a 2-group, we have $\beta(x) = x^2$ for each $x \in H^1(G, \mathbb{F}_2)$, so Theorem 1.1 gives $(\sigma_G)^2 = 0$ where

$$\sigma_G = \prod_{x \in H^1(G, \mathbb{F}_2) - \{0\}} x.$$

The class σ_G itself is zero for many 2-groups, but there are also examples of 2-groups where σ_G is not zero. For example, if $G \cong \mathbb{Z}/4 \times \mathbb{Z}/2$, then σ_G is not zero. In this note, we show that such examples are limited to abelian 2-groups. Our main result is the following:

Theorem 1.2. *If G is a nonabelian 2-group, then $\sigma_G = 0$.*

This theorem has many applications in group cohomology. Our main motivation for studying this version of Serre's theorem is a calculation related to the Stiefel-Whitney classes of the regular representation of a 2-group. As a consequence of Theorem 1.2, we obtain that the top Stiefel-Whitney class of the augmentation module for the regular representation of

a nonabelian 2-group is zero. We also give some other applications related to essential cohomology. All these applications are explained in Section 4.

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1 although there are some subtle differences. In the proof of Theorem 1.1, Serre uses the fact that if G is a p-group which is not elementary abelian, then there exist some two dimensional relations of the form

$$(1) \qquad \sum_k a_k \beta(x_k) + \sum_{i<j} c_{ij} x_i x_j = 0$$

where $\{x_1, \ldots, x_n\}$ is a basis for $H^1(G, \mathbb{F}_p)$. But the existence of such relations are no longer enough to prove Theorem 1.2. Some complications arise when G is a nonabelian 2-group which has no two dimensional relations with some of the c_{ij}'s nonzero. For example, the modular group

$$M_{2^n} = \langle a, b \mid a^2 = b^{2^{n-1}} = 1, \ aba = b^{2^{n-2}+1} \rangle$$

of order 2^n with $n \geq 4$ is such a group (see Remark 3.3). To overcome this problem, we prove the following:

Proposition 1.3. *Let G be a 2-group and let $\{x_1, \ldots, x_n\}$ be a basis for $H^1(G, \mathbb{F}_2)$. If G is nonabelian, then there exists a nontrivial relation of the form*

$$(2) \qquad \sum_{i<j} c_{ij}(x_i^2 x_j + x_j^2 x_i) = 0$$

in $H^3(G, \mathbb{F}_2)$.

To prove Proposition 1.3, we consider group cohomology in integer coefficients and show that the inflation map

$$\text{inf} : H^3(G/\Phi(G), \mathbb{Z}) \to H^3(G, \mathbb{Z})$$

has a nontrivial kernel when G is a nonabelian p-group (here $\Phi(G)$ denotes the Frattini subgroup of G). We can also state this in the following way:

Proposition 1.4. *Let G be a p-group. Then, the following are equivalent:*
(i) G is abelian.
(ii) The map $\text{inf} : H^2(G/\Phi(G), \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Q}/\mathbb{Z})$ is injective.

This is similar to a criterion given by Serre in [6] for a p-group to be elementary abelian (see Proposition 3.4).

This note is organized as follows: In Section 2, we go through Serre's arguments to show how Theorem 1.2 follows from Proposition 1.3. Propositions 1.3 and 1.4 are proved in Section 3. In Section 4, we describe some consequences of Theorem 1.2.

2. Proof of the Main Theorem

Let G be a 2-group and let $\{x_1, \ldots, x_n\}$ be a basis for $H^1(G, \mathbb{F}_p)$. We can express σ_G as the product

$$\sigma_G = \prod_{(a_1, \ldots, a_n) \in (\mathbb{F}_2)^n - \{0\}} (a_1 x_1 + \cdots + a_n x_n).$$

Note that σ_G can also be expressed as the determinant of the matrix A defined by

$$A_{ij} = x_j^{2^{i-1}} \quad \text{for} \ \ 1 \le i, j \le n.$$

This is because both σ_G and $\det(A)$ have the same degree and the same zero set as polynomials in $x_1, \ldots, x_n$ over the algebraic closure of $\mathbb{F}_2$. Note that both polynomials are zero at a point $\alpha \in (\overline{\mathbb{F}}_2)^n$ if and only if the coordinates of α satisfy a linear equation of the form $a_1 x_1 + \cdots + a_n x_n = 0$ for some $(a_1, \ldots, a_n) \in (\mathbb{F}_2)^n - \{0\}$.

Let M denote the n-dimensional free module over the cohomology algebra $H^*(G, \mathbb{F}_2)$. For $i = 1, \ldots, n$, let e_i denote the vector

$$e_i = (x_i, x_i^2, \ldots, x_i^{2^{n-1}})$$

in M. Since the column vector given by e_i is the i-th column of A, we have

$$\det(A) = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

in the n-th exterior power $\wedge^n M \cong H^*(G, \mathbb{F}_p)$. Suppose there is a nontrivial relation

$$\sum_{i<j} c_{ij}(e_i \wedge e_j) = 0 \tag{3}$$

in $M \wedge M$, say with $c_{12} \ne 0$, then by taking exterior product with $e_3 \wedge \cdots \wedge e_n$, we get $\det(A) = 0$. So, to complete the proof of Theorem 1.2, we need to show that there is a nontrivial relation in $M \wedge M$.

Lemma 2.1 (cf. Serre [7]). *Let G be a 2-group. If there is a nontrivial relation of the form*

$$\sum_{i<j} c_{ij}(x_i^2 x_j + x_i x_j^2) = 0$$

in $H^3(G, \mathbb{F}_2)$, then there exists a nontrivial relation also in $M \wedge M$.

Proof. Let c_{ij} be the coefficients of a nontrivial relation in $H^3(G, \mathbb{F}_2)$. Define

$$L_{kl} = \sum_{i<j} c_{ij}(x_i^{2^k} x_j^{2^l} + x_i^{2^l} x_j^{2^k})$$

for $k, l \ge 0$. Note that there is a relation in $M \wedge M$ of the form given in Equation 3 if and only if $L_{kl} = 0$ for all $k, l \ge 0$. The class L_{00} is zero by trivial reasons and L_{10} is zero by assumption. For $m \ge 2$, we have

$$L_{m0} = Sq^{2^{m-1}} \cdots Sq^4 Sq^2 (L_{10}).$$

So, $L_{m0} = 0$ for all $m \ge 0$. Now, suppose k and l are such that $k = l + m$ for some $m \ge 0$. Then, $L_{kl} = (L_{m0})^{2^l} = 0$. Since $L_{kl} = L_{lk}$, we conclude that $L_{kl} = 0$ for all $k, l \ge 0$. $\qquad \square$

This shows that the proof of Theorem 1.2 is complete once we prove Proposition 1.3.

3. Proof of Proposition 1.3

As before, we assume that G is a 2-group. Let $\{x_1, \ldots, x_n\}$ be a basis for $H^1(G, \mathbb{F}_p)$. Recall that the inflation map

$$\mathrm{inf} : H^*(G/\Phi(G), \mathbb{F}_2) \to H^*(G, \mathbb{F}_2)$$

is bijective in dimension one. For each $x_i \in H^1(G, \mathbb{F}_2)$, we denote the associated class in $H^1(G/\Phi(G), \mathbb{F}_2)$ also by x_i. Note that to prove Proposition 1.3, it is enough to show that there is an element of the form

$$\sum_{i<j} c_{ij}(x_i^2 x_j + x_i x_j^2)$$

in $H^3(G/\Phi(G), \mathbb{F}_2)$ which lies in the kernel of the inflation map. We show this using integral cohomology. The quotient group $G/\Phi(G)$ is an elementary abelian 2-group and the integral cohomology rings of elementary abelian 2-groups are well understood. In particular, it is known that $H^3(G/\Phi(G), \mathbb{Z})$ is generated by $\{u_{ij} \mid 1 \leq i < j \leq n\}$ where the mod 2-reduction of u_{ij} is equal to $x_i^2 x_j + x_i x_j^2$ in $H^3(G, \mathbb{F}_2)$ (see, for example, [2]). Consider the following diagram:

$$
\begin{array}{ccc}
H^3(G/\Phi(G), \mathbb{Z}) & \xrightarrow{\ \mathrm{inf}\ } & H^3(G, \mathbb{Z}) \\
\downarrow{\scriptstyle \mathrm{mod}\ 2} & & \downarrow{\scriptstyle \mathrm{mod}\ 2} \\
H^3(G/\Phi(G), \mathbb{F}_2) & \xrightarrow{\ \mathrm{inf}\ } & H^3(G, \mathbb{F}_2).
\end{array}
$$

If the kernel of the inflation map in integer coefficients is nonzero, then there is a nontrivial class $\sum_{i<j} c_{ij} u_{ij}$ in $H^3(G/\Phi(G), \mathbb{Z})$ whose mod 2 reduction $\sum_{i<j} c_{ij}(x_i^2 x_j + x_i x_j^2)$ lies in the kernel of the mod 2 inflation map. Hence the proof of Proposition 1.3 is complete once we prove the following:

Proposition 3.1. *If G is a nonabelian p-group, then the kernel of the inflation map*

$$\mathrm{inf} : H^3(G/\Phi(G), \mathbb{Z}) \to H^3(G, \mathbb{Z})$$

is nontrivial.

Proof. The statement is equivalent to the statement that the kernel of the inflation map

$$\mathrm{inf} : H^2(G/\Phi(G), \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Q}/\mathbb{Z})$$

is nontrivial. Consider the 5-term exact sequence

$$0 \to H^1(G/\Phi(G)) \xrightarrow{\mathrm{inf}} H^1(G) \xrightarrow{\mathrm{res}} H^1(\Phi(G))^G \xrightarrow{d_2} H^2(G/\Phi(G)) \xrightarrow{\mathrm{inf}} H^2(G)$$

where all the cohomology groups are with coefficients in $\mathbb{Q}/\mathbb{Z}$. Note that

$$H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G/[G,G], \mathbb{Q}/\mathbb{Z}) = (G/[G,G])^*$$

where $(G/[G,G])^*$ denotes the dual group of the abelian group $G/[G,G]$. The above 5-term exact sequence gives an exact sequence of the form

$$0 \to (G/\Phi(G))^* \xrightarrow{\mathrm{inf}} (G/[G,G])^* \longrightarrow \left(\left(\Phi(G)/[\Phi(G), \Phi(G)] \right)^* \right)^G \longrightarrow \ker(\mathrm{inf}) \to 0.$$

From this it is easy to see that the cokernel of the first inflation map is $\left(\Phi(G)/[G,G]\right)^*$. Also note that

$$\left(\left(\Phi(G)/[\Phi(G),\Phi(G)]\right)^*\right)^G = \left(\operatorname{Hom}\left(\Phi(G),\mathbb{Q}/\mathbb{Z}\right)\right)^G = \operatorname{Hom}_G\left(\Phi(G),\mathbb{Q}/\mathbb{Z}\right)$$
$$= \operatorname{Hom}\left(\Phi(G)/[\Phi(G),G],\mathbb{Q}/\mathbb{Z}\right) = \left(\Phi(G)/[\Phi(G),G]\right)^*.$$

Thus, the kernel of inflation map in degree 2 is isomorphic to the group

$$\left([G,G]/[\Phi(G),G]\right)^*.$$

Now, the result follows from the following group theoretical lemma. $\qquad\square$

Lemma 3.2. *If G is a nonabelian p-group, then $[G,G] \neq [\Phi(G),G]$.*

Proof. Let G be a minimal counterexample to this lemma. Then, G is a nonabelian p-group with $[G,G] = [\Phi(G),G]$. Let H be a normal subgroup of G such that $H \leq [G,G]$ with index p. Note that G/H is a nonabelian p-group satisfying

$$[G/H,G/H] = [G,G]/H = [\Phi(G),G]/H = [\Phi(G)/H,G/H] = [\Phi(G/H),G/H].$$

So, by the minimality of G, we must have $H = 1$. This gives that $[G,G] = [\Phi(G),G]$ is a cyclic group of order p. In a p-group, a normal subgroup of order p is always central, so we have $[G,G] \leq Z(G)$. Under these conditions, $[a^p,b] = [a,b]^p = 1$ for every $a,b \in G$. Thus, the subgroup $G^{(p)}$ generated by p-th powers is also a central subgroup of G. Note that $\Phi(G) = [G,G]G^{(p)}$, so we obtain that $\Phi(G) \leq Z(G)$ which is in contradiction with our assumption since $[\Phi(G),G] = [G,G]$ is nontrivial. $\qquad\square$

This completes the proof of Proposition 1.3 and hence the proof of Theorem 1.2.

Remark 3.3. In the proof of Theorem 1.1 given in [7], Steenrod operations are applied to a relation of the form

$$\sum_k a_k\beta(x_k) + \sum_{i<j} c_{ij}x_ix_j = 0$$

to produce higher dimensional relations which eventually forces $\prod\beta(x)$ to be zero. But, the argument with Steenrod operations works only when some of the c_{ij}'s are nonzero. Because of this, the case where all c_{ij}'s are zero is considered separately. In that case, one has $\sum_k a_k\beta(x_k) = 0$ which gives $\beta(\sum_k a_kx_k) = 0$. So, Theorem 1.1 still holds if G has no relations with some c_{ij}'s nonzero. However, this causes a problem with the proof of Theorem 1.2. One may expect that if G is nonabelian, then there should exist at least one relation of the above form with some c_{ij}'s nonzero, but unfortunately this is not true. The modular group

$$M_{2^n} = \langle a,b \mid a^2 = b^{2^{n-1}} = 1,\ aba = b^{2^{n-2}+1}\rangle$$

of order 2^n with $n \geq 4$ has no relation of the above form with some c_{ij}'s nonzero. The only two dimensional relation that M_{2^n} has in its mod 2 cohomology is $\beta(u) = 0$ where $u \in H^1(M_{2^n},\mathbb{F}_2)$ is the class defined by the homomorphism $u(a^ib^j) = j \mod 2$.

Note that if G is abelian, then $[G,G] = [\Phi(G),G] = 1$, and hence the kernel of the inflation map is trivial. So, the proof of Proposition 1.4 is also complete. Proposition 1.4 has formal similarities to the following result by Serre.

Proposition 3.4 (Proposition 3 in [6])**.** *Let G be a p-group. Then, the following are equivalent:*
(i) G is elementary abelian.
(ii) The inflation map $\inf : H^2(G/\Phi(G), \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ is injective.

We can interpret Proposition 1.4 as a similar criterion for being abelian. Another way to interpret Proposition 3.1 is to say that the image of the differential d_2 is nontrivial in the Hochschield-Serre spectral sequence for the extension

$$0 \to \Phi(G) \to G \to G/\Phi(G) \to 0$$

of a nonabelian 2-group G. In this sense, Proposition 3.1 is a generalization of Proposition 7.2 in [9].

Remark 3.5. There is an alternative approach to the proof Proposition 1.3 which was suggested to us by the anonymous referee. Since it gives more insight to this proposition, we briefly explain it here. Consider the quotient group $F = G/[\Phi(G), G]$ which fits into a central extension of the form $1 \to \Phi(F) \to F \to E \to 1$ where $E = G/\Phi(G)$. By Lemma 3.2, F is nonabelian. Since $\Phi(F)$ is central, F has a nonabelian quotient Q which fits into a central extension of the form

$$\mathcal{E} : 1 \to C \to Q \to E \to 1$$

where C is cyclic. By taking further quotients if necessary, we can assume that Q/C_2 is abelian where C_2 is the unique subgroup of C of order 2. Let $\alpha \in H^2(E, C)$ be the extension class for $\mathcal{E}$. Consider the long exact sequence

$$\cdots \to H^2(G, \mathbb{Z}/2) \to H^2(E, \overline{C}) \to H^2(E, C) \xrightarrow{d} H^3(E, \mathbb{Z}/2) \to \cdots$$

associated to the short exact sequence $0 \to \mathbb{Z}/2 \to \overline{C} \to C \to 0$ where $\overline{C}$ is a cyclic group of order $2|C|$. We claim that $d(\alpha)$ is nonzero. Assume the contrary that $d(\alpha) = 0$. Then, the extension $\mathcal{E}$ will lift to a central extension of the form $1 \to \overline{C} \to \overline{Q} \to E \to 1$. Note that since the commutator subgroup of $\overline{Q}$ is central, we have $[a, b]^2 = [a^2, b] = 1$ for every $a, b \in \overline{Q}$. This gives that $Q = \overline{Q}/(\mathbb{Z}/2)$ is abelian which contradicts our starting assumption. This proves $d(\alpha)$ is nonzero. It is clear that inflation of $d(\alpha)$ to G is zero, because $\inf_E^G(\alpha) = 0$. So, it remains to show that $d(\alpha)$ is in the desired form. For this it is enough to show that $d(\alpha)$ is in the image of the Bockstein operator. Since Q/C_2 is abelian, there is a class $\gamma \in H^2(E, C)$ such that $d(\gamma) = 0$ and that $\pi_*(\gamma) = \pi_*(\alpha)$ under the homomorphism $\pi_* : H^2(E, C) \to H^2(E, C/C_2)$. So, $\alpha' = \alpha - \gamma$ is in the image of $j_* : H^2(E, C_2) \to H^2(E, C)$ and $d(\alpha) = d(\alpha')$. If $\alpha' = j_*(u)$, then an easy diagram chase shows that $d(\alpha') = \beta(u)$. This completes the proof that $d(\alpha)$ is in the desired form, and hence completes the argument.

4. Consequences of the main theorem

The converse of Theorem 1.2 does not hold since there are some abelian groups with $\sigma_G = 0$. For example, when $G = \mathbb{Z}/4 \times \mathbb{Z}/4$, we have

$$H^*(G, \mathbb{F}_2) \cong \wedge(x_1, x_2) \otimes \mathbb{F}_2[y_1, y_2],$$

so $\sigma_G = x_1 x_2 (x_1 + x_2) = 0$. In general, if G is an abelian group with more than two factors with exponent greater or equal to 4, then $\sigma_G = 0$. On the other hand, it is easy to see that $\sigma_G \neq 0$ when $G \cong \mathbb{Z}/2^k \times (\mathbb{Z}/2)^n$ for some $n \geq 0$ and $k \geq 1$. Thus, we conclude:

Corollary 4.1. *Let G be a 2-group. Then, $\sigma_G \neq 0$ if and only if $G \cong \mathbb{Z}/2^k \times (\mathbb{Z}/2)^n$ for some $n \geq 0$ and $k \geq 1$.*

This gives a complete description of 2-groups with $\sigma_G \neq 0$. The element σ_G is also defined for p-groups with $p > 2$, but it is easy to see that $\sigma_G = 0$ for all p-groups in this case.

Corollary 4.1 suggests another way to prove Theorem 1.2. Since abelian 2-groups with $\sigma_G \neq 0$ are in a special form, a minimal counterexample to Theorem 1.2 would be a nonabelian 2-group G which fits into an extension of the form

$$1 \to \mathbb{Z}/2 \to G \to \mathbb{Z}/2^k \times (\mathbb{Z}/2)^n \to 1$$

for some $n \geq 0$ and $k \geq 1$. One can argue directly that such groups have some nontrivial three dimensional relations of the form given in Proposition 1.3. The proof we give in Section 3 is more conceptual and explains why such three dimensional relations exist.

4.1. **The top Stiefel-Whitney class of an augmented regular representation.** Our main motivation for obtaining the vanishing product in Theorem 1.2 is a calculation related to the Stiefel-Whitney classes of regular representations. Let

$$I_G = \ker\{\varepsilon : \mathbb{R}G \longrightarrow \mathbb{R}\}$$

denote the augmentation ideal of the regular representation of G over real numbers. We are interested in calculating the top Stiefel-Whitney class of I_G.

This calculation is important for two reasons. First, this calculation is related to a question asked by Reiner and Webb. In [5], Reiner and Webb define a cohomology class ζ_G as the extension class of the subset complex of G and ask for which groups ζ_G is nonzero. We showed in [3] that the mod 2-reduction of ζ_G is the top-Stiefel Whitney class of I_G. So, calculation of the top Stiefel-Whitney class of I_G has direct consequences for the class ζ_G. Another reason for calculating the top Stiefel-Whitney class of I_G is a theorem of Turygin which says that there are Borsuk-Ulam type theorems for group actions on products of spheres if the top Stiefel-Whitney class of I_G pulls back to a nonzero class in the associated equivariant cohomology (see [8] for details).

Now, we explain how Theorem 1.2 applies to the calculation of Stiefel-Whitney classes. Let $w_{\text{top}}(I_G)$ denote the top Stiefel-Whitney class of I_G. Note that

$$I_G = I_{G/\Phi(G)} \oplus \text{ind}_{\Phi(G)}^G (I_{\Phi(G)}).$$

So, by the multiplicative property of the top Stiefel-Whitney class, we have

$$w_{\text{top}}(I_G) = w_{\text{top}}(I_{G/\Phi(G)}) \cdot w_{\text{top}}(\text{ind}_{\Phi(G)}^G (I_{\Phi(G)})).$$

Recall that

$$I_{G/\Phi(G)} = \bigoplus_{\xi \in \text{Hom}(G, \mathbb{R}^\times) - \{0\}} \xi$$

and the first Stiefel-Whitney class $w_1(\xi) \in H^1(G, \mathbb{F}_2)$ is the class associated to ξ under the following isomorphisms

$$H^1(G, \mathbb{F}_2) \cong \mathrm{Hom}(G, \mathbb{Z}/2) \cong \mathrm{Hom}(G, \mathbb{R}^\times).$$

So, we have

$$w_{\mathrm{top}}(I_{G/\Phi(G)}) = \prod_{\xi \in \mathrm{Hom}(G, \mathbb{R}^\times) - \{0\}} w_1(\xi) = \sigma_G.$$

As a consequence of Theorem 1.2, we obtain the following

Corollary 4.2. *If G is a nonabelian 2-group, then $w_{\mathrm{top}}(I_G) = 0$.*

This was also proved in [3] using different methods.

4.2. **Inflated essential classes.** The essential cohomology ideal of a p-group G is defined as the ideal

$$\mathcal{E}ss^*(G) = \ker\{\prod_{H < G} \mathrm{res}^G_H : H^*(G, \mathbb{F}_p) \to \prod_{H < G} H^*(H, \mathbb{F}_p)\}.$$

Note that if N is a normal subgroup of G such that $N \leq \Phi(G)$, then

$$\mathrm{inf}^G_{G/N}\Big(\mathcal{E}ss^*(G/N)\Big) \subseteq \mathcal{E}ss^*(G).$$

We would like to know if these inflated essential classes give some interesting essential classes in $\mathcal{E}ss^*(G)$. We consider the case where G is 2-group and $N = \Phi(G)$. Note that

$$H^*(G/\Phi(G), \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_k]$$

where $k = \dim\big(H^1(G, \mathbb{F}_2)\big)$. It is easy to see that the essential ideal of an elementary abelian 2-group is the principle ideal generated by the product of all nontrivial one dimensional classes. So, $\mathrm{inf}^G_{G/\Phi(G)}(u)$ is divisible by σ_G. Another corollary of Theorem 1.2 is the following:

Corollary 4.3. *If G is a nonabelian 2-group, then*

$$\mathrm{inf}^G_{G/\Phi(G)}\Big(\mathcal{E}ss^*(G/\Phi(G))\Big) = 0.$$

This corollary has some further applications in terms of essential classes coming from relative cohomology classes. For a G-set X, there is a X-relative version of the usual group cohomology. The usual cohomology algebra of a p-group G with coefficients in a field $k = \mathbb{F}_p$ is isomorphic to the algebra $\mathrm{Ext}^*_{kG}(k, k)$ of the equivalence classes of k by k extensions of kG-modules. If we require the extensions to be X-split (i.e., split after tensoring with permutation module kX), then we get a X-relative version of the usual $\mathrm{Ext}^*_{kG}(k, k)$ which is usually denoted by $\mathrm{XExt}^*_{kG}(k, k)$. There is an obvious map

$$\varphi : \mathrm{XExt}^*_{kG}(k, k) \to \mathrm{Ext}^*_{kG}(k, k)$$

which takes an X-relative extension to a usual kG-module extension. It is shown in [1] that if G is p-group and X is the disjoint union of all G-sets of the form G/H with H a maximal subgroup, then the image of φ lies in $\mathrm{inf}^G_{G/\Phi(G)}\Big(\mathcal{E}ss^*(G/\Phi(G))\Big)$. As a consequence of Corollary 4.3, it is obtained in [1] that φ is a zero map when G is a nonabelian 2-group.

4.3. **Minh's theorem on essential cohomology.** After proving Theorem 1.2, we noticed that in [4] Minh uses the statement of Theorem 1.2 to prove the $p = 2$ case of the following theorem:

Theorem 4.4 (Minh [4])**.** *Let G be a p-group. If $u \in \mathcal{E}ss^*(G)$, then $u^p = 0$.*

Minh refers to Serre's Theorem for the proof of Theorem 1.2 but, as we explained in Remark 3.3, the proof of Theorem 1.2 does not directly follow from Serre's arguments in [7]. So, the $p = 2$ case of the above theorem can be considered as another application of Theorem 1.2.

REFERENCES

[1] F. Altunbulak, *Relative extension classes and essential cohomology*, Ph.D. thesis, in preparation.

[2] D. Benson and J. F. Carlson, *The cohomology of extraspecial groups*, Bull. London Math. Soc. **24** (1992), 209–235.

[3] A. Güçlükan and E. Yalçın, *The Euler class of a subset complex*, preprint 2007.

[4] P. A. Minh, *Essential mod-p cohomology classes of p-groups: an upper bound for nilpotency degrees*, Bull. London Math. Soc. **32** (2000), 285–291.

[5] V. Reiner and P. Webb, *The combinatorics of the bar resolution in group cohomology*, J. Pure Appl. Algebra **190** (2004), 291–327.

[6] J. P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology **3** (1965), 413–420.

[7] J. P. Serre, *Une relation dans la cohomologie des p-groupes*, C.R. Acad. Sci. Paris **304** (1987), 587–590.

[8] Y. A. Turygin, *A Borsuk-Ulam theorem for $(\mathbb{Z}_p)^k$-actions on products of (mod p) homology spheres*, Topology Appl. **154** (2007), 455–461.

[9] E. Yalçın, *Group actions and group extensions*, Trans. Amer. Math. Soc. **352** (2000), 2689–2700.

DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06800 BILKENT, ANKARA, TURKEY
E-mail address: yalcine@fen.bilkent.edu.tr