

A Tate cohomology sequence for generalized Burnside rings

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Abstract

We generalize the fundamental theorem for Burnside rings to the mark morphism of plus constructions defined by Boltje. The main observation is the following: If D is a restriction functor for a finite group G , then the mark morphism $\varphi : D_+ \rightarrow D^+$ is the same as the norm map of the Tate cohomology sequence (over conjugation algebra for G) after composing with a suitable isomorphism of D^+ . As a consequence, we obtain an exact sequence of Mackey functors

$$0 \rightarrow \widehat{\text{Ext}}_{\gamma}^{-1}(\rho, D) \rightarrow D_+ \xrightarrow{\varphi} D^+ \rightarrow \widehat{\text{Ext}}_{\gamma}^0(\rho, D) \rightarrow 0$$

where ρ denotes the restriction algebra and γ denotes the conjugation algebra for G . Then, we show how one can calculate these Tate groups explicitly using group cohomology and give some applications to integrality conditions.

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1 Introduction

The Burnside ring of a finite group G is defined as the Grothendieck ring of the isomorphism classes of G -sets where addition and multiplication are given by disjoint union and cartesian product respectively. As a group, the Burnside ring $B(G)$ of G is a free abelian group with basis given by the set $\{[G/H] \mid H \in Cl(G)\}$ where $Cl(G)$ denotes a set representatives of conjugacy classes of subgroups of G . The ring structure in terms of this basis is given by the so called double coset formula which is a decomposition formula for the product $[G/H][G/K]$ in terms of the basis elements of $B(G)$.

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In order to understand the structure of the Burnside ring, one often considers the *mark homomorphism* $\varphi : B(G) \rightarrow C(G)$ where $C(G)$ is the ring of superclass functions on G . A superclass function $f \in C(G)$ is a function from subgroups of G to integers which is constant on conjugacy classes of subgroups. The set of superclass functions $C(G)$ forms a ring in the obvious way. The mark homomorphism $\varphi : B(G) \rightarrow C(G)$ is defined as the linear extension of the assignment which takes a G -set X to the superclass function $f_X : K \mapsto |X^K|$. Here K is a subgroup of G and X^K denotes the set of K -fixed points in X . The fundamental result concerning the mark homomorphism is the following theorem which is due to Dress [8] (see also [3] or [7]).

Theorem 1.1 (Fundamental theorem). *Let G be a finite group. Then, there is an exact sequence of abelian groups*

$$0 \longrightarrow B(G) \xrightarrow{\varphi} C(G) \xrightarrow{\psi} \bigoplus_{K \in Cl(G)} \mathbb{Z}/|W_G(K)|\mathbb{Z} \longrightarrow 0$$

where φ is the mark homomorphism and $W_G(H) := N_G(H)/H$ is the Weyl group of H in G .

There are several generalizations of the Burnside ring, some of which are the crossed Burnside ring, the monomial Burnside ring, and the cohomological Burnside ring. In each case, there is an analog of the ring of class functions and the mark homomorphism, which satisfies an analogous exact sequence (see, for example, [10] and [9]). A systematic approach to generalized Burnside rings is given by Boltje, in the context of the theory of canonical induction formulae. The main tool of this theory is the lower and upper plus constructions which, in special cases, produce the Burnside ring Mackey functor as well as its generalizations. To understand these constructions, we need to introduce some more terminology.

Let G be a finite group and R be a noetherian commutative ring with unity. A *Mackey functor* for G over R is a quadruple (M, t, c, r) where M is a family of R -modules $M(H)$ for each subgroup $H \leq G$, called the coordinate module at H . Between these modules there are families of three types of maps, called transfer maps, conjugation maps, and restriction maps, denoted by t, c , and r respectively. These maps are subject to certain relations, including the Mackey formula (see [12]). We denote the functor category of Mackey functors by $\text{Mack}_R(G)$.

One can also consider restriction functors (that is a triple (M, c, r) with the above notation) and conjugation functors (that is a couple (M, c)). As above, we have the categories $\text{Res}_R(G)$ and $\text{Con}_R(G)$. Now the lower plus construction, denoted by $-_+$, is a functor $\text{Res}_R(G) \rightarrow \text{Mack}_R(G)$ and the upper plus construction, denoted by $-^+$, is a functor $\text{Con}_R(G) \rightarrow \text{Mack}_R(G)$ (see [4]). If one starts with a restriction functor D and applies the plus constructions to D , then there is a natural morphism of Mackey functors $\varphi : D_+ \rightarrow D^+$ which we call, following Boltje [4], the *mark morphism* for D . Now our main theorem is the following.

Theorem 1.2 (Fundamental Theorem for plus constructions). *Let D be a restriction functor over a Noetherian ring R . Then there is an exact sequence of Mackey functors*

$$0 \longrightarrow \widehat{\text{Ext}}_{\gamma}^{-1}(\rho, D) \longrightarrow D_+ \xrightarrow{\varphi} D^+ \longrightarrow \widehat{\text{Ext}}_{\gamma}^0(\rho, D) \longrightarrow 0$$

where ρ and γ denote the restriction algebra and the conjugation algebra for G , and φ is the mark morphism of plus constructions.

The idea behind the proof is simple: It is well-known that Mackey functors can be realized as modules over the *Mackey algebra* $\mu_R(G)$ (see [12]). Similarly, we can consider the *restriction algebra* $\rho_R(G)$ and the *conjugation algebra* $\gamma_R(G)$. It is shown in [6] that the lower plus construction $-_+$ is just induction Ind_ρ^μ and the upper plus construction $-^+$ is the composition of inflation Inf_γ^τ and coinduction Coind_τ^μ . Here $\tau_R(G)$ is the *transfer algebra* whose modules are called *transfer functors*. A transfer functor is a triple (M, t, c) like a restriction functor. The proof of the main theorem depends on these equivalences given in [6] which we briefly explain in Section 2. The key argument in the proof is to show that after composing it with a suitable isomorphism, the mark morphism $\varphi : D_+ \rightarrow D^+$ given in Theorem 1.2 is the same as the norm map ν of the following Tate cohomology sequence

$$0 \rightarrow \widehat{\text{Ext}}_\gamma^{-1}(\rho, C) \longrightarrow \rho^* \otimes_\gamma C \xrightarrow{\nu} \text{Hom}_\gamma(\rho, C) \longrightarrow \widehat{\text{Ext}}_\gamma^0(\rho, C) \rightarrow 0$$

once we take $C = \text{Res}_\gamma^\ell D$. Note that since γ is Morita equivalent to a direct sum of group algebras, the Tate cohomology is defined in this case. The exactness of the above sequence is proved in Lemma 5.8 of [1].

In Section 4, we show that the Ext groups seen in the Theorem 1.2 can be calculated in terms of group cohomology, i.e., as the direct sum of Tate cohomology of certain finite groups. We do these calculations in Section 4. When $R = \mathbb{Z}$ and D is the constant restriction functor, we recover Theorem 1.1.

Another application of Theorem 1.2 is to the integrality conditions. Note that there are various choices for ψ in Theorem 1.1 which make the sequence exact. For example, we can take ψ as the map whose K -th coordinate is given by

$$\psi_K(f) = \sum_{K \leq L} \mu(K, L) f(L) \pmod{|W_G(K)|}.$$

For an element $f \in C(G)$, the condition that $f \in \text{im} \varphi$ is called the integrality condition. Different definitions of ψ give us different integrality conditions. Recently Boltje [5] found some interesting integrality conditions which seem to have some group cohomology flavor. One of Boltje's results gives that the integrality of a superclass function can be decided by checking it on the Sylow p -subgroups of the Weyl group $W_G(K) = N_G(K)/K$ (with a changed sum). In Section 5, we explain how this follows from the Tate cohomology interpretation of the cokernel of the mark homomorphism and a well-known detection result in group cohomology.

Our main theorem on integrality conditions is Theorem 5.1 which is very similar to Theorem 2.2 in [5]. The only difference is that Boltje assumes that the functor D has a stable basis and writes his conditions in terms of stabilizers of the basis elements. Here, we remove the condition that D has a stable-basis and prove a version of Boltje's theorem which does not depend on any basis choice.

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2 Preliminaries

Throughout the paper, let G be a finite group and R be a noetherian commutative ring with unity. Consider the free algebra on the variables c_H^g, r_K^H, t_K^H for each $K \leq H \leq G$ and each $g \in G$. We define the *Mackey algebra* $\mu_R(G)$ for G over R , written μ , as the quotient of this algebra by the ideal generated by the following six relations. Let $L \leq K \leq H \leq G$ and $h \in H$ and $g, g' \in G$, then

- (1) $c_H^h = r_H^H = t_H^H$
- (2) $c_{gH}^g c_H^g = c_H^{g'g}$ and $r_L^K r_K^H = r_L^H$ and $t_K^H t_L^K = t_L^H$
- (3) $c_K^g r_K^H = r_{gK}^g c_H^g$ and $c_H^g t_K^H = t_{gK}^g c_K^g$
- (4) $r_J^H t_K^H = \sum_{x \in J \backslash H/K} t_{J \cap xK}^J c_{J^x \cap K}^x r_{J^x \cap K}^K$ for $J \leq H$ (Mackey Relation)
- (5) $\sum_{H \leq G} c_H = 1$ where $c_H := c_H^1$.
- (6) All other products of generators are zero.

It is known that, letting H and K run over the subgroups of G , letting g run over the double coset representatives $HgK \subset G$, and letting L run over representatives of the subgroups of $H^g \cap K$ up to conjugacy, then the elements $t_{gL}^H c_L^g r_L^K$ run (without repetition) over the elements of an R -basis for the Mackey algebra $\mu_R(G)$ (cf.[12, Section 3]).

We denote by $\rho_R(G)$, called the *restriction algebra* for G over R , the subalgebra of the Mackey algebra generated by c_H^g and r_K^H for $K \leq H \leq G$ and $g \in G$. We denote by $\tau_R(G)$ the *transfer algebra* for G over R the subalgebra generated by c_H^g and t_K^H for $K \leq H \leq G$ and $g \in G$. The *conjugation algebra*, denoted $\gamma_R(G)$, is the subalgebra generated by the elements c_H^g . When there is no ambiguity, we write $\mu = \mu_R(G)$, $\rho = \rho_R(G)$, $\tau = \tau_R(G)$, and $\gamma = \gamma_R(G)$. Evidently, the restriction algebra ρ has generators $c_J^g r_J^K$, the transfer algebra τ has generators $c_K^g t_J^K$ and the conjugation algebra γ has generators c_J^g .

We define a *Mackey functor* for G over R to be a $\mu_R(G)$ -module. Similarly, we define a *restriction functor*, a *transfer functor*, and a *conjugation functor* as a $\rho_R(G)$ -module, a $\tau_R(G)$ -module, and a $\gamma_R(G)$ -module, respectively. In order to show that this definition of a Mackey functor is equivalent to the definition given in the previous section, let M be a $\mu_R(G)$ -module. We define the coordinate module at H of the corresponding Mackey functor as $c_H M$ and define the maps via the action of the Mackey algebra. Conversely, given a Mackey functor (M, t, c, r) , the corresponding $\mu_R(G)$ -module is given by $\bigoplus_{H \leq G} M(H)$ and the action of the Mackey algebra is induced by the maps t, c , and r . It is straightforward to check that this gives an equivalence (see also [12]). Similar comments apply to the other three functors.

Now consider the triangle of functors given below, called the *Mackey triangle*, which was introduced in [6]:

$$\begin{array}{ccccc}
 & & \mu & & \\
 & \nearrow & & \nwarrow & \\
 & \tau & & \rho & \\
 \swarrow & & & & \searrow \\
 \gamma & & & & \gamma \\
 \hline & & & & \hline \\
 \gamma & & & & \gamma
 \end{array}$$

Here the square in the middle consists of trivial inclusions and the surjections $\tau \rightarrow \gamma$ and $\rho \rightarrow \gamma$ are the canonical projection maps with the kernel consisting of proper transfer and restriction maps. The Mackey triangle gives rise to some equivalences between certain functors between module categories of these algebras. In the following theorem, we collect together some of these equivalences that we will use later in the proof of our main theorem. Proofs of these propositions can be found in [6].

Proposition 2.1 (Coşkun [6]). *The following natural equivalences hold.*

1. $\text{Res}_\tau^\mu \text{Ind}_\rho^\mu \cong \text{Ind}_\gamma^\tau \text{Res}_\gamma^\rho$.
2. $\text{Res}_\rho^\mu \text{Coind}_\tau^\mu \cong \text{Coind}_\gamma^\rho \text{Res}_\gamma^\tau$.
3. $\text{Res}_\gamma^\tau \text{inf}_\gamma^\tau \cong \text{id}_\gamma$.

As mentioned in the introduction, we have the following characterization of the plus constructions.

Proposition 2.2 (Coşkun [6]). *Under the equivalence $\mu_{R(G)}\text{mod} \cong \text{Mack}_R(G)$ of categories, the following equivalences of functors hold.*

1. $\text{Ind}_\rho^\mu \cong -_+$.
2. $\text{Coind}_\tau^\mu \text{inf}_\gamma^\tau \cong -^+$.

Note that all the equivalences in these propositions are canonical. Indeed, the composite functor in the third part of Proposition 2.1 is equal to the identity functor. In the first two parts, we use the canonical isomorphism $\tau \otimes_\gamma \rho \rightarrow \mu$ given in [6, Theorem 3.2] and the canonical adjunction isomorphism for induction and coinduction. Also the equivalences for the plus constructions are natural and canonical. Throughout the paper, we fix all these canonical isomorphisms and regard these functors as equal.

Now, we explain the mark homomorphism of plus constructions. Let D be a ρ -module. Recall that the coordinate module $D_+(H)$ at H of D_+ is given by

$$D_+(H) = \left(\bigoplus_{L \leq H} D(L) \right)_H$$

and the module $D^+(H)$ is given by

$$D^+(H) = \left(\prod_{L \leq H} D(L) \right)^H.$$

We denote the elements of $D^+(H)$ as tuples $(x_K)_{K \leq H}$ where $x_K \in D(K)$ and the elements in $D_+(H)$ as $t_K^H \otimes a$ where $K \leq H$ and $a \in D(K)$. For a detailed information on plus constructions and the definition of restriction, transfer, and conjugation maps, we refer the reader to [4]. Note that these maps induce from the usual actions of Mackey algebra on induction and coinduction under the identification given in Proposition 2.2. In fact, this is why we use the notation $t_K^H \otimes a$ for basis elements instead of the notation $[K, a]_H$ used by Boltje in [4] and [5]. It makes it easier to calculate the action of Mackey algebra on $D_+(H)$.

The *mark morphism* for D is the map

$$\varphi : D_+ \rightarrow D^+$$

given, for subgroups $K \leq H$ of G and an element $t_K^H \otimes a$ of $D_+(H)$, by

$$\varphi_H(t_K^H \otimes a) = \left(\sum_{h \in H/K, L \leq hK} r_L^{hK}(^h a) \right)_{L \leq H}.$$

Note that when D is the constant ρ -module, that is when $D(H) = R$ for any $H \leq G$ and any restriction map is the identity map, the mark morphism for D coincides with the usual mark homomorphism.

Finally we need the following result concerning the γ - γ -bimodule $\rho^* := \text{Hom}_\gamma(\rho, \gamma)$. Note that as a right γ -module, ρ^* is the γ -dual of the γ -module ρ . Hence the right γ -module structure is the usual one and ρ^* becomes a γ - γ -bimodule via the action of γ on the left given by right multiplication. First it is easy to see that, as a γ - γ -bimodule, the restriction algebra decomposes as

$$\rho = \bigoplus_{K \leq_G G, L \leq_K K} \gamma r_L^K \gamma.$$

Hence the γ - γ -bimodule ρ has basis $\{r_L^K : L \leq_K K, K \leq_G G\}$. Therefore the module ρ^* has basis $\{\tilde{r}_L^K : L \leq_K K, K \leq_G G\}$ which is completely determined by the following four properties. Let $L \leq K \leq G$, and $N \leq M \leq G$ then

1. $\tilde{r}_L^K(r_N^M) = 0$ unless $M = {}^g K$ for some $g \in G$ and $N = {}_M {}^g L$.
2. $\tilde{r}_L^K(r_{gL}^{{}^g K}) = (c_K^g \tilde{r}_L^K c_L^{g^{-1}})(r_L^K)$ for any $g \in G$.
3. $\tilde{r}_L^K(r_{kL}^K) = c_L^k(\tilde{r}_L^K(r_L^K))$ for any $k \in K$.
4. $\tilde{r}_L^K(r_L^K) = \sum_{x \in N_K(L)/L} c_L^x$.

Now we can prove the characterization of the module ρ^* .

Lemma 2.3. *The γ - γ -bimodules ρ^* and τ are isomorphic.*

Proof. As the γ - γ -bimodule ρ , the γ - γ -bimodule τ has basis $\{t_L^K : L \leq_K K, K \leq_G G\}$. It is straightforward to check that the correspondence $\tilde{r}_L^K \mapsto t_L^K$ associating the above basis of ρ^* to the basis of τ is an isomorphism of γ - γ -bimodules. \square

3 Proof of the main theorem

In this section, we prove Theorem 1.2 stated in the introduction. We prove the theorem in two steps. First we prove it for the *diagonal* part of the mark morphism from which our main theorem follows. Here the diagonal part of the mark morphism is the map

$$\mathcal{N}^D : \text{Res}_\gamma^\mu D_+ \rightarrow \text{Res}_\gamma^\mu D^+$$

$$\text{given by } \mathcal{N}_H^D(t_K^H \otimes a) = \left(\sum_{h \in H/K, L = {}^h K} {}^h a \right)_{L \leq H}.$$

Proposition 3.1. *Let D be a ρ -module. There is an exact sequence of γ -modules*

$$0 \rightarrow \widehat{\text{Ext}}_\gamma^{-1}(\rho, D) \longrightarrow \text{Res}_\gamma^\mu D_+ \xrightarrow{\mathcal{N}^D} \text{Res}_\gamma^\mu D^+ \longrightarrow \widehat{\text{Ext}}_\gamma^0(\rho, D) \rightarrow 0$$

where \mathcal{N}^D is the diagonal part of the mark morphism for D .

Proof. By Proposition 2.1, we have

$$\text{Res}_\gamma^\mu D_+ = \text{Res}_\gamma^\tau \text{Ind}_\gamma^\tau \text{Res}_\gamma^\rho D$$

and

$$\text{Res}_\gamma^\mu D^+ = \text{Res}_\gamma^\mu \text{Coind}_\gamma^\mu \text{inf}_\gamma^\tau \text{Res}_\gamma^\rho D = \text{Res}_\gamma^\rho \text{Coind}_\gamma^\rho \text{Res}_\gamma^\rho D.$$

Also, by Lemma 2.3, we have $\rho_\gamma^* \cong \tau_\gamma$ as γ - γ -bimodules. So, we have

$$\text{Res}_\gamma^\tau \text{Ind}_\gamma^\tau \text{Res}_\gamma^\rho D \cong \rho^* \otimes_\gamma \text{Res}_\gamma^\rho D$$

as left γ -modules. Therefore $\text{Res}_\gamma^\mu D_+ \cong \rho^* \otimes_\gamma C$ and $\text{Res}_\gamma^\mu D^+ \cong \text{Hom}_\gamma(\rho, C)$ where $C = \text{Res}_\gamma^\rho D$. Now, by Lemma 5.8 in [1], the sequence

$$0 \rightarrow \widehat{\text{Ext}}_\gamma^{-1}(\rho, C) \longrightarrow \rho^* \otimes_\gamma C \xrightarrow{\nu} \text{Hom}_\gamma(\rho, C) \longrightarrow \widehat{\text{Ext}}_\gamma^0(\rho, C) \rightarrow 0$$

is exact where ν is given by $\nu(\alpha \otimes c)(x) = \alpha(x)c$ for $\alpha \in \rho^* = \text{Hom}_\gamma(\rho, \gamma)$, $c \in C$, and $x \in \rho$. Note that $\text{Res}_\gamma^l \rho$ satisfies the condition of the Lemma 5.8 in [1], that is $\text{Res}_\gamma^l \rho$ has finite Gorenstein dimension, by Example 3.3(2) in [1].

Hence it remains to show that the morphism ν coincides with the diagonal part of the mark morphism \mathcal{N}^D . It suffices to show that for any $H \leq G$, we have $\mathcal{N}_H^D = \nu_H$. Fix $H \leq G$. The map $\nu_H : \rho^* \otimes_\gamma C(H) \rightarrow \text{Hom}_\gamma(\rho, C)(H)$ is given by $\nu_H(\tilde{r}_K^H \otimes c)(r_L^H) = \tilde{r}_K^H(r_L^H)c$. Note that the map $\nu_H(\tilde{r}_K^H \otimes c)$ is determined uniquely by its values at the elements r_L^H for $L \leq H$ since $\text{Hom}_\gamma(\rho, C)(H) = \text{Hom}_\gamma(\rho c_H, C)$ and it commutes with conjugation maps. Moreover since $\tilde{r}_K^H(r_L^H)c = 0$ unless $L =_H K$, the map is determined by its values at the elements r_L^H with $L = {}^h K$ for some $h \in H$. But in this case $r_L^H = c_K^h r_K^H$. Thus the map $\nu_H(\tilde{r}_K^H \otimes c)$ is determined by its value at K and we have

$$\nu_H(\tilde{r}_K^H \otimes c)(r_K^H) = \sum_{h \in N_H(K)/K} {}^h c.$$

On the other hand, it is easy to see that $\mathcal{N}_H^D(\tilde{r}_K^H \otimes c)$, which is equal to $\mathcal{N}_H^D(t_K^H \otimes c)$, is also determined by its K -th coordinate and

$$\mathcal{N}_H^D(t_K^H \otimes c) = \sum_{h \in H/K, K = {}^h K} {}^h c = \sum_{h \in N_H(K)/K} {}^h c.$$

Therefore $\mathcal{N}_H^D = \nu_H$ as required. \square

Now we prove our main theorem.

Proof of Theorem 1.2. It is enough to prove the exactness for each G . But, then it is enough to prove exactness by first restricting the entire sequence to γ . Once we prove the exactness of the sequence, the Mackey functor structure of the Ext groups will be evident. Indeed, let $K \leq H$ be subgroups of G . Then we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\text{Ext}}_\gamma^{-1}(\rho, D)(H) & \longrightarrow & D_+(H) & \xrightarrow{\varphi^H} & D^+(H) & \longrightarrow & \widehat{\text{Ext}}_\gamma^0(\rho, D)(H) & \longrightarrow & 0 \\ & & & & \downarrow r_K^H & & \downarrow r_K^H & & & & \\ 0 & \longrightarrow & \widehat{\text{Ext}}_\gamma^{-1}(\rho, D)(K) & \longrightarrow & D_+(K) & \xrightarrow{\varphi^K} & D^+(K) & \longrightarrow & \widehat{\text{Ext}}_\gamma^0(\rho, D)(K) & \longrightarrow & 0 \end{array}$$

By diagram chasing, there are well-defined maps $\widehat{\text{Ext}}_\gamma^i(\rho, D)(H) \rightarrow \widehat{\text{Ext}}_\gamma^i(\rho, D)(K)$ for $i = -1, 0$. Considering a similar diagram, we also obtain well-defined transfer and conjugation maps. It is straightforward to check that these maps satisfy the relations (1)-(6) of the previous section. Hence via these maps, the R -modules $\widehat{\text{Ext}}_\gamma^i(\rho, D)$ with $i = -1, 0$ become Mackey functors, as required.

In order to prove exactness after restricting to γ , consider the following triangle

$$\begin{array}{ccc} & \text{Res}_\gamma^\mu D^+ & \\ \varphi^D \nearrow & & \searrow \eta^D \\ \text{Res}_\gamma^\mu D_+ & \xrightarrow{\mathcal{N}^D} & \text{Res}_\gamma^\mu D^+ \end{array}$$

where φ^D and \mathcal{N}^D are as defined above and $\eta^D : \text{Res}_\gamma^\mu D^+ \rightarrow \text{Res}_\gamma^\mu D^+$ is defined as follows. Given $H \leq G$ and $(x_L)_{L \leq H} \in D^+(H)$. Then

$$\eta_H^D((x_L)_{L \leq H}) = \left(\sum_{L \leq K} \mu(L, K) r_L^K x_K \right)_{L \leq H}.$$

Note that the morphism η^D is introduced, in a special case, in [9]. We claim that the above triangle commutes, that is, $\eta^D \circ \varphi^D = \mathcal{N}^D$. To prove this, recall that by [4, Section 2.1], given $H \leq G$, any element $x \in D_+(H)$ has the form

$$x = \sum_{K \leq H, v_K \in D(K)} t_K^H \otimes v_K.$$

Since the morphisms η^D, φ^D and \mathcal{N}^D are all linear, it suffices to prove the claim for an arbitrary element of the form $t_K^H \otimes v$. By the definition of the mark morphism given in the previous section, for a given subgroup $L \leq H$, we have

$$\begin{aligned} ((\eta_H \circ \varphi_H)(t_K^H \otimes v))_L &= \sum_{L \leq N} \mu(L, N) r_L^N \sum_{g \in H/K, N \leq {}^g K} r_N^{gK} {}^g v \\ &= \sum_{g \in H/K} r_L^{gK} {}^g v \sum_{L \leq N \leq {}^g K} \mu(L, N) \\ &= \sum_{g \in H/K} r_L^{gK} {}^g v [{}^g K = L] \end{aligned}$$

where $[a = b]$ is 1 if $a = b$ and zero otherwise. Therefore putting $L = K$, we obtain

$$((\eta_H \circ \varphi_H)(t_K^H \otimes v))_K = \sum_{g \in H/K, K = {}^g K} {}^g v = (\mathcal{N}_H(t_K^H \otimes v))_K$$

as required. It is proved, in a special case, in [9] that η^D is an isomorphism. Using the same argument, we can prove that the morphism η^D is an isomorphism in general. Now, the proof of Theorem 1.2 follows from Lemma 3.2 below. \square

Lemma 3.2. *Let Λ be a ring and let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be maps of Λ -modules and denote the composition by $\zeta : A \rightarrow C$. If β is an isomorphism, then $\ker \alpha \cong \ker \zeta$ and $\operatorname{coker} \alpha \cong \operatorname{coker} \zeta$.*

Proof. By Proposition 4.5 in [2], in this situation, there exist unique maps making the following diagram commute:

$$\begin{array}{ccccc} & & \ker \beta & \longrightarrow & \operatorname{coker} \alpha \\ & \nearrow & \searrow & \nearrow & \searrow \\ \ker \zeta & \longrightarrow & A & \xrightarrow{\zeta} & C & \longrightarrow & \operatorname{coker} \zeta \\ & \nwarrow & \nearrow & \nwarrow & \nearrow & & \\ & & \ker \alpha & & \operatorname{coker} \beta & & \end{array}$$

Moreover the outer perimeter sequence is exact. When β is an isomorphism, we have $\ker \beta = \operatorname{coker} \beta = 0$, so we obtain the desired result. \square

It is easy to see that over a field of characteristic zero, say over \mathbb{Q} , the conjugation algebra γ is semisimple. So over \mathbb{Q} , the Ext groups in Theorem 1.2 vanish. Thus the mark morphism is an isomorphism in this case. We can find the inverse of the mark morphism easily. Let D be a $\rho_{\mathbb{Q}}(G)$ -module. Then by the above proof, the mark morphism φ^D satisfies the equality $\eta^D \circ \varphi^D = \mathcal{N}^D$. Therefore we get

$$(\varphi^D)^{-1} = (\mathcal{N}^D)^{-1} \circ \eta^D.$$

Now the inverse of \mathcal{N}^D is given by

$$(\mathcal{N}_H^D)^{-1}((x_K)_{K \leq H}) = \sum_{K \leq_H H} \frac{|K|}{|N_H(K)|} t_K^H \otimes x_K = \sum_{K \leq H} \frac{|K|}{|H|} t_K^H \otimes x_K$$

where H is a subgroup of G and $(x_K)_{K \leq H} \in D^+(H)$. Then by composing with η^D , we get

$$\begin{aligned} (\varphi_H^D)^{-1}((x_K)_{K \leq H}) &= (\mathcal{N}_H^D)^{-1} \left(\left(\sum_{L \leq K} \mu(L, K) r_L^K x_K \right)_{L \leq H} \right) \\ &= \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) (t_L^H \otimes r_L^K x_K). \end{aligned}$$

Note that the inverse $(\varphi^D)^{-1}$ of the mark morphism is a scalar multiple of the map

$$\sigma_H^D : (x_K)_{K \leq H} \mapsto \sum_{L \leq K \leq H} |L| \mu(L, K) (t_L^H \otimes r_L^K x_K)$$

which is usually referred to as an almost inverse to φ^D (cf. [4, Proposition 2.4]) because of the following property.

Corollary 3.3 (Boltje [4]). *Let D be a $\rho_R(G)$ -module and $H \leq G$. Then*

$$\sigma_H^D \circ \varphi_H^D = |H| \text{id}_{D_+(H)} \quad \text{and} \quad \varphi_H^D \circ \sigma_H^D = |H| \text{id}_{D_+(H)}.$$

4 Calculation of the Tate Ext groups

In this section, we calculate the Tate Ext groups appearing in Theorem 1.2 in terms of the Tate Ext groups for group algebras. This result will generalize fundamental theorems for several generalized Burnside rings.

Theorem 4.1. *Let C be a γ -module. Then there is an isomorphism of R -modules*

$$\widehat{\text{Ext}}_\gamma^i(\rho, C) \cong \bigoplus_{K \leq G} \bigoplus_{H \leq_K K} \widehat{H}^i(W_K(H), C(H))$$

for each i , where $\widehat{H}^i(G, N)$ denotes the i -th Tate cohomology group of G with coefficients in N .

Proof. First note that any γ -module C has the following decomposition

$$C \cong \bigoplus_{H \leq_G G} C_{H, C(H)}$$

where $C_{H, C(H)}$ is the γ -submodule of C generated by the H -th coordinate module of C . Since induction and coinduction are additive, it suffices to prove the theorem for the γ -module $C_{H, V}$ where $V := C(H)$. Since, up to conjugation, $C_{H, V}$ has a unique non-zero coordinate module, which is H , the following isomorphism holds

$$\widehat{\text{Ext}}_\gamma^i(\rho, C_{H, V}) \cong \widehat{\text{Ext}}_{RW_G(H)}^i(c_H \rho, V)$$

where $c_H = c_H^1$ and $c_H\rho$ is the H -th coordinate of the γ -module ρ . Indeed this follows easily since the conjugation algebra γ is Morita equivalent to the algebra $\bigoplus_{H \leq G} RW_G(H)$ and the cohomology functor is Morita invariant.

Now the $W_G(H)$ -module $c_H\rho$ can be decomposed as

$$c_H\rho = \bigoplus_{K \leq G} c_H\rho c_K$$

by multiplying from the right by the unit $1 = \sum_{K \leq G} c_K$ of the conjugation algebra γ . This implies the following decomposition.

$$\widehat{\text{Ext}}_\gamma^i(\rho, C_{H,V}) \cong \bigoplus_{K \leq G} \widehat{\text{Ext}}_{RW_G(H)}^i(c_H\rho c_K, V).$$

To complete the proof, it remains to decompose the modules $c_H\rho c_K$ for each $K \leq G$. So let us fix $K \leq G$ for the rest of the proof and find the K -th coordinate of the Ext group. It is easy to see that the R -basis of the Mackey algebra given in Section 2 gives the R -basis $\{c_{H^x}^x r_{H^x}^K : H, K \leq G, x \in H \backslash G/K, H^x \leq K\}$ of the restriction algebra ρ . Using this basis, we get

$$c_H\rho c_K \cong \bigoplus_{x \in H \backslash G/K, H^x \leq K} Rc_{H^x}^x r_{H^x}^K = \bigoplus_{x \in G/K, H^x \leq K} Rc_{H^x}^x r_{H^x}^K$$

Here the second equality holds because $H \leq {}^xK$ implies that $HxK = xK$. Thus it is clear from the above decomposition that the $RW_G(H)$ -module $c_H\rho c_K$ is a permutation module with the permutation basis

$$X = X_H^K = \{xK \in G/K : H^x \leq K\}$$

where $W_G(H)$ acts on the left by multiplication.

Now we are to find the $W_G(H)$ orbits of the set X and the stabilizers of the orbits. It is easy to see that $x, x' \in X$ are in the same orbit if the subgroups H^x and $H^{x'}$ are K -conjugate. To find the stabilizers, let $x \in X$ and $n \in N_G(H)$. Then n is in the stabilizer of x in $N_G(H)$ if and only if the equality $nxK = xK$ holds. But $nxK = xK$ holds if and only if $n \in {}^xK$. Hence we get

$$\text{stab}_{N_G(H)}(x) = N_G(H) \cap {}^xK = N_{xK}(H).$$

Therefore we obtain

$$X \cong \bigoplus_{x \in T(H,K)} N_G(H)/N_{xK}(H)$$

where $T(H, K) = N_G(H) \backslash X$ is a set of representatives of $N_G(H)$ -orbits of X . As a result we get

$$\widehat{\text{Ext}}_{RW_G(H)}^i(c_H\rho c_K, V) \cong \bigoplus_{x \in T(H,K)} \widehat{\text{Ext}}_{RW_G(H)}^i(\text{Ind}_{N_{xK}(H)}^{N_G(H)} R, V).$$

Now by Shapiro's Lemma, we obtain

$$\widehat{\text{Ext}}_{RW_G(H)}^i(c_H\rho c_K, V) \cong \bigoplus_{x \in T(H,K)} \widehat{\text{Ext}}_{RW_{xK}(H)}^i(R, V).$$

Finally notice that the index set $T(H, K)$ above is counting the K -conjugacy classes of H . Indeed consider the set $\{N_G(H)x \in G/N_G(H)\}/K$ where K acts on $\{N_G(H)x \in$

$N_G(H)\backslash G\}$ by right multiplication. It is clear that the K -classes of the G -classes of the subgroup H is parameterized by this set. Moreover the correspondence $N_G(H)(xK) \rightarrow (N_G(H)x)K$ gives a bijection between $T(H, K)$ and $\{N_G(H)x \in N_G(H)\backslash G\}/K$. Hence we can rewrite the last isomorphism as

$$\widehat{\text{Ext}}_{RW_G(H)}^i(c_H \rho c_K, V) \cong \bigoplus_{L \leq_K K, L=G/H} \widehat{\text{Ext}}_{RW_K(L)}^i(R, C(L)).$$

Now the result follows as $\widehat{H}^i(G, N) := \widehat{\text{Ext}}_{RG}^i(R, N)$ for any group G and any RG -module N . \square

Now we can make the maps in Theorem 1.2 more explicit. For simplicity, assume that $R = \mathbb{Z}$ and for each $H \leq G$, the R -module $D_+(H)$ has no non-trivial $|H|$ -torsion. Recall that by Theorem 1.2, we have the following exact sequence.

$$0 \longrightarrow \widehat{\text{Ext}}_{\gamma}^{-1}(\rho, D) \longrightarrow D_+ \xrightarrow{\varphi} D^+ \longrightarrow \widehat{\text{Ext}}_{\gamma}^0(\rho, D) \longrightarrow 0.$$

Now by Theorem 4.1, we get $\widehat{\text{Ext}}_{\gamma}^{-1}(\rho, D)$ is zero and for each $K \leq G$, we obtain

$$\widehat{\text{Ext}}_{\gamma}^0(\rho, D)(K) \cong \bigoplus_{H \leq_K K} \widehat{H}^0(W_K(H), D(H)).$$

So, for each $K \leq G$, there is an exact sequence of abelian groups

$$0 \longrightarrow D_+(K) \xrightarrow{\varphi_K} D^+(K) \xrightarrow{\psi_K} \bigoplus_{H \leq_K K} \widehat{H}^0(W_K(H), D(H)) \longrightarrow 0$$

where ψ_K is given by composition $q_K \circ \eta_K$ where η_K is as given in Section 3 and q_K is the quotient map in the following sequence

$$0 \longrightarrow D_+(K) \xrightarrow{\mathcal{N}_K} D^+(K) \xrightarrow{q_K} \bigoplus_{H \leq_K K} \widehat{H}^0(W_K(H), D(H)) \longrightarrow 0.$$

Note that there are isomorphisms of $W_G(K)$ -modules

$$D_+(K) \cong \bigoplus_{H \leq_K K} D(H)_{W_K(H)}$$

and

$$D^+(K) \cong \bigoplus_{H \leq_K K} D(H)^{W_K(H)}.$$

Therefore the quotient map in the above sequence is the sum of the canonical surjections

$$q_K^H : D(H)^{W_K(H)} \rightarrow \widehat{H}^0(W_K(H), D(H)).$$

So we have proved the following.

Corollary 4.2. *For each $K \leq G$, the following sequence is exact.*

$$0 \longrightarrow D_+(K) \xrightarrow{\varphi_K} D^+(K) \xrightarrow{q_K \circ \eta_K} \bigoplus_{H \leq_K K} \widehat{H}^0(W_K(H), D(H)) \longrightarrow 0$$

where $q_K := \sum_{H \leq_K K} q_K^H$ and q_K^H is the usual quotient map of the Tate cohomology sequence for group algebras.

There is an easy corollary of Theorem 4.1. Note that the Tate cohomology groups $\widehat{H}^i(G, V)$, for $i = -1, 0$ are annihilated by the order $|G|$ of G . Therefore we obtain the following well-known result.

Corollary 4.3 (Thévenaz [11]). *The kernel and cokernel of the mark morphism is annihilated by the integer $\prod_{H \leq G} |N_G(H) : H|$. In particular, φ is an isomorphism if and only if $|G|$ is invertible in R .*

When the ρ -module D has a G -stable basis, it is possible to decompose the Ext groups further. Now, we discuss how this can be done. Let D be a ρ -module with a G -stable basis $\mathcal{B} = (\mathcal{B}_H)_{H \leq G}$, that is, \mathcal{B}_H is an R -basis for $D(H)$ and for any $g \in G$, we have $c_H^g(\mathcal{B}_H) = \mathcal{B}_{gH}$ (cf. [4, Definition 7.1]). It is clear that the later condition gives an $N_G(H)$ -action on the set \mathcal{B}_H . Note further that H acts trivially on \mathcal{B}_H since c_H^h is equal to the identity for any element h of H . Therefore we obtain the following isomorphism of $RW_G(H)$ -modules

$$D(H) \cong R\mathcal{B}_H.$$

Now the $W_G(H)$ -set \mathcal{B}_H decomposes into transitive $W_G(H)$ -sets as

$$\mathcal{B}_H \cong \bigoplus_{\phi \in [W_G(H) \backslash \mathcal{B}_H]} W_G(H)/W_G(H, \phi)$$

where the sum is over representatives of orbits of $W_G(H)$ on \mathcal{B}_H and the group $W_G(H, \phi) = N_H(H, \phi)/H$ is the stabilizer in $W_G(H)$ of the basis element $\phi \in \mathcal{B}_H$. Thus we can write the above isomorphism as

$$D(H) \cong \bigoplus_{\phi \in [W_G(H) \backslash \mathcal{B}_H]} RW_G(H)/W_G(H, \phi) \cong \bigoplus_{\phi \in [W_G(H) \backslash \mathcal{B}_H]} \text{Ind}_{W_G(H, \phi)}^{W_G(H)} R.$$

Therefore we have proved the following.

Corollary 4.4. *Let D be a ρ -module with a G -stable basis $\mathcal{B} = (\mathcal{B}_H)_{H \leq G}$. Then, there is an isomorphism of R -modules*

$$\widehat{\text{Ext}}_\gamma^i(\rho, D)(G) \cong \bigoplus_{H \leq G} \bigoplus_{\phi \in [W_G(H) \backslash \mathcal{B}_H]} \widehat{H}^i(W_G(H, \phi), R).$$

5 Integrality conditions

Throughout this section, assume that $R = \mathbb{Z}$. In this section, we find integrality conditions for the elements of the Mackey functor D^+ . An element $f \in D^+(G)$ is called integral if it is in the image of the mark morphism φ_G . Thus we are to find conditions on f to be in the image of φ_G . Note that in [5], Boltje found integrality conditions when the ρ -module D has a G -stable basis. We generalize his result to arbitrary D , as follows.

Theorem 5.1. *Let $f = (f_K)_{K \leq G} \in D^+(G)$. Then the following are equivalent.*

1. f is in the image of φ_G .
2. The equality

$$q_Q^H \left(\sum_{H \leq K \leq Q} \mu(H, K) r_H^K(f_K) \right) = 0 \quad (*)$$

holds for any $H \leq Q \leq G$.

3. The equality (*) holds for any $H \leq G$ and for $Q = N_G(H)$.
4. The equality (*) holds for any $H \leq G$ and for any $H \leq P \leq N_G(H)$ such that P/H is a Sylow subgroup of $N_G(H)/H$.

Proof. The proof is very similar to the proof of Theorem 2.2 in [5]. We include it for the convenience of the reader. It is clear that (2) implies (3) and (4). We shall show that (1) implies (2) and each (3) and (4) implies (1). Note that the equality (*) is just the H -th component of $q_Q \circ \eta_Q(r_Q^G f)$. But then (1) implies (2) trivially since if f is in the image of φ_G then $r_Q^G f$ is in the image of φ_Q . So it remains to show that each (3) and (4) implies (1).

First suppose that the equality (*) is satisfied for each H and for $Q = N_G(H)$. Let $\{H_1, H_2, \dots, H_n\}$ be a set of representatives of G -conjugacy classes of subgroups of G of maximal order with f_{H_i} non-zero. We prove the claim by induction on $o(f) := |H_i|$. It is trivial if $f = 0$. Otherwise applying the condition (3) for H_i , we get $f_{H_i} = \mathcal{N}_{W_G(H_i)}(a_i)$ for some $a_i \in D(H_i)$. Now consider the element

$$f' = f - \sum_{H_i} \varphi_G(t_{H_i}^G \otimes a_i) .$$

It is clear that $\varphi_G(t_{H_i}^G \otimes a_i)$ is non-zero only at subgroups of G which are conjugate to a subgroup of H_i , and it is f_{H_i} at H_i . Therefore $o(f') < o(f)$ and by induction f' is in the image of φ_G . Hence, f is integral.

Finally assume that the condition (4) holds. We argue by induction on $o(f)$ again. Let H_i be as above and fix an $i \in \{1, 2, \dots, n\}$. It suffices to show that $q_{N_G(H_i)}^{H_i}(f_{H_i}) = 0$. There is nothing to prove if $H_i = N_G(H_i)$. Otherwise the equality (*) applied to $H = H_i$ and P being a subgroup as in part (4) gives that $q_P^{H_i}(f_{H_i}) = 0$. But this holds for all Sylow subgroups of $N_G(H_i)/H_i$. Since the restriction map

$$\widehat{H}^0(W_G(H_i), D(H_i)) \rightarrow \bigoplus_{P \in \text{Syl}(W_G(H_i))} \widehat{H}^0(P, D(H_i))$$

is injective, the result follows. \square

The integrality conditions lie in the group $\text{Ext}_\gamma^0(\rho, D)$. But it is possible to describe these conditions via the group $\text{Ext}_\gamma^{-1}(\rho, (\mathbb{Q}/\mathbb{Z})D)$ in the following way. As shown in Section 3, the mark morphism is an isomorphism over \mathbb{Q} . So given a subgroup H of G and an element $(x_K)_{K \leq H}$ in $D^+(H)$, there exists an element X_H in $\mathbb{Q}D_+(H)$ such that $\varphi_H(X_H) = (x_K)_{K \leq H}$. Now the element $(x_K)_{K \leq H}$ is called integral if X_H is in $D_+(H)$. Now consider the following commutative diagram

$$\begin{array}{ccccccc} \widehat{\text{Ext}}_\gamma^{-1}(\rho, D)(H) & \longrightarrow & D_+(H) & \xrightarrow{\varphi_H} & D^+(H) & \xrightarrow{\psi_H} & \widehat{\text{Ext}}_\gamma^0(\rho, D)(H) \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}D_+(H) & \xrightarrow{\varphi_H} & \mathbb{Q}D^+(H) & \longrightarrow & 0 \\ & & \downarrow \pi_H & & \downarrow \pi_H & & \\ \widehat{\text{Ext}}_\gamma^{-1}(\rho, (\mathbb{Q}/\mathbb{Z})D)(H) & \longrightarrow & (\mathbb{Q}/\mathbb{Z})D_+(H) & \xrightarrow{\varphi_H} & (\mathbb{Q}/\mathbb{Z})D^+(H) & & \end{array}$$

with exact rows. Note that since $(x_K)_{K \leq H}$ is in $D^+(H)$, we have $\varphi_H \circ \pi_H(X_H) = 0$. Hence $\pi_H(X_H)$ comes from an element, say ζ_H , in $\widehat{\text{Ext}}_\gamma^{-1}(\rho, (\mathbb{Q}/\mathbb{Z})D)(H)$. By the commutativity of the diagram, $(x_K)_{K \leq H}$ is integral if and only if ζ_H is zero. Note that ζ_H is the class associated to $\psi_H(x_H)$ under the isomorphism

$$\widehat{\text{Ext}}_\gamma^{-1}(\rho, (\mathbb{Q}/\mathbb{Z})D) \cong \widehat{\text{Ext}}_\gamma^0(\rho, D)$$

induced by the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$. This illustrates one of the ways we can use homological algebra to understand integrality conditions. We also tried to find some applications of our Tate sequence to canonical induction formulae. We think there should exist some homological conditions which are expressible in terms of Tate cohomology and that are equivalent to the integrality of a chosen canonical induction formula. But, we were not able to find such a condition. We leave this as an open problem.

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