

# HIGHER LIMITS OVER THE FUSION ORBIT CATEGORY

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ABSTRACT. The fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  of a discrete group  $G$  over a collection  $\mathcal{C}$  is the category whose objects are the subgroups  $H$  in  $\mathcal{C}$ , and whose morphisms  $H \rightarrow K$  are given by the  $G$ -maps  $G/H \rightarrow G/K$  modulo the action of the centralizer group  $C_G(H)$ . We show that the higher limits over  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  can be computed using the hypercohomology spectral sequences coming from the Dwyer  $G$ -spaces for centralizer and normalizer decompositions for  $G$ .

If  $G$  is the discrete group realizing a saturated fusion system  $\mathcal{F}$ , then these hypercohomology spectral sequences give two spectral sequences that converge to the cohomology of the centric orbit category  $\mathcal{O}^c(\mathcal{F})$ . This allows us to apply our results to the sharpness problem for the subgroup decomposition of a  $p$ -local finite group. We prove that the subgroup decomposition for every  $p$ -local finite group is sharp (over  $\mathcal{F}$ -centric subgroups) if it is sharp for every  $p$ -local finite group with nontrivial center. We also show that for every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , the subgroup decomposition is sharp if and only if the normalizer decomposition is sharp.

## 1. INTRODUCTION

A *saturated fusion system*  $\mathcal{F}$  over a finite  $p$ -group  $S$  is a category whose objects are subgroups of  $S$  and whose morphisms are injective group homomorphisms satisfying certain axioms (see Definitions 2.1 and 2.2). The main example of a saturated fusion system is the fusion system of a finite group defined over one of its Sylow  $p$ -subgroups whose morphisms are the group homomorphisms induced by conjugations in  $G$ . However, fusion systems that do not come from finite groups also exist.

A subgroup  $P \leq S$  is called  *$\mathcal{F}$ -centric* if  $C_S(Q) \leq Q$  for every subgroup  $Q$  isomorphic to  $P$  in  $\mathcal{F}$ . We denote by  $\mathcal{F}^c$  the full subcategory of  $\mathcal{F}$  whose objects are the  $\mathcal{F}$ -centric subgroups of  $S$ . The *orbit category*  $\mathcal{O}(\mathcal{F})$  of a fusion system  $\mathcal{F}$  is the category whose objects are the subgroups in  $S$  and whose morphisms are given by

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) := \mathrm{Inn}(Q) \backslash \mathrm{Mor}_{\mathcal{F}}(P, Q).$$

The orbit category over the collection of  $\mathcal{F}$ -centric subgroups is denoted by  $\mathcal{O}^c(\mathcal{F})$  and is called the *centric orbit category of  $\mathcal{F}$* .

A  *$p$ -local finite group* is a triple  $(S, \mathcal{F}, \mathcal{L})$  where  $S$  is a finite  $p$ -group,  $\mathcal{F}$  is a saturated fusion system over  $S$ , and  $\mathcal{L}$  is a centric linking system associated to  $\mathcal{F}$ . A centric linking system  $\mathcal{L}$  associated to a fusion system  $\mathcal{F}$  is a category whose objects are the  $\mathcal{F}$ -centric subgroups of  $S$ , and the morphisms are defined such that there is a quotient functor  $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$ . As part of the structure of  $\mathcal{L}$ , there exist also distinguished monomorphisms  $\delta_P : P \rightarrow \mathrm{Aut}_{\mathcal{L}}(P)$  defined for every  $P \in \mathcal{F}^c$  satisfying certain properties (see [6, Def 1.7]). In Section 10 we give a more recent definition of a linking system using the transporter category  $\mathcal{T}_S^{\mathcal{F}^c}$  (see Definition 10.9). It

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*Key words and phrases.* Fusion systems, higher limits, orbit category,  $p$ -local finite group, cohomology of small categories.

2020 *Mathematics Subject Classification.* Primary: 55R35; Secondary: 18G10, 20D20, 55N91, 18G40.

has been proved by Chermak [9] that for every saturated fusion system  $\mathcal{F}$ , there exists a unique centric linking system  $\mathcal{L}$  associated to  $\mathcal{F}$  (see [2, §III.4] for details).

The classifying space of a  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is defined to be the Bousfield-Kan  $p$ -completion of the geometric realization  $|\mathcal{L}|_p^\wedge$  of the category  $\mathcal{L}$ . There is a subgroup homology decomposition for a  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , similar to the subgroup decomposition for finite groups, introduced by Broto, Levi, and Oliver [6, Prop 2.2]. They showed that for every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , there is a homotopy equivalence

$$|\mathcal{L}| \simeq \operatorname{hocolim}_{\mathcal{O}^c(\mathcal{F})} \tilde{B} \quad (1.1)$$

where  $\tilde{B} : \mathcal{O}^c(\mathcal{F}) \rightarrow \operatorname{Top}$  is a functor such that  $\tilde{B}(P)$  is homotopy equivalent to the classifying space  $BP$  for every  $P \in \mathcal{F}^c$ . The Bousfield-Kan spectral sequence associated to the above homology decomposition gives a spectral sequence

$$E_2^{s,t} = \lim_{\mathcal{O}^c(\mathcal{F})}^s H^t(-; \mathbb{F}_p) \Rightarrow H^{s+t}(|\mathcal{L}|; \mathbb{F}_p)$$

where  $H^t(-; \mathbb{F}_p)$  denotes the contravariant functor  $\mathcal{O}^c(\mathcal{F}) \rightarrow \mathbb{F}_p\text{-Mod}$  that sends an  $\mathcal{F}$ -centric subgroup  $P \leq S$  to its group cohomology  $H^t(P; \mathbb{F}_p)$  in  $\mathbb{F}_p$ -coefficients.

**Definition 1.1.** The subgroup decomposition for  $(S, \mathcal{F}, \mathcal{L})$  is said to be *sharp* if the associated Bousfield-Kan spectral sequence collapses at the  $E_2$ -page to the vertical axis, i.e., if  $E_2^{s,t} = 0$  for all  $s > 0$  and for all  $t \geq 0$ .

Note that if the subgroup decomposition is sharp, then the edge homomorphism

$$H^*(|\mathcal{L}|; \mathbb{F}_p) \rightarrow E_2^{0,*} = \lim_{P \in \mathcal{O}^c(\mathcal{F})} H^*(P; \mathbb{F}_p)$$

is an isomorphism. The limit term on the right-hand side is called the *cohomology of the fusion system*  $\mathcal{F}$  and it is denoted by  $H^*(\mathcal{F}; \mathbb{F}_p)$ . When this isomorphism holds, we say that the Cartan-Eilenberg theorem holds for  $(S, \mathcal{F}, \mathcal{L})$ .

Diaz and Park [11, Thm B] proved that the subgroup decomposition is sharp if  $\mathcal{F}$  is a fusion system realized by a finite group. Broto, Levi, and Oliver [6, Thm 5.8] proved that the Cartan-Eilenberg theorem holds for every saturated fusion system. These two results suggest that the subgroups decomposition is sharp for every saturated fusion system. This is stated as a conjecture by Diaz and Park [11] and as a question by Ashbacher and Oliver [2, Ques 7.12].

**Conjecture 1.2.** Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. Then for every  $n \geq 0$  and every  $i \geq 1$ ,

$$\lim_{\mathcal{O}^c(\mathcal{F})}^i H^n(-; \mathbb{F}_p) = 0.$$

This conjecture is the main motivation for us to study the higher limits over the centric orbit category  $\mathcal{O}^c(\mathcal{F})$ . To compute the higher limits over  $\mathcal{O}^c(\mathcal{F})$ , we propose to use an ambient discrete group  $G$  that realizes the fusion system  $\mathcal{F}$ . By theorems of Leary and Stancu [20] and Robinson [30], for every (saturated) fusion system  $\mathcal{F}$ , there is a discrete group  $G$  (possibly infinite) with a finite Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} \cong \mathcal{F}_S(G)$ . For a discrete group  $G$  and a collection  $\mathcal{C}$  of subgroups of  $G$  (always assumed to be closed under conjugation), the following categories are defined:

- (1) The *orbit category*  $\mathcal{O}_{\mathcal{C}}(G)$  of  $G$  is the category whose objects are subgroups  $H \in \mathcal{C}$ , and whose morphisms  $\operatorname{Mor}_{\mathcal{O}_{\mathcal{C}}(G)}(H, K)$  are given by  $G$ -maps  $G/H \rightarrow G/K$ .

- (2) The *fusion category*  $\mathcal{F}_{\mathcal{C}}(G)$  of  $G$  is the category whose objects are the subgroups  $H \in \mathcal{C}$ , and whose morphisms  $H \rightarrow K$  are given by conjugation maps  $c_g : H \rightarrow K$  for an element in  $g \in G$ .

For every  $H, K \in \mathcal{C}$ , let  $N_G(H, K) := \{g \in G \mid gHg^{-1} \leq K\}$ . The category whose objects are subgroups  $H \in \mathcal{C}$  and whose morphisms from  $H$  to  $K$  are given by  $N_G(H, K)$  is called the *transporter category* of  $G$  and is denoted by  $\mathcal{T}_{\mathcal{C}}(G)$ . Both the orbit category and the fusion category can be viewed as the quotient category of the transporter category (see Section 2.3 for details).

**Definition 1.3.** The *fusion orbit category*  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  of a discrete group  $G$  over a collection  $\mathcal{C}$  is the category whose objects are subgroups  $H \in \mathcal{C}$ , and whose morphisms are given by

$$\text{Mor}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}(H, K) := K \backslash N_G(H, K) / C_G(H)$$

for every  $H, K \in \mathcal{C}$ .

The fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  is a quotient category of both the orbit category  $\mathcal{O}_{\mathcal{C}}(G)$  and the fusion category  $\mathcal{F}_{\mathcal{C}}(G)$ . If  $G$  is a discrete group realizing a saturated fusion system  $\mathcal{F}$  and if we take  $\mathcal{C}$  to be the collection of all  $p$ -subgroups in  $G$  that are conjugate to a  $\mathcal{F}$ -centric subgroup in  $S$ , then  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  is equivalent to  $\mathcal{O}^c(\mathcal{F})$  as categories (see Lemma 2.5). This allows us to calculate the higher limits over  $\mathcal{O}^c(\mathcal{F})$  as the higher limits over  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  for a discrete group  $G$  realizing  $\mathcal{F}$ . The idea of using an ambient discrete group for proving theorems for abstract fusion systems was also used by Libman in [22].

Let  $G$  be a discrete group and  $\mathcal{C}$  be a collection of subgroups of  $G$ . To compute the higher limits of fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ , we consider the hypercohomology spectral sequences coming from certain  $G$ -spaces. Let  $X$  be a  $G$ -CW-complex and  $R$  be a commutative ring with unity. Associated with  $X$ , there is a chain complex of  $RO_{\mathcal{C}}(G)$ -modules  $C_*(X^?; R)$  defined by

$$H \rightarrow C_*(X^H; R) \quad \text{and} \quad (f : G/H \rightarrow G/K) \rightarrow (f^* : C_*(X^K; R) \rightarrow C_*(X^H; R))$$

for every  $H \in \mathcal{C}$ . When the collection  $\mathcal{C}$  is large enough to include all the isotropy subgroups of  $X$ , the complex  $C_*(X^?; R)$  is a chain complex of projective  $RO_{\mathcal{C}}(G)$ -modules. The Bredon cohomology of the space  $X$  is defined using this chain complex.

**Definition 1.4.** Let  $\mathcal{O}(G)$  denote the orbit category of  $G$  over all subgroups of  $G$ , and let  $M$  be an  $RO(G)$ -module. The (*ordinary*) *Bredon cohomology* of  $X$  with coefficients in  $M$  is defined by

$$H_{\mathcal{O}(G)}^*(X^?; M) := H^*(\text{Hom}_{RO(G)}(C_*(X^?; R), M)).$$

If the isotropy subgroups of  $X$  do not lie in  $\mathcal{C}$ , then it is still possible to define the Bredon cohomology using hypercohomology. In this case the Bredon cohomology is defined by

$$H_{\mathcal{O}_{\mathcal{C}}(G)}^*(X^?; M) := H^*(\text{Hom}_{RO_{\mathcal{C}}(G)}(\text{Tot}^{\oplus}(P_{*,*}); M))$$

where  $P_{*,*}$  is a Cartan-Eilenberg resolution of the complex  $C_*(X^?; R)$  as a chain complex of  $RO_{\mathcal{C}}(G)$ -modules. This definition of the Bredon cohomology is due to Symonds [34].

The definitions above can be modified to obtain chain complexes over the fusion orbit category and to define a fusion orbit category version of the Bredon cohomology. For a  $G$ -CW-complex  $X$ , let  $C_*(C_G(?) \backslash X^?; R)$  denote the chain complex of  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules defined by

$$\begin{aligned} H &\rightarrow C_*(C_G(H) \backslash X^H; R) \\ (f : G/H \rightarrow G/K) &\rightarrow (f^* : C_*(C_G(K) \backslash X^K; R) \rightarrow C_*(C_G(H) \backslash X^H; R)) \end{aligned}$$

for every  $H, K \in \mathcal{C}$ . These chain complexes are introduced by Lück in [24] and [25] to give a more efficient way to compute equivariant Chern characters.

As in the orbit category case, if the isotropy subgroups of  $X$  lie in  $\mathcal{C}$ , then  $C_*(C_G(?) \backslash X^?; R)$  is a chain complex of projective  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. When  $\mathcal{C}$  does not include all the isotropy subgroups of  $X$ , in general the complex  $C_*(C_G(?) \backslash X^?; R)$  is not a chain complex of projective  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. In this case we define the fusion orbit category version of the Bredon cohomology using hypercohomology.

**Definition 1.5.** Let  $P_{*,*}$  be a Cartan-Eilenberg resolution of the chain complex  $C_*(C_G(?) \backslash X^?; R)$  as a chain complex of  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. The *fusion Bredon cohomology* of a  $G$ -CW-complex  $X$  is defined by

$$H_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(X^?; M) := H^*(\mathrm{Hom}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}(\mathrm{Tot}^{\oplus}(P_{*,*}); M)).$$

When the family  $\mathcal{C}$  includes all isotropy subgroups of  $X$ , for every  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ , there is an isomorphism

$$H_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(X^?; M) \cong H_{\mathcal{O}_{\mathcal{C}}(G)}^*(X^?; \mathrm{Res}_{pr} M)$$

where  $pr : \mathcal{O}_{\mathcal{C}}(G) \rightarrow \overline{\mathcal{F}}_{\mathcal{C}}(G)$  is the projection functor (see Proposition 5.5). However in general the fusion Bredon cohomology is not isomorphic to the Bredon cohomology (see Example 5.6).

There are two spectral sequences converging to the fusion Bredon cohomology of  $X$  coming from the two different ways of filtering the double complex  $\mathrm{Hom}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}(P_{*,*}, M)$  (see Proposition 5.7). One of these spectral sequences can be used to prove that if  $X$  is a  $G$ -CW-complex such that for every  $H \in \mathcal{C}$ , the orbit space  $C_G(H) \backslash X^H$  is  $R$ -acyclic, then for any  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ ,

$$H_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(X^?; M) \cong H^*(\overline{\mathcal{F}}_{\mathcal{C}}(G); M).$$

In this case, the second spectral sequence gives a spectral sequence that converges to the  $H^*(\overline{\mathcal{F}}_{\mathcal{C}}(G); M)$  (see Theorem 5.8). We apply this spectral sequence to the Dwyer spaces for the homology decompositions of  $G$  (see Section 6 for the definitions of Dwyer spaces). We show that for the following choices of  $X$  and the collection  $\mathcal{C}$ , the condition on  $X$  given in Theorem 5.8 holds (see Propositions 6.7 and 6.12):

- (1)  $X = X_{\mathcal{E}}^{\alpha} = EA_{\mathcal{E}}$  is the Dwyer space for the centralizer decomposition over the collection  $\mathcal{E}$  of all nontrivial elementary abelian  $p$ -subgroups in  $G$ , and  $\mathcal{C}$  is any collection of nontrivial  $p$ -subgroups in  $G$ .
- (2)  $X = X_{\mathcal{C}}^{\delta} = |\mathcal{C}|$  is the Dwyer space for the normalizer decomposition over  $\mathcal{C}$ , and  $\mathcal{C}$  is any collection of nontrivial  $p$ -subgroups of  $G$  closed under taking products (see Definition 6.11).

As a consequence, we obtain two spectral sequences that converge to the cohomology of fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  (see Propositions 7.7 and 7.11). These spectral sequences can be considered as the centralizer and normalizer decompositions for the cohomology of fusion orbit category.

If we take  $G$  to be a discrete group realizing a saturated fusion system  $\mathcal{F}$ , then the hypercohomology spectral sequences that we constructed give two spectral sequences that converge to the cohomology of the centric orbit category  $\mathcal{O}^c(\mathcal{F})$ . We now explain these spectral sequences.

Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $\mathcal{F}^e$  denote the full subcategory of  $\mathcal{F}$  whose objects are the collection of all nontrivial elementary abelian  $p$ -subgroups of  $S$  which are fully  $\mathcal{F}$ -centralized. For every  $E \in \mathcal{F}^e$ , let  $C_{\mathcal{F}}(E)$  denote the centralizer fusion system over  $C_S(E)$  as defined in Definition 8.1.

**Theorem 1.6.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ ,  $R$  be a commutative ring with unity, and  $M$  be an  $R\mathcal{O}^c(\mathcal{F})$ -module. For every  $j \geq 0$ , let  $\mathcal{H}_{M, C_{\mathcal{F}}}^j : \mathcal{F}^e \rightarrow R\text{-Mod}$  denote the functor defined in Lemma 9.3 such that for every  $E \in \mathcal{F}^e$ ,*

$$\mathcal{H}_{M, C_{\mathcal{F}}}^j(E) = H^j(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M).$$

*Then, there is a spectral sequence*

$$E_2^{s,t} = \lim_{\mathcal{F}^e}^s \mathcal{H}_{M, C_{\mathcal{F}}}^t \Rightarrow H^{s+t}(\mathcal{O}^c(\mathcal{F}); M).$$

We call this spectral sequence the centralizer decomposition for the cohomology of the centric orbit category. We have a similar spectral sequence involving normalizer fusion systems that can be considered as the normalizer decomposition which we now describe.

Let  $\mathcal{F}$  be a saturated fusion system over  $S$  and let  $\overline{sd}(\mathcal{F}^c)$  denote the poset category of  $\mathcal{F}$ -conjugacy classes of chains  $\sigma = (P_0 < P_1 < \cdots < P_n)$  of  $\mathcal{F}$ -centric subgroups  $P_i$  of  $S$ . For every fully  $\mathcal{F}$ -normalized chain  $\sigma$ , let  $N_{\mathcal{F}}(\sigma)$  denote the normalizer fusion system of  $\sigma$  as defined in Definition 10.2.

**Theorem 1.7.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and  $M$  be an  $\mathbb{Z}_{(p)}\mathcal{O}(\mathcal{F})$ -module. For every  $j \geq 0$ , let  $\mathcal{H}_{M, N_{\mathcal{F}}}^j : \overline{sd}(\mathcal{F}^c)^{op} \rightarrow \mathbb{Z}_{(p)}\text{-Mod}$  denote the functor defined in Lemma 10.7 such that for every  $[\sigma] \in \overline{sd}(\mathcal{F}^c)$ ,*

$$\mathcal{H}_{M, N_{\mathcal{F}}}^j([\sigma]) = H^j(\mathcal{O}^c(N_{\mathcal{F}}(\sigma)); \text{Res}_{\mathcal{O}^c(N_{\mathcal{F}}(\sigma))}^{\mathcal{O}(\mathcal{F})} M).$$

*Then, there is a spectral sequence*

$$E_2^{s,t} = \lim_{\overline{sd}(\mathcal{F}^c)}^s \mathcal{H}_{M, N_{\mathcal{F}}}^t \Rightarrow H^{s+t}(\mathcal{O}^c(\mathcal{F}); \text{Res}_{\mathcal{O}^c(\mathcal{F})}^{\mathcal{O}(\mathcal{F})} M).$$

We apply these spectral sequences to the sharpness problem stated at the beginning. A subgroup  $Q \leq S$  is *central* in  $\mathcal{F}$  if  $C_{\mathcal{F}}(Q) = \mathcal{F}$ . The product of all central subgroups in  $\mathcal{F}$  is called the center of  $\mathcal{F}$  and denoted by  $Z(\mathcal{F})$ . Using the centralizer spectral sequence constructed in Theorem 1.6, we prove the following:

**Theorem 1.8.** *If the subgroup decomposition is sharp for every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  with  $Z(\mathcal{F}) \neq 1$ , then it is sharp for every  $p$ -local finite group.*

This reduces the sharpness problem for  $p$ -local finite groups to the ones with nontrivial center. The main ingredient for the proof of Theorem 1.8 is the sharpness of the centralizer decomposition for  $p$ -local finite groups. The centralizer decomposition for a  $p$ -local finite group is introduced by Broto, Levi, and Oliver in [6, Thm 2.6] and the proof of the sharpness of the centralizer decomposition can be found in the proof of [6, Thm 5.8].

Next we consider the normalizer decomposition of  $p$ -local finite groups introduced by Libman [21]. Applying the spectral sequence in Theorem 1.7, we show that for every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , there is an isomorphism

$$\lim_{\mathcal{O}^c(\mathcal{F})}^i H^n(-; \mathbb{F}_p) \cong \lim_{\overline{sd}(\mathcal{F}^c)}^i H^n(N_{\mathcal{F}}(-); \mathbb{F}_p)$$

(see Theorem 10.16). The higher limits on the right are isomorphic to the  $E_2^{i,n}$ -term in the Bousfield-Kan spectral sequence for the normalizer decomposition for  $(S, \mathcal{F}, \mathcal{L})$ . As a consequence we obtain the following theorem.

**Theorem 1.9.** *For every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , the subgroup decomposition is sharp if and only if the normalizer decomposition is sharp (over the collection of  $\mathcal{F}$ -centric subgroups).*

In the last section of the paper, we consider the Dwyer space  $X := X_{\mathcal{C}}^{\beta}$  for the subgroup decomposition. When  $G$  is an infinite group, it is not true in general that for every  $H \in \mathcal{C}$ , the orbit space  $C_G(H) \backslash X^H$  is  $R$ -acyclic (see Example 11.6). However we show that this holds when  $G$  is a finite group,  $R = \mathbb{Z}_{(p)}$ , and  $\mathcal{C}$  is a collection of  $p$ -subgroups of  $G$  closed under taking  $p$ -overgroups (see Proposition 11.2). As a consequence, we obtain the following.

**Theorem 1.10.** *If  $G$  is a finite group and  $\mathcal{C}$  is a collection of  $p$ -subgroups of  $G$  closed under taking  $p$ -overgroups, then for every  $\mathbb{Z}_{(p)}\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ ,*

$$H^*(\overline{\mathcal{F}}_{\mathcal{C}}(G); M) \cong H^*(\mathcal{O}_{\mathcal{C}}(G); \text{Res}_{pr} M)$$

where the isomorphism is induced by the projection functor  $pr : \mathcal{O}_{\mathcal{C}}(G) \rightarrow \overline{\mathcal{F}}_{\mathcal{C}}(G)$ .

Using Theorem 1.10, we also prove the following vanishing result.

**Theorem 1.11.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$  realized by a finite group and  $\mathcal{C}$  be a collection of subgroups of  $S$  closed under taking  $p$ -overgroups. If  $\mathcal{C}$  includes all  $\mathcal{F}$ -centric and  $\mathcal{F}$ -radical subgroups in  $S$ , then for all  $n \geq 0$  and  $i > 0$ ,*

$$\lim_{\mathcal{O}(\mathcal{F}_{\mathcal{C}})}^i H^n(-; \mathbb{F}_p) = 0.$$

This generalizes the vanishing result proved by Diaz and Park in [11, Thm B] for fusion systems realized by finite groups and for the collection of  $\mathcal{F}$ -centric subgroups. We end the paper with an explicit example of an infinite group  $G$  where Theorem 1.10 no longer holds for the collection of all  $p$ -subgroups (see Example 11.6). For the infinite group  $G$  in this example, the subgroup decomposition for  $BG$  is not sharp, but Conjecture 1.2 still holds for the fusion system  $\mathcal{F} = \mathcal{F}_S(G)$ .

*Notation:* Throughout the paper,  $G$  is a discrete group and  $p$  is a prime number. When there exists a Sylow  $p$ -subgroup of  $G$ , it is denoted by  $S$ . In homological algebra sections, we work over an arbitrary commutative ring  $R$  with unity. When we say  $\mathcal{C}$  is a collection of subgroups in  $G$ , we always assume that  $\mathcal{C}$  is closed under conjugation. The subcategory of the fusion system  $\mathcal{F}$  with object set  $\mathcal{C}$  is denoted by  $\mathcal{F}_{\mathcal{C}}$ . The full subcategory of  $\mathcal{F}$  generated by the collection of  $\mathcal{F}$ -centric subgroups in  $S$  is denoted by  $\mathcal{F}^c$ , and the orbit category of  $\mathcal{F}$  defined over  $\mathcal{F}$ -centric subgroups is denoted by  $\mathcal{O}^c(\mathcal{F})$ .

**Acknowledgements:** We would like to thank the referee for a careful reading of the paper and for many corrections and valuable suggestions. In particular current versions of Lemma 4.6 and Lemma 8.13 were suggested by the referee along with many other comments that improved the paper substantially.

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## 2. FUSION SYSTEMS AND THE ORBIT CATEGORY

In this section we introduce some basic definitions related to fusion systems and orbit categories. For further details on this material, we refer the reader to [1], [2], and [10].

### 2.1. Fusion systems.

**Definition 2.1.** A *fusion system*  $\mathcal{F}$  over a finite  $p$ -group  $S$  is a category whose objects are subgroups of  $S$  and whose set of morphisms  $\text{Mor}_{\mathcal{F}}(P, Q)$  between two subgroups  $P, Q \leq S$  satisfies the following properties:

- (1)  $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q) \subseteq \text{inj}(P, Q)$  where  $\text{Hom}_S(P, Q)$  denotes the set of all conjugation maps  $c_s : P \rightarrow Q$  induced by an element in  $S$ .
- (2) If  $\varphi : P \rightarrow Q$  is a morphism in  $\mathcal{F}$ , then  $\varphi^{-1} : \varphi P \rightarrow P$  is a morphism in  $\mathcal{F}$ .

Two subgroups  $P$  and  $Q$  are  $\mathcal{F}$ -conjugate if there is an isomorphism  $\varphi : P \rightarrow Q$  in  $\mathcal{F}$ . In this case we write  $P \sim_{\mathcal{F}} Q$ . A subgroup  $P \leq S$  is *fully  $\mathcal{F}$ -normalized* if  $|N_S(P)| \geq |N_S(P')|$  for every  $P' \sim_{\mathcal{F}} P$ . A subgroup  $P \leq S$  is *fully  $\mathcal{F}$ -centralized* if  $|C_S(P)| \geq |C_S(P')|$  for every  $P' \sim_{\mathcal{F}} P$ . We say  $P$  is *fully  $\mathcal{F}$ -automized* if  $\text{Aut}_S(P)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ . A subgroup  $Q \leq S$  is called  *$\mathcal{F}$ -receptive* if every morphism  $\varphi : P \rightarrow Q$  in  $\mathcal{F}$  extends to a morphism  $\tilde{\varphi} : N_{\varphi} \rightarrow S$  where

$$N_{\varphi} := \{x \in N_S(P) \mid \exists y \in N_S(\varphi(P)) \text{ such that } c_x = \varphi^{-1} \circ c_y \circ \varphi\}.$$

**Definition 2.2.** [1, Part I, Prop 2.5] A fusion system  $\mathcal{F}$  over  $S$  is *saturated* if it satisfies the following properties:

- (1) If  $P \leq S$  is fully  $\mathcal{F}$ -normalized, then  $P$  is fully  $\mathcal{F}$ -centralized and fully  $\mathcal{F}$ -automized.
- (2) If  $P \leq S$  is fully  $\mathcal{F}$ -centralized, then  $P$  is  $\mathcal{F}$ -receptive.

A subgroup  $P \leq S$  is called  *$\mathcal{F}$ -centric* if  $C_S(Q) \leq Q$  for every  $Q \leq S$  such that  $Q \sim_{\mathcal{F}} P$ . We denote by  $\mathcal{F}^c$  the full subcategory of  $\mathcal{F}$  whose object set is the set of  $\mathcal{F}$ -centric subgroups in  $S$ .

**Definition 2.3.** The *orbit category*  $\mathcal{O}(\mathcal{F})$  of the fusion system  $\mathcal{F}$  is the quotient category of  $\mathcal{F}$  whose morphisms are defined by

$$\text{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \text{Inn}(Q) \backslash \text{Mor}_{\mathcal{F}}(P, Q).$$

One can similarly define the orbit category  $\mathcal{O}(\mathcal{F}^c)$  of the subcategory  $\mathcal{F}^c$ . We define the centric orbit category  $\mathcal{O}^c(\mathcal{F})$  to be the full subcategory of  $\mathcal{O}(\mathcal{F})$  generated by  $\mathcal{F}$ -centric subgroups in  $S$ . Note that  $\mathcal{O}^c(\mathcal{F}) \cong \mathcal{O}(\mathcal{F}^c)$ .

**2.2. Realization of fusion systems.** For a discrete group  $G$ , a finite  $p$ -subgroup  $S$  of  $G$  is called a *Sylow  $p$ -subgroup* of  $G$ , if for every finite  $p$ -subgroup  $P$ , there is an element  $g \in G$  such that  $gPg^{-1} \leq S$ . In general, discrete groups do not have Sylow  $p$ -subgroups even when the orders of finite subgroups are bounded from above. A simple example of such a group is  $G = C_2 * C_2$ . If  $G$  has a Sylow  $p$ -subgroup  $S$ , then the fusion system  $\mathcal{F}_S(G)$  is defined to be the category whose objects are subgroups of  $S$ , and whose morphisms  $P \rightarrow Q$  are the group homomorphisms induced by conjugation by elements of  $G$ .

If  $G$  is a finite group and  $S$  is a Sylow  $p$ -subgroup of  $G$ , then  $\mathcal{F}_S(G)$  is saturated (see [10, Thm 4.12]). In general this is not true for fusion systems induced by infinite groups. A fusion system  $\mathcal{F}$  over  $S$  is said to be *realized by  $G$* , if  $G$  has a Sylow  $p$ -subgroup isomorphic to  $S$  and there is an isomorphism of categories  $\mathcal{F} \cong \mathcal{F}_S(G)$ . We have the following realization theorem for fusion systems.

**Theorem 2.4** (Leary-Stancu [20], Robinson [30]). *Every fusion system  $\mathcal{F}$  is realized by a (possibly infinite) discrete group  $G$  constructed as the fundamental group of a graph of groups  $(\mathcal{G}, Y)$  with finite vertex and edge groups  $G_v$  and  $G_e$  over a finite graph  $Y$ .*

The graph  $Y$  and the vertex groups  $G_v$  are described explicitly in each construction. Leary and Stancu [20] constructs  $G$  as an iterated HNN-extension (with only one vertex) and their construction works for any fusion system (even if it is not necessarily saturated). Robinson model [30] is a generalized amalgamation with vertex groups given by a family of finite groups  $G_i$  which generate the fusion system. By this we mean that for each  $i$ , there is a monomorphism  $\varphi_i : S_i \rightarrow S$  from a Sylow  $p$ -subgroup  $S_i$  of  $G_i$  to  $S$  such that  $\mathcal{F}$  is generated by the images of  $\mathcal{F}_{S_i}(G_i)$  under the map induced by  $\varphi_i$ . By Alperin's fusion theorem [10, Thm 4.52] and by the model theorem for constrained fusion systems [5], a collection of finite groups  $\{G_i\}$  that generates  $\mathcal{F}$  can always be found when  $\mathcal{F}$  is a saturated fusion system.

**2.3. Fusion orbit category.** Let  $G$  be a discrete group and  $\mathcal{C}$  be a collection of subgroups of  $G$ . The transporter category  $\mathcal{T}_{\mathcal{C}}(G)$  is the category whose objects are subgroups  $H \in \mathcal{C}$ , and whose morphisms are given by

$$\text{Mor}_{\mathcal{T}_{\mathcal{C}}(G)}(H, K) := N_G(H, K) = \{g \in G \mid gHg^{-1} \leq K\}.$$

The orbit category  $\mathcal{O}_{\mathcal{C}}(G)$  is the category whose objects are subgroups  $H \in \mathcal{C}$ , and whose morphisms are given by  $G$ -maps  $G/H \rightarrow G/K$ . A  $G$ -map  $f : G/H \rightarrow G/K$  such that  $f(H) = g^{-1}K$  can be identified with the coset  $Kg$  satisfying  $gHg^{-1} \leq K$ . This gives an identification of morphism sets

$$\text{Mor}_{\mathcal{O}_{\mathcal{C}}(G)}(H, K) = K \backslash N_G(H, K)$$

where the  $K$ -action of  $N_G(H, K)$  is defined by the left multiplication.

The fusion category  $\mathcal{F}_{\mathcal{C}}(G)$  is the category whose objects are the subgroups  $H \in \mathcal{C}$ , and whose morphisms are given by the group homomorphisms  $c_g : H \rightarrow K$  defined by  $c_g(h) = ghg^{-1}$  for some  $g \in G$ . Each conjugation map  $c_g : H \rightarrow K$  can be identified with the coset  $gC_G(H)$  where  $g \in N_G(H, K)$ , hence there is a bijection

$$\text{Mor}_{\mathcal{F}_{\mathcal{C}}(G)}(H, K) = N_G(H, K)/C_G(H).$$

From these identifications, we see that both  $\mathcal{O}_{\mathcal{C}}(G)$  and  $\mathcal{F}_{\mathcal{C}}(G)$  are isomorphic to quotient categories of the transporter category  $\mathcal{T}_{\mathcal{C}}(G)$ . The fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  is defined in Definition 1.3 with morphism set

$$\text{Mor}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}(H, K) = K \backslash N_G(H, K)/C_G(H).$$

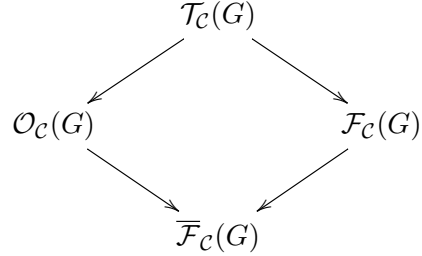


Hence there are bijections

$$\text{Mor}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}(H, K) = \text{Mor}_{\mathcal{O}_{\mathcal{C}}(G)}(H, K)/C_G(H)$$

$$\text{Mor}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}(H, K) = \text{Inn}(K)\backslash\text{Mor}_{\mathcal{F}_{\mathcal{C}}(G)}(H, K)$$

where  $\text{Inn}(K)$  denotes the set of conjugations  $c_k : K \rightarrow K$  induced by elements  $k \in K$ . These bijections show that the fusion orbit category is a quotient category of both the orbit category and the fusion category. We can summarize these in the following diagram:



where each arrow is the projection functor to the corresponding quotient category.

Let  $\mathcal{F}$  be a fusion system over  $S$ , and  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} \cong \mathcal{F}_S(G)$ . If we take  $\mathcal{C}$  to be the collection of all  $p$ -subgroups of  $G$ , then the fusion category  $\mathcal{F}_{\mathcal{C}}(G)$  is equivalent (as a category) to the fusion system  $\mathcal{F}$ . If  $\mathcal{C}$  is the collection of all  $p$ -subgroups of  $G$  that are conjugate to an  $\mathcal{F}$ -centric subgroup in  $S$ , then  $\mathcal{F}_{\mathcal{C}}(G)$  is equivalent to  $\mathcal{F}^c$  as categories. We show in Lemma 8.9 that the collection of all  $p$ -subgroups of  $G$  that are conjugate to an  $\mathcal{F}$ -centric subgroup in  $S$  is equal to the collection of all  $p$ -centric subgroups in  $G$ .

**Lemma 2.5.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  realizing  $\mathcal{F}$ . Let  $\mathcal{C}$  be the collection of all  $p$ -subgroups of  $G$  that are conjugate to an  $\mathcal{F}$ -centric subgroup in  $S$ . Then the orbit category  $\mathcal{O}^c(\mathcal{F})$  is equivalent, as categories, to the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ .*

*Proof.* By definition of  $\mathcal{C}$ , every object in  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  is isomorphic to an  $\mathcal{F}$ -centric subgroup in  $S$ . For  $\mathcal{F}$ -centric subgroups  $P, Q \leq S$ ,

$$\text{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P, Q) = \text{Inn}(Q)\backslash\text{Mor}_{\mathcal{F}^c}(P, Q) = \text{Inn}(Q)\backslash\text{Mor}_{\mathcal{F}_{\mathcal{C}}(G)}(P, Q) = \text{Mor}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}(P, Q).$$

Hence these two categories are equivalent. □

By Proposition 3.9, equivalent categories have isomorphic cohomology groups, hence the higher limits over  $\mathcal{O}^c(\mathcal{F})$  are isomorphic to the higher limits over  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ .

### 3. MODULES OVER THE ORBIT CATEGORY AND INDUCTION

In this section, we introduce the preliminaries on modules over small categories and discuss the restriction and induction functors induced by a functor between two categories. We refer the reader to [23, Chp 9] and [17] for more details on homological algebra over orbit categories.

**3.1. Cohomology of a small category.** Let  $\mathbf{C}$  be a nonempty small category and  $R$  be a commutative ring with unity. A (right)  $RC$ -module  $M$  is a contravariant functor  $M : \mathbf{C} \rightarrow R\text{-mod}$ , and the  $RC$ -module homomorphism  $\varphi : M_1 \rightarrow M_2$  is defined as a natural transformation of functors. The category of  $RC$ -modules is an abelian category so the usual notions of kernel, cokernel, and exact sequence exist and they are defined objectwise. For example, a short exact sequence of  $RC$ -modules  $M_1 \rightarrow M_2 \rightarrow M_3$  is exact if for every  $x \in \text{Ob}(\mathbf{C})$  the sequence of  $R$ -modules

$$M_1(x) \rightarrow M_2(x) \rightarrow M_3(x)$$

is exact. For  $x \in \text{Ob}(\mathbf{C})$ , we define the  $RC$ -module  $RMor_{\mathbf{C}}(? , x)$  as the module which takes  $y \in \text{Ob}(\mathbf{C})$  to the free  $R$ -module  $RMor_{\mathbf{C}}(y, x)$ . For any  $x \in \mathbf{C}$  and for any  $RC$ -module  $M$ , there is an isomorphism

$$\text{Hom}_{RC}(RMor_{\mathbf{C}}(? , x), M) \cong M(x).$$

This proves that for every  $x \in \text{Ob}(\mathbf{C})$ , the  $RC$ -module  $RMor_{\mathbf{C}}(? , x)$  is projective. Using these projective modules one can show that for every  $RC$ -module  $M$ , there is a projective module  $P$  and a surjective  $RC$ -module homomorphism  $P \rightarrow M$ . Hence there are enough projectives in the category of  $RC$ -modules. There are also enough injectives in this category (see [36, p. 43]).

For  $RC$ -modules  $M$  and  $N$ , we define the  $n$ -th ext-group to be

$$\text{Ext}_{RC}^n(M, N) := H^n(\text{Hom}_{RC}(P_*, N))$$

where  $P_* \rightarrow M$  is a projective resolution of  $M$  as an  $RC$ -module. By the balancing theorem for ext-groups, the ext-group  $\text{Ext}_{RC}^n(M, N)$  can also be calculated as the  $n$ -th cohomology of the cochain complex  $\text{Hom}_{RC}(M, I^*)$  where  $I^*$  is an injective co-resolution of  $N$  as an  $RC$ -module (see [36, Thm 2.7.6]).

**Definition 3.1.** The *constant functor*  $\underline{R}$  over the category  $\mathbf{C}$  is the  $RC$ -module that sends every object  $x \in \text{Ob}(\mathbf{C})$  to  $R$  and every morphism in  $\mathbf{C}$  to the identity map  $\text{id}_R : R \rightarrow R$ . If  $M$  is an  $RC$ -module, then for every  $n \geq 0$ , the  *$n$ -th cohomology group of  $\mathbf{C}$  with coefficients in  $M$*  is defined to be

$$H^n(\mathbf{C}; M) := \text{Ext}_{RC}^n(\underline{R}, M).$$

For any  $RC$ -module  $M$ , the limit of  $M$  over  $\mathbf{C}$  is defined by

$$\lim_{x \in \mathbf{C}} M(x) := \{(m_x) \in \prod_{x \in \mathbf{C}} M(x) \mid M(\alpha)(m_y) = m_x \text{ for every } \alpha \in \text{Mor}_{\mathbf{C}}(x, y)\}.$$

The functor  $M \rightarrow \lim_{x \in \mathbf{C}} M$  is left exact. The  $n$ -th right derived functor of  $\lim_{\mathbf{C}}(-)$  is called the  *$n$ -th higher limit of  $M$* , and it is denoted by  $\lim_{\mathbf{C}}^n M$ . Since

$$\text{Hom}_{RC}(\underline{R}; M) \cong \lim_{\mathbf{C}} M,$$

there is an isomorphism

$$\lim_{\mathbf{C}}^n M \cong H^n(\mathbf{C}, M)$$

of right derived functors for every  $n \geq 0$ . Throughout the paper we will replace the higher limits  $\lim_{\mathbf{C}}^n M$  of an  $RC$ -module  $M$  with the cohomology groups  $H^n(\mathbf{C}; M)$  without further explanation.

When  $M : \mathbf{C} \rightarrow R$  is a covariant functor, then we say  $M$  is a left  $RC$ -module. For left  $RC$ -modules, we can define projective resolutions and ext-groups in the same way that we defined them for the right  $RC$ -modules. In fact, a left  $RC$ -module is a right  $RC^{op}$ -module, so all the definitions above can be repeated easily. The  $n$ -th cohomology  $H^n(\mathbf{C}; M)$  of  $\mathbf{C}$  with coefficients

in a left  $RC$ -module  $M$  is defined to be the ext-group  $\text{Ext}_{RC}^n(\underline{R}, M)$  over the category of left  $RC$ -modules. The  $n$ -th higher limits is defined in a similar way and we have  $\lim_{\mathbf{C}}^n M \cong H^n(\mathbf{C}; M)$ .

**Remark 3.2.** In the literature, the cohomology groups of a category  $\mathbf{C}$  are sometimes defined only for a left  $RC$ -module and the cohomology groups of  $\mathbf{C}$  with coefficients in a right  $RC$ -module  $M$  are denoted by  $H^*(\mathbf{C}^{op}; M)$ . In this paper, most of our cohomology groups are with coefficients in a right  $RC$ -module so it becomes inconvenient for us to use this convention. We denote both of the cohomology groups of  $\mathbf{C}$  with coefficients in a left  $RC$ -module and a right  $RC$ -module by  $H^*(\mathbf{C}; M)$ . Similarly we will always denote the higher limits of both covariant and contravariant functors  $M : \mathbf{C} \rightarrow R\text{-Mod}$  over  $\mathbf{C}$  by  $\lim_{\mathbf{C}}^i M$ .

**Remark 3.3.** The empty category  $\emptyset$  is a category with no objects and no morphisms. For every category  $\mathbf{C}$ , there is a unique functor  $F : \emptyset \rightarrow \mathbf{C}$ , i.e. the empty category  $\emptyset$  is the initial object in the category of small categories. If  $\mathbf{C}$  is the empty category then the only  $RC$ -module is the zero module which is both injective and projective. Applying the definitions, we see that if  $\mathbf{C} = \emptyset$ , then  $\lim_{\mathbf{C}} M = 0$  and  $H^i(\mathbf{C}, M) = \lim_{\mathbf{C}}^i M = 0$  for all  $i$ .

**3.2. Restriction and induction via a functor.** Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor between two nonempty small categories. The restriction functor

$$\text{Res}_F : R\mathbf{D}\text{-Mod} \rightarrow RC\text{-Mod}$$

is defined by composition with  $F$ , i.e.,  $\text{Res}_F M = M \circ F$ . The induction functor

$$\text{Ind}_F : RC\text{-Mod} \rightarrow R\mathbf{D}\text{-Mod}$$

is defined to be the functor which is left adjoint of the restriction functor. The left adjoint of the restriction functor exists because the category of  $R$ -modules is cocomplete. We explain this in detail below using Kan extensions.

For an  $RC$ -module  $M$ , let  $LK_{F^{op}}(M)$  denote the left Kan extension of  $M : \mathbf{C}^{op} \rightarrow R\text{-Mod}$  along the functor  $F^{op} : \mathbf{C}^{op} \rightarrow \mathbf{D}^{op}$ . There is a formula for left Kan extensions using colimits over a comma category. We first recall the definition of a comma category.

**Definition 3.4.** Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor and  $d \in \text{Ob}(\mathbf{D})$ . The *comma category*  $d \setminus F$  is the category whose objects are the pairs  $(c, f)$  where  $c \in \text{Ob}(\mathbf{C})$  and  $f \in \text{Mor}_{\mathbf{D}}(d, F(c))$ , and whose morphisms  $(c, f) \rightarrow (c', f')$  are given by the morphisms  $\varphi : c \rightarrow c'$  in  $\mathbf{C}$  such that  $f' = F(\varphi) \circ f$ .

The comma category  $F/d$  is defined in a similar way as the category whose objects are the pairs  $(c, f)$  where  $c \in \text{Ob}(\mathbf{C})$  and  $f \in \text{Mor}_{\mathbf{D}}(F(c), d)$ . Note that  $F^{op}/d = (d \setminus F)^{op}$ , and there is a functor  $\pi_d : F^{op}/d \rightarrow \mathbf{C}^{op}$  defined by  $(c, f) \rightarrow c$ . For every  $d \in \text{Ob}(\mathbf{D})$ , we have

$$LK_{F^{op}}(M)(d) = \text{colim}_{F^{op}/d} (M \circ \pi_d) = \text{colim}_{(d \setminus F)^{op}} (M \circ \pi_d)$$

(see [29, Thm 6.2.1]). Since the colimits exists in the category of  $R$ -modules, the left Kan-extension exists (see [29, Cor 6.2.6]). When it exists, the Kan extension functor  $LK_{F^{op}}(-)$  is left adjoint to the functor defined by precomposing with  $F^{op}$ , i.e., to the restriction functor  $\text{Res}_F(-)$  (see [29, Prop 6.1.5]). Hence we can take  $\text{Ind}_F(-)$  to be the functor  $LK_{F^{op}}(-)$ .

In [23, 9.15], the induction functor  $\text{Ind}_F(-)$  is defined by using a tensor product with the  $RC$ - $R\mathbf{D}$ -bimodule  $RMor_{\mathbf{D}}(?, F(?)) : \mathbf{C} \times \mathbf{D}^{op} \rightarrow R\text{-Mod}$  defined by

$$(c, d) \rightarrow RMor_{\mathbf{D}}(d, F(c))$$

on objects. This gives an explicit formula for  $\text{Ind}_F M$  that can be described as follows: for every  $x \in \text{Ob}(\mathbf{D})$ ,

$$(\text{Ind}_F M)(x) = \left( \bigoplus_{y \in \text{Ob}(\mathbf{C})} M(y) \otimes_R \text{RMor}_{\mathbf{D}}(x, F(y)) \right) / J$$

where  $J$  is the ideal generated by the elements of the form  $m \otimes f - m' \otimes f'$  where  $m \in M(y)$ ,  $f \in \text{Mor}_{\mathbf{D}}(x, F(y))$ ,  $m' \in M(y')$ , and  $f' \in \text{Mor}_{\mathbf{D}}(x', F(y'))$  such that there is a morphism  $\varphi \in \text{Mor}_{\mathbf{C}}(y, y')$  satisfying

$$M(\varphi)(m') = m \quad \text{and} \quad f' = F(\varphi) \circ f.$$

Since the tensor product is adjoint to the hom-functor, we have the following:

**Lemma 3.5.** *Let  $\text{Ind}_F : \mathbf{RC}\text{-Mod} \rightarrow \mathbf{RD}\text{-Mod}$  be the functor defined by the formula given above. Then for every  $\mathbf{RC}$ -module  $M$  and  $\mathbf{RD}$ -module  $N$ , there is a natural isomorphism*

$$\text{Hom}_{\mathbf{RC}}(\text{Ind}_F M, N) \cong \text{Hom}_{\mathbf{RD}}(M, \text{Res}_F N).$$

*Proof.* See [23, 9.22]. □

Note that these two different descriptions of  $\text{Ind}_F M$  coincide by the uniqueness of the left adjoints. We can also see this by proving that the two formulas for  $(\text{Ind}_F M)(x)$  given by colimits and by using the definition of tensor products give isomorphic modules for every  $x \in \text{Ob}(\mathbf{D})$ .

**Lemma 3.6.** *The restriction functor  $\text{Res}_F(-)$  preserves exact sequences, hence its left adjoint the induction functor  $\text{Ind}_F(-)$  takes projectives to projectives.*

*Proof.* This is clear from the definition of the restriction functor. The second statement follows from the adjointness of the restriction and induction functor. □

In general the induction functor  $\text{Ind}_F(-)$  is not an exact functor. When  $\text{Ind}_F(-)$  is an exact functor, there is a version of Shapiro's isomorphism for ext-groups over the corresponding module categories.

**Proposition 3.7.** *Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor such that the associated induction functor  $\text{Ind}_F : \mathbf{RC}\text{-Mod} \rightarrow \mathbf{RD}\text{-Mod}$  is exact. Then for every  $\mathbf{RC}$ -module  $M$  and for every  $\mathbf{RD}$ -module  $N$ , there is an isomorphism*

$$\text{Ext}_{\mathbf{RD}}^*(\text{Ind}_F M, N) \cong \text{Ext}_{\mathbf{RC}}^*(M, \text{Res}_F N),$$

called Shapiro's isomorphism for the functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ .

*Proof.* Let  $P_* \rightarrow M$  be a projective resolution of  $M$  as an  $\mathbf{RC}$ -module. Since the induction functor takes projectives to projectives, the induced module  $\text{Ind}_F P_i$  is a projective  $\mathbf{RD}$ -module for every  $i$ . By assumption the induction functor  $\text{Ind}_F(-)$  is an exact functor, therefore we can conclude that  $\text{Ind}_F P_* \rightarrow \text{Ind}_F M$  is a projective resolution of  $\text{Ind}_F M$  as an  $\mathbf{RD}$ -module. This gives

$$\begin{aligned} \text{Ext}_{\mathbf{RD}}^*(\text{Ind}_F M, N) &\cong H^*(\text{Hom}_{\mathbf{RD}}(\text{Ind}_F P_*, N)) \\ &\cong H^*(\text{Hom}_{\mathbf{RC}}(P_*, \text{Res}_F N)) \cong \text{Ext}_{\mathbf{RC}}^*(M, \text{Res}_F N). \end{aligned}$$

□

**Remark 3.8.** If  $\emptyset$  is the empty category and  $F : \emptyset \rightarrow \mathbf{C}$  is the unique functor between  $\emptyset$  and a category  $\mathbf{C}$ , then  $\text{Res}_F : \mathbf{RC}\text{-Mod} \rightarrow R\emptyset\text{-Mod}$  is the functor which takes every  $\mathbf{RC}$ -module to the zero module.  $\text{Ind}_F : R\emptyset\text{-Mod} \rightarrow \mathbf{RC}\text{-Mod}$  is the functor which sends the zero module to the zero module. It is easy to see that both Lemma 3.5 and Proposition 3.7 hold for this functor  $F$ .

We will use the empty category in the paper when we are discussing the restriction and induction functors for the (fusion) orbit categories. We also will refer to the following well-known result for the equivalent categories.

**Proposition 3.9.** *If  $F : \mathbf{C} \rightarrow \mathbf{D}$  is an equivalence of two small categories, then for every  $\mathbf{RD}$ -module  $M$ , there is an isomorphism*

$$H^n(\mathbf{D}; M) \cong H^n(\mathbf{C}; \text{Res}_F M)$$

induced by  $F$ .

*Proof.* See [1, Prop 5.6]. □

#### 4. INDUCTION FROM A SUBGROUP

Let  $\mathbf{G}$  denote the category with one object whose set of endomorphisms is a group given by the group  $G$ . Then an  $\mathbf{RG}$ -module is the same as an  $RG$ -module. If  $H \leq G$  is a subgroup of  $G$ , then the inclusion map  $H \rightarrow G$  induces a functor  $F : \mathbf{H} \rightarrow \mathbf{G}$ , and the corresponding induction functor  $\text{Ind}_F : RH\text{-Mod} \rightarrow RG\text{-Mod}$  coincides with the usual induction of a module defined by  $\text{Ind}_H^G M = M \otimes_H RG$ . Since  $RG$  is free as an  $RH$ -module,  $\text{Ind}_H^G(-)$  is an exact functor. The exactness of the induction gives an isomorphism

$$\text{Ext}_{RG}^*(\text{Ind}_H^G N, M) \cong \text{Ext}_{RH}^*(N, \text{Res}_H^G M)$$

called *Shapiro's isomorphism* for ext-groups. We will show below that Shapiro's isomorphism also holds for ext-groups over the (fusion) orbit category.

**4.1. Induction for the orbit category.** Let  $\mathcal{C}$  be a collection of subgroups in  $G$ , and let  $H$  be a subgroup of  $G$ . We will consider the orbit category of  $H$  over the collection

$$\mathcal{C}|_H := \{K \in \mathcal{C} \mid K \leq H\}.$$

We denote the orbit category  $\mathcal{O}_{\mathcal{C}|_H}(H)$  by  $\mathcal{O}_{\mathcal{C}}(H)$  to simplify the notation. Let

$$i_H^G : \mathcal{O}_{\mathcal{C}}(H) \rightarrow \mathcal{O}_{\mathcal{C}}(G)$$

be the functor which takes a subgroup  $K \in \mathcal{C}|_H$  to itself in  $\mathcal{C}$ , and takes an  $H$ -map  $f : H/K \rightarrow H/L$  to the  $G$ -map  $i_H^G(f) : G/K \rightarrow G/L$  defined by  $i_H^G(f)(gK) = gf(K)$ . We denote the induction functor associated to  $i_H^G$  by  $\text{Ind}_{\mathcal{O}_{\mathcal{C}}(H)}^{\mathcal{O}_{\mathcal{C}}(G)}$ . If  $\mathcal{C}|_H$  is the empty set, then  $\mathcal{O}_{\mathcal{C}}(H)$  is the empty category  $\emptyset$ , and  $i_H^G$  is the unique functor  $\emptyset \rightarrow \mathcal{O}_{\mathcal{C}}(G)$ . In this case the restriction and induction functors are defined as described in Remarks 3.3 and 3.8.

There is an explicit formula for the induced module  $\text{Ind}_{\mathcal{O}_{\mathcal{C}}(H)}^{\mathcal{O}_{\mathcal{C}}(G)} M$  which is due to Symonds [34, p. 266] (see also Lemma 3.1 in [17]).

**Proposition 4.1** (Symonds [34]). *Let  $M$  be an  $R\mathcal{O}_{\mathcal{C}}(H)$ -module. Then for every  $K \in \mathcal{C}$ ,*

$$(\text{Ind}_{\mathcal{O}_{\mathcal{C}}(H)}^{\mathcal{O}_{\mathcal{C}}(G)} M)(K) \cong \bigoplus_{x^{-1}H \in (G/H)^K} M({}^x K).$$

Note that the indexing set  $(G/H)^K$  for the above direct sum can be identified with

$$\text{Map}_G(G/K, G/H) = \text{Mor}_{\mathcal{O}(G)}(K, H)$$

where  $\mathcal{O}(G)$  denotes the orbit category of  $G$  over all subgroups of  $G$ . For every  $G$ -map  $f : G/L \rightarrow G/K$ , the induced  $R$ -module homomorphism

$$f^* : (\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} M)(K) \rightarrow (\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} M)(L)$$

can be described using the map  $\text{Mor}_{\mathcal{O}(G)}(K, H) \rightarrow \text{Mor}_{\mathcal{O}(G)}(L, H)$  induced by  $f$ . On the summands one uses inclusion and conjugation maps between summands (see [34, p. 266] for details). As a consequence, we conclude the following.

**Proposition 4.2** ([34, Lemma 2.9]). *The induction functor*

$$\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} : R\mathcal{O}_C(H)\text{-Mod} \rightarrow R\mathcal{O}_C(G)\text{-Mod}$$

is exact. Hence, for every  $R\mathcal{O}_C(H)$ -module  $N$  and  $R\mathcal{O}_C(G)$ -module  $M$ , there is an isomorphism

$$\text{Ext}_{R\mathcal{O}_C(G)}^*(\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} N, M) \cong \text{Ext}_{R\mathcal{O}_C(H)}^*(N, \text{Res}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} M).$$

*Proof.* The first sentence follows from Proposition 4.1. The second part is a consequence of Proposition 3.7.  $\square$

Let  $R[G/H^?]$  denote the  $R\mathcal{O}_C(G)$ -module defined by  $K \rightarrow R[(G/H)^K]$  for every  $K \in \mathcal{C}$ . If  $K \in \mathcal{C}$  such that  $(G/H)^K = \emptyset$ , then  $R[(G/H)^K] = 0$ . If there is no  $K \in \mathcal{C}$  such that  $K \leq H$ , i.e. if  $\mathcal{C}|_H$  is the empty collection, then  $R[G/H^?] = 0$  as an  $R\mathcal{O}_C(G)$ -module. As a consequence of Proposition 4.1, we obtain the following:

**Proposition 4.3.** [34, Lemma 2.7] *Let  $\underline{R}$  denote the constant functor for  $\mathcal{O}_C(H)$ . Then, there is an isomorphism of  $R\mathcal{O}_C(G)$ -modules*

$$\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} \underline{R} \cong R[G/H^?].$$

*Proof.* By Proposition 4.5, for every  $K \in \mathcal{C}$ , we have

$$(\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} \underline{R})(K) \cong \bigoplus_{x^{-1}H \in (G/H)^K} R \cong R[(G/H)^K].$$

For every  $G$ -map  $f : G/L \rightarrow G/K$ , the induced maps

$$f^* : (\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} \underline{R})(K) \rightarrow (\text{Ind}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} \underline{R})(L)$$

and  $f^* : R[(G/H)^K] \rightarrow R[(G/H)^L]$  commutes with the isomorphisms given above.  $\square$

As a consequence we have the following.

**Corollary 4.4.** *For every  $R\mathcal{O}_C(G)$ -module  $M$ , there is an isomorphism*

$$\text{Ext}_{R\mathcal{O}_C(G)}^n(R[G/H^?], M) \cong H^n(\mathcal{O}_C(H); \text{Res}_{\mathcal{O}_C(H)}^{\mathcal{O}_C(G)} M).$$

*Proof.* This follows from Propositions 4.2 and 4.3, and from the definition of the cohomology of a category  $\mathbf{C}$  as the ext-group over  $R\mathbf{C}$ .  $\square$

**4.2. Induction for the fusion orbit category.** Let  $\mathcal{C}$  be a collection of subgroups in  $G$ , and  $H$  be a subgroup of  $G$ . We denote by  $\overline{\mathcal{F}}_{\mathcal{C}}(H)$  the fusion orbit category of  $H$  over the collection  $\mathcal{C}|_H$ . Let

$$j_H^G : \overline{\mathcal{F}}_{\mathcal{C}}(H) \rightarrow \overline{\mathcal{F}}_{\mathcal{C}}(G)$$

denote the functor which takes a subgroup  $K \in \mathcal{C}|_H$  to itself in  $\mathcal{C}$ , and takes a conjugation map  $c_h : L \rightarrow K$  modulo  $\text{Inn}(K)$  to itself in  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ . The induction functor associated to  $j_H^G$  will be denoted by  $\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)}$ . The case where  $\mathcal{C}|_H$  is the empty collection is handled with the empty category  $\emptyset$  as in the case of the orbit category.

**Proposition 4.5.** *For every  $R\overline{\mathcal{F}}_{\mathcal{C}}(H)$ -module  $M$  and for every subgroup  $K \in \mathcal{C}$ ,*

$$\left(\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} M\right)(K) \cong \bigoplus_{[f] \in \text{Mor}_{\overline{\mathcal{F}}(G)}(K, H)} M(f(K))$$

where  $\overline{\mathcal{F}}(G)$  denotes the fusion orbit category defined on all subgroups of  $G$ . In particular, the induction functor

$$\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} : R\overline{\mathcal{F}}_{\mathcal{C}}(H)\text{-Mod} \rightarrow R\overline{\mathcal{F}}_{\mathcal{C}}(G)\text{-Mod}$$

is an exact functor.

To prove Proposition 4.5, we first prove some lemmas. If  $F : \mathbf{C} \rightarrow \mathbf{D}$  is a functor and  $d \in \text{Ob}(\mathbf{D})$ , then the comma category  $d \backslash F$  is the category whose objects are the pairs  $(c, f)$  where  $c \in \text{Ob}(\mathbf{C})$  and  $f : d \rightarrow F(c)$  is a morphism in  $\mathbf{D}$  (see Definition 3.4). If  $\mathbf{C}$  is a subcategory of  $\mathbf{D}$  and  $j : \mathbf{C} \rightarrow \mathbf{D}$  is the inclusion functor, then we can assume that the objects of  $d \backslash j$  are the morphisms  $f : d \rightarrow c$  in  $\mathbf{D}$  instead of the pairs  $(c, f)$ . A morphism  $f_1 \rightarrow f_2$  in  $d \backslash j$  is a morphism  $\varphi : c_1 \rightarrow c_2$  in  $\mathbf{C}$  such that  $\varphi \circ f_1 = f_2$ .

**Lemma 4.6.** *Let  $\mathbf{C}$  be a subcategory of a small category  $\mathbf{D}$  and let  $j : \mathbf{C} \rightarrow \mathbf{D}$  be the inclusion functor. Assume that*

- (1) *Every morphism  $d \rightarrow c$  in  $\mathbf{D}$  with  $d \in \text{Ob}(\mathbf{D})$  and  $c \in \text{Ob}(\mathbf{C})$  factors as  $d \xrightarrow{\cong} c' \xrightarrow{C} c$ , an isomorphism in  $\mathbf{D}$  followed by a morphism in  $\mathbf{C}$ .*
- (2) *Given any  $c, c', c'' \in \text{Ob}(\mathbf{C})$ , if  $c \xrightarrow{D} c' \xrightarrow{C} c''$  are morphisms such that  $C \in \mathbf{C}$ ,  $D \in \mathbf{D}$ , and  $C \circ D$  is a morphism in  $\mathbf{C}$  then  $D \in \mathbf{C}$ .*

Then for any  $d \in \text{Ob}(\mathbf{D})$ ,  $d \backslash j = \coprod_{i \in I} \mathbf{E}_i$  where  $I$  is a set of representatives for the  $\mathbf{C}$ -isomorphism classes in the set of all  $\mathbf{D}$ -isomorphisms  $d \rightarrow c$ ,  $\mathbf{E}_i$  is the component of  $d \backslash j$  containing  $i \in I$  and furthermore  $i$  is an initial object in  $\mathbf{E}_i$ .

*Proof.* We can write  $\mathbf{E} := d \backslash j$  as a disjoint union  $\coprod_{\alpha \in A} \mathbf{E}_{\alpha}$  of its connected components over some indexing set  $A$ . By (1), every morphism  $f : d \rightarrow c$  in  $\mathbf{D}$  factors as  $d \xrightarrow{i} c' \xrightarrow{f'} c$  where  $i$  is an isomorphism in  $\mathbf{D}$  and  $f'$  is a morphism in  $\mathbf{C}$ . The morphism  $f'$  defines a morphism  $i \rightarrow f$  in  $\mathbf{E}$ , hence  $i$  and  $f$  lie in the same component of  $\mathbf{E}$ . This shows that in every component  $\mathbf{E}_{\alpha}$  there is an object  $i : d \rightarrow c$  which is a  $\mathbf{D}$ -isomorphism. Choose a  $\mathbf{D}$ -isomorphism  $i_{\alpha} : d \rightarrow c$  in each component  $\mathbf{E}_{\alpha}$ .

If  $\varphi : (f_1 : d \rightarrow c_1) \rightarrow (f_2 : d \rightarrow c_2)$  is a morphism in  $\mathbf{E}$ , then there is a commuting diagram

$$\begin{array}{ccc}
 & d & \\
 i_1 \swarrow & & \searrow i_2 \\
 c'_1 & \xrightarrow{\psi} & c'_2 \\
 f_1 \downarrow & & \downarrow f'_2 \\
 c_1 & \xrightarrow{\varphi} & c_2
 \end{array}$$

where  $\psi := i_2 \circ i_1^{-1}$  is an isomorphism in  $\mathbf{D}$ . Since  $\varphi$  is in  $\mathbf{C}$ , the composition  $\varphi \circ f'_1$  is in  $\mathbf{C}$ . By condition (2) applied to  $c'_1 \xrightarrow{\psi} c'_2 \xrightarrow{f'_2} c_2$ , we obtain that  $\psi$  is in  $\mathbf{C}$ . By applying the condition (2) to  $c'_2 \xrightarrow{\psi^{-1}} c'_1 \xrightarrow{f_1} c_1$ , we conclude that  $\psi^{-1}$  is also in  $\mathbf{C}$ , hence  $\psi$  is an isomorphism in  $\mathbf{C}$ .

If there is a zigzag of morphisms between  $f_1 : d \rightarrow c_1$  and  $f_n : d \rightarrow c_n$  in  $\mathbf{E}$ , then applying the argument above to each morphism in the zigzag, we obtain an  $\mathbf{E}$ -isomorphism between  $i_1 : d \rightarrow c'_1$  and  $i_n : d \rightarrow c'_n$ . Combining this isomorphism with  $f'_n$  gives a morphism from  $i_1$  to  $f_n$  in  $\mathbf{E}$ . This shows that if  $f : d \rightarrow c$  lies in the component  $\mathbf{E}_\alpha$ , then there is a morphism  $\varphi : i_\alpha \rightarrow f$  in  $\mathbf{E}$ . If  $\varphi_1, \varphi_2 : i_\alpha \rightarrow f$  are two such morphisms, then  $\varphi_1 \circ i_\alpha = f = \varphi_2 \circ i_\alpha$  gives  $\varphi_1 = \varphi_2$ . So the morphism  $\varphi$  is unique. This proves that  $i_\alpha$  is an initial object in  $\mathbf{E}_\alpha$ .

Two  $\mathbf{D}$ -isomorphisms  $i_1 : d \rightarrow c_1$  and  $i_2 : d \rightarrow c_2$  lie in the same component of  $\mathbf{E}$  if and only if there is a  $\mathbf{C}$ -isomorphism  $\psi : c_1 \rightarrow c_2$  such that  $i_2 = \psi \circ i_1$ . So, the index set  $A$  can be taken as the  $\mathbf{C}$ -isomorphism classes of  $\mathbf{D}$ -isomorphisms.  $\square$

Using Lemma 4.6, we prove the following.

**Lemma 4.7.** *Let  $\mathbf{C}$  be a subcategory of a small category  $\mathbf{D}$  satisfying the conditions in Lemma 4.6. Let*

$$\text{Ind}_{\mathbf{C}}^{\mathbf{D}} : \mathbf{RC}\text{-Mod} \rightarrow \mathbf{RD}\text{-Mod}$$

denote the induction functor induced by the inclusion  $j : \mathbf{C} \rightarrow \mathbf{D}$ . Then for every  $\mathbf{RC}$ -module  $M$  and for every  $d \in \text{Ob}(\mathbf{D})$ , we have

$$(\text{Ind}_{\mathbf{C}}^{\mathbf{D}} M)(d) \cong \bigoplus_{(i:d \rightarrow c_i) \in I} M(c_i)$$

where  $I$  is a set of representatives for the  $\mathbf{C}$ -isomorphism classes in the set of all  $\mathbf{D}$ -isomorphisms  $d \rightarrow c$ .

*Proof.* By the definition of the induction functor via left Kan extensions given in Section 3.2, we have

$$(\text{Ind}_{\mathbf{C}}^{\mathbf{D}} M)(d) = \text{colim}_{(c,f) \in (d \setminus j)^{op}} (M \circ \pi_d).$$

By Lemma 4.6,  $d \setminus j = \coprod_{i \in I} \mathbf{E}_i$  where  $\mathbf{E}_i$  is the component of  $d \setminus j$  containing the isomorphism  $i : d \rightarrow c_i$ . Since  $i$  is an initial object for the  $\mathbf{E}_i$ , it is a terminal object for  $\mathbf{E}_i^{op}$ , hence for every  $d \in \text{Ob}(\mathbf{D})$  we have,

$$(\text{Ind}_{\mathbf{C}}^{\mathbf{D}} M)(d) = \bigoplus_{i \in I} \text{colim}_{(c,f) \in \mathbf{E}_i^{op}} (M \circ \pi_d \circ inc_i) \cong \bigoplus_{i \in I} M(c_i)$$

where  $inc_i : \mathbf{E}_i \rightarrow d \setminus j$  is the inclusion map. This completes the proof.  $\square$



*Proof of Proposition 4.5.* We claim that if we take  $\mathbf{C} = \overline{\mathcal{F}}_{\mathcal{C}}(H)$  and  $\mathbf{D} = \overline{\mathcal{F}}_{\mathcal{C}}(G)$ , then the conditions given in Lemma 4.6 are satisfied.

Let  $[f] : K \rightarrow L$  be a morphism in  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  represented by a morphism  $f : K \rightarrow L$  in  $\mathcal{F}_{\mathcal{C}}(G)$  such that  $L \leq H$ . Then  $f$  factors as  $K \xrightarrow{i} f(K) \xrightarrow{f'} L$  where the first map is an isomorphism in  $\mathcal{F}_{\mathcal{C}}(G)$  and the second map is the inclusion map  $f' : f(K) \hookrightarrow L$  which is a morphism in  $\mathcal{F}_{\mathcal{C}}(H)$ . Then  $[f] = [f'] \circ [i]$  where  $[i]$  is an isomorphism in  $\overline{\mathcal{F}}(G)$  and  $[f']$  is in  $\overline{\mathcal{F}}_{\mathcal{C}}(H)$ . Hence condition (1) in Lemma 4.6 holds for these categories.

Let  $U \xrightarrow{[\varphi]} U' \xrightarrow{[\psi]} U''$  be a sequence of morphisms such that  $U, U', U'' \leq H$ ,  $[\varphi] \in \overline{\mathcal{F}}_{\mathcal{C}}(G)$ , and  $[\psi] \in \overline{\mathcal{F}}_{\mathcal{C}}(H)$ . Suppose that  $\psi \circ \varphi$  is in  $\mathcal{F}_{\mathcal{C}}(H)$ . Let  $g \in G$  and  $h_1, h_2 \in H$  are such that  $\varphi = c_g$ ,  $\psi = c_{h_1}$ , and  $\psi \circ \varphi = c_{h_2}$ , then  $h_1 g = h_2 z$  for some  $z \in C_G(U)$ . This gives  $g = h_1^{-1} h_2 z$ , hence  $\varphi = c_g$  is in  $\mathcal{F}_{\mathcal{C}}(H)$ . We conclude that condition (2) in Lemma 4.6 holds for these categories.

Applying Lemma 4.7 to  $\mathbf{C}$  and  $\mathbf{D}$ , we obtain that for every  $K \in \mathcal{C}$ ,

$$(\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} M)(K) \cong \bigoplus_{[f] \in I} M(f(K))$$

where  $I$  is a set of representatives for the  $\overline{\mathcal{F}}_{\mathcal{C}}(H)$ -isomorphism classes in the set of all  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -isomorphisms  $[f] : K \rightarrow f(K)$  with  $f(K) \leq H$ . The equivalence relation is given by  $[f_1] \sim [f_2]$  if there is an isomorphism  $[c_h] : f_1(K) \rightarrow f_2(K)$  in  $\overline{\mathcal{F}}_{\mathcal{C}}(H)$  such that  $[f_2] = [c_h] \circ [f_1]$ . This shows that the index set  $I$  can be taken as the set of morphisms  $\text{Mor}_{\overline{\mathcal{F}}(G)}(K, H)$  which is the orbit set of  $\text{Mor}_{\mathcal{F}(G)}(K, H)$  under the action of  $\text{Inn}(H)$ .  $\square$

The following is an easy consequence of Propositions 3.7 and 4.5. We call this isomorphism the Shapiro's isomorphism for the fusion orbit category.

**Proposition 4.8.** *For every  $R\overline{\mathcal{F}}_{\mathcal{C}}(H)$ -module  $N$  and every  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ , there is an isomorphism*

$$\text{Ext}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} N, M) \cong \text{Ext}_{R\overline{\mathcal{F}}_{\mathcal{C}}(H)}^*(N, \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} M).$$

As a consequence of Proposition 4.5, we also have the following.

**Proposition 4.9.** *There is an isomorphism of  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules*

$$\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} \underline{R} \cong R[C_G(?) \setminus (G/H)^?].$$

*Proof.* By Proposition 4.5, we have

$$\text{Ind}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} \underline{R} \cong R[\text{Mor}_{\overline{\mathcal{F}}(G)}(?, H)] \cong R[C_G(?) \setminus \text{Mor}_{\mathcal{O}(G)}(?, H)] \cong R[C_G(?) \setminus (G/H)^?]$$

as  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. The induced maps also coincide because of the description of the isomorphism.  $\square$

We conclude the following.

**Corollary 4.10.** *For every  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ ,*

$$\text{Ext}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(R[C_G(?) \setminus (G/H)^?], M) \cong H^*(\overline{\mathcal{F}}_{\mathcal{C}}(H); \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(H)}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} M).$$

*Proof.* This follows from Propositions 4.8 and 4.9.  $\square$

We use this isomorphism throughout the paper to replace the ext-groups on the left with the cohomology of the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(H)$ .

**4.3. Induction via the projection map.** Let  $pr : \mathcal{O}_{\mathcal{C}}(G) \rightarrow \overline{\mathcal{F}}_{\mathcal{C}}(G)$  denote the *projection functor* that takes every subgroup  $H \in \mathcal{C}$  to itself and takes a morphism  $f : L \rightarrow K$  defined by  $f(L) = g^{-1}K$  to the morphism  $KgC_G(L)$  in  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ . The restriction functor  $\text{Res}_{pr}$  takes an  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$  to an  $R\mathcal{O}_{\mathcal{C}}(G)$ -module via composition with the functor  $pr$ . In particular  $(\text{Res}_{pr}M)(K) = M(K)$  for every  $K \in \mathcal{C}$ . If  $M' = \text{Res}_{pr}M$  for some  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -Module  $M$ , then for every  $K \in \mathcal{C}$ , the centralizer  $C_G(K)$  acts trivially on  $M'(K)$ . Such an  $R\mathcal{O}_{\mathcal{C}}(G)$ -module is called a *conjugation invariant  $R\mathcal{O}_{\mathcal{C}}(G)$ -module* in [17], and called a *geometric coefficient system* in [34]. For the induction functor we have the following observation.

**Lemma 4.11.** *Let  $M$  be an  $R\mathcal{O}_{\mathcal{C}}(G)$ -module. Then for every  $K \in \mathcal{C}$ ,*

$$(\text{Ind}_{pr}M)(K) \cong M(K)_{C_G(K)} := M / \langle m - c_x m \mid m \in M(K), x \in C_G(K) \rangle$$

as an  $RN_G(K)$ -module.

*Proof.* By the formula for the induction functor given in Section 3.2, for every  $K \in \mathcal{C}$ , we have

$$(\text{Ind}_{pr}M)(K) = \text{colim}_{(K \setminus pr)^{op}} (M \circ \pi_K).$$

The comma category  $\mathbf{D} := K \setminus pr$  is the category whose objects are the pairs  $(L, f)$  where  $L \in \mathcal{C}$  and  $f : K \rightarrow L$  is a morphism in  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ . Let  $\mathbf{C}$  be the full subcategory of  $\mathbf{D}$  with one object  $(K, \text{id}_K)$ . The automorphisms of  $(K, \text{id}_K)$  in  $\mathbf{D}$  are given by the morphisms  $Kx : G/K \rightarrow G/K$  in  $\mathcal{O}_{\mathcal{C}}(G)$  that are sent to the identity map under the projection functor  $pr$ . So the automorphism group of  $(K, \text{id}_K)$  is the group  $KC_G(K)/K = \{Kx \mid x \in C_G(K)\} \leq N_G(K)/K$ . We claim that the inclusion functor  $j : \mathbf{C} \rightarrow \mathbf{D}$  defines an equivalence of categories. To see this, let us fix a morphism  $\hat{f} : K \rightarrow L$  in  $\mathcal{O}_{\mathcal{C}}(G)$  for every  $f : K \rightarrow L$  in  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ . For every  $\varphi : (L_1, f_1) \rightarrow (L_2, f_2)$  in  $\mathbf{D}$ , there is a unique  $\hat{\varphi} \in \text{Aut}_{\mathbf{D}}(K, \text{id}_K)$  such that  $\varphi \circ \hat{f}_1 = \hat{f}_2 \circ \hat{\varphi}$ . Define  $\pi : \mathbf{D} \rightarrow \mathbf{C}$  to be the functor that sends each object  $(L, f)$  in  $\mathbf{D}$  to  $(K, \text{id}_K)$  in  $\mathbf{C}$ , and each morphism  $\varphi : (L_1, f_1) \rightarrow (L_2, f_2)$  to  $\hat{\varphi}$  in  $\text{Aut}_{\mathbf{D}}(K, \text{id}_K)$ . It is easy to see that  $\pi \circ j = \text{id}_{\mathbf{C}}$  and  $j \circ \pi \simeq \text{id}_{\mathbf{D}}$ , so we can conclude that  $\mathbf{C}$  and  $\mathbf{D}$  are equivalent categories. This gives

$$(\text{Ind}_{pr}M)(K) = \text{colim}_{(K \setminus pr)^{op}} (M \circ \pi_d) \cong \text{colim}_{\mathbf{C}^{op}} (M \circ \pi_d \circ j) \cong M(K)_{C_G(K)}.$$

This completes the proof.  $\square$

By Lemma 4.11, it is easy to see that

$$\text{Ind}_{pr} \text{Res}_{pr} = \text{id}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)\text{-mod}}.$$

For  $H \leq G$ , there is a commuting diagram of functors

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{C}}(H) & \xrightarrow{pr} & \overline{\mathcal{F}}_{\mathcal{C}}(H) \\ \downarrow i_H^G & & \downarrow j_H^G \\ \mathcal{O}_{\mathcal{C}}(G) & \xrightarrow{pr} & \overline{\mathcal{F}}_{\mathcal{C}}(G) \end{array}$$

where  $i_H^G$  and  $j_H^G$  are the functors defined in the previous section. For every pair of composable functors  $F_1 : \mathbf{C} \rightarrow \mathbf{D}$  and  $F_2 : \mathbf{D} \rightarrow \mathbf{E}$ , we have  $\text{Ind}_{F_2 \circ F_1} = \text{Ind}_{F_2} \text{Ind}_{F_1}$ . In particular, we have

$$\text{Ind}_{j_H^G} \text{Ind}_{pr} = \text{Ind}_{pr} \text{Ind}_{i_H^G}.$$

Thus for every  $R\overline{\mathcal{F}}_{\mathcal{C}}(H)$ -module  $M$ ,

$$\text{Ind}_{j_H^G} M = \text{Ind}_{j_H^G} \text{Ind}_{pr} \text{Res}_{pr} M = \text{Ind}_{pr} \text{Ind}_{i_H^G} \text{Res}_{pr} M.$$

This equality can be used to give a second proof of Proposition 4.5 as a consequence of Proposition 4.1. This explains the similarities between the formulas in Propositions 4.1 and 4.5.

We now state a lemma which also appears in [25, §3] (for a covariant version, see [24, §3]).

**Lemma 4.12** (Lück [25, §3]). *Let  $pr : \mathcal{O}_{\mathcal{C}}(G) \rightarrow \overline{\mathcal{F}}_{\mathcal{C}}(G)$  be the projection functor defined above. Then, for every subgroup  $H$  of  $G$ , there is an isomorphism*

$$\mathrm{Ind}_{pr}(R[G/H^?]) \cong R[C_G(?)\backslash(G/H)^?]$$

of  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. In particular, for every  $G$ -set  $X$  and for every  $R\overline{\mathcal{F}}_G$ -module  $N$ ,

$$\mathrm{Hom}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}(R[C_G(?)\backslash X^?], N) \cong \mathrm{Hom}_{R\mathcal{O}_{\mathcal{C}}(G)}(R[X^?], \mathrm{Res}_{pr}N).$$

*Proof.* For every  $K \in \mathcal{C}$ , we have

$$\mathrm{Ind}_{pr}(R[G/H^?])(K) = R[(G/H)^K]_{C_G(K)} \cong R[C_G(K)\backslash(G/H)^K].$$

The second formula follows from the adjointness of induction and restriction functors.  $\square$

## 5. FUSION BREDON COHOMOLOGY

In [34, p. 281], Symonds defines Bredon cohomology over an arbitrary collection  $\mathcal{C}$ . The main aim of this section is to extend Symonds' definition to chain complexes over the fusion orbit category. We also show that there is a hypercohomology spectral sequence converging to the fusion Bredon cohomology of a  $G$ -CW-complex.

Throughout,  $\mathcal{C}$  is an arbitrary collection of subgroups in the group  $G$ . For all the statements that hold for both orbit category  $\mathcal{O}_{\mathcal{C}}(G)$  and the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ , we use  $\mathbf{O}_{\mathcal{C}}(G)$  to denote  $\mathcal{O}_{\mathcal{C}}(G)$  or  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ . Similarly  $\mathbf{O}(G)$  is used to denote  $\mathcal{O}(G)$  or  $\overline{\mathcal{F}}(G)$ .

**5.1. Yoneda functors.** Given a category  $\mathbf{D}$  and a full subcategory  $\mathbf{C} \subseteq \mathbf{D}$ , define a functor

$$\tilde{Y}_{\mathbf{C}}^{\mathbf{D}} : \mathbf{C}^{op} \times \mathbf{D} \rightarrow R\text{-Mod}, \quad \tilde{Y}_{\mathbf{C}}^{\mathbf{D}}(c, d) = R[\mathrm{Mor}_{\mathbf{D}}(c, d)].$$

This gives a functor

$$Y_{\mathbf{C}}^{\mathbf{D}} : \mathbf{D} \rightarrow R\mathbf{C}\text{-Mod}, \quad Y_{\mathbf{C}}^{\mathbf{D}}(d) = R[\mathrm{Mor}_{\mathbf{D}}(-, d)].$$

If  $M$  is an  $R\mathbf{C}$ -module, then  $\mathrm{Ext}_{R\mathbf{C}}^j(-, M)$  is a contravariant functor from  $R\mathbf{C}\text{-mod}$  to  $R\text{-Mod}$ , so

$$\mathrm{Ext}_{R\mathbf{C}}^j(Y_{\mathbf{C}}^{\mathbf{D}}, M)$$

is an  $R\mathbf{D}$ -module. Applying this construction to the categories  $\mathbf{O}_{\mathcal{C}}(G) \subseteq \mathbf{O}(G)$ , we obtain that for every  $R\mathbf{O}_{\mathcal{C}}(G)$ -module  $M$  and for all  $j \geq 0$ ,

$$\tilde{M}^j := \mathrm{Ext}_{R\mathbf{O}_{\mathcal{C}}(G)}^j(Y_{\mathbf{O}_{\mathcal{C}}(G)}^{\mathbf{O}(G)}, M)$$

is an  $R\mathbf{O}(G)$ -module. For any collection  $\mathcal{D}$ , we can regard  $\tilde{M}^j$  as an  $R\mathbf{O}_{\mathcal{D}}(G)$ -module by precomposing with the inclusion  $\mathbf{O}_{\mathcal{D}}(G) \subseteq \mathbf{O}(G)$ .

For every  $H \leq G$ , we have

$$Y_{\mathcal{O}_{\mathcal{C}}(G)}^{\mathcal{O}(G)}(H) \cong R[G/H^?] \quad \text{and} \quad Y_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^{\overline{\mathcal{F}}(G)}(H) \cong R[C_G(?)\backslash(G/H)^?].$$

By Shapiro's lemma proved in Propositions 4.3 and 4.9, for every  $H \leq G$ , we have an isomorphism of  $R$ -modules

$$\tilde{M}^j(H) \cong H^j(\mathbf{O}_{\mathcal{C}}(H); \mathrm{Res}_{\mathbf{O}_{\mathcal{C}}(H)}^{\mathbf{O}_{\mathcal{C}}(G)} M). \quad (5.1)$$

We can use these isomorphisms to define an  $R\mathbf{O}(G)$ -module as follows:

**Lemma 5.1.** *For every  $\mathbf{RO}_C(G)$ -module  $M$ , and for every integer  $j \geq 0$ , there is an  $\mathbf{RO}(G)$ -module  $\overline{\mathcal{H}}_M^j$  such that for every  $H \leq G$ ,*

$$\overline{\mathcal{H}}_M^j(H) = H^j(\mathbf{O}_C(H); \text{Res}_{\mathbf{O}_C(H)}^{\mathbf{O}_C(G)} M).$$

*For every  $f : G/K \rightarrow G/H$ , the induced map  $\overline{\mathcal{H}}_M^j(f) : \overline{\mathcal{H}}_M^j(H) \rightarrow \overline{\mathcal{H}}_M^j(K)$  is defined in such a way that the Shapiro's isomorphism given in (5.1) induces an isomorphism of  $\mathbf{RO}(G)$ -modules*

$$\overline{\mathcal{H}}_M^j \cong \tilde{M}^j.$$

*Proof.* For each subgroup  $H \leq G$ , let  $\varphi_H : \tilde{M}^j(H) \xrightarrow{\cong} H^j(\mathbf{O}_C(H); \text{Res}_{\mathbf{O}_C(H)}^{\mathbf{O}_C(G)} M)$  denote the canonical homomorphism that gives the Shapiro's isomorphism. For every  $f : G/K \rightarrow G/H$ , let  $\overline{\mathcal{H}}_M^j(f) : \overline{\mathcal{H}}_M^j(H) \rightarrow \overline{\mathcal{H}}_M^j(K)$  denote the unique homomorphism which makes the following diagram commute

$$\begin{array}{ccc} \tilde{M}^j(H) & \xrightarrow{\varphi_H \cong} & \overline{\mathcal{H}}_M^j(H) \\ \tilde{M}^j(f) \downarrow & & \downarrow \overline{\mathcal{H}}_M^j(f) \\ \tilde{M}^j(K) & \xrightarrow{\varphi_K \cong} & \overline{\mathcal{H}}_M^j(K). \end{array}$$

It is clear from the commutativity of the above diagram that  $\overline{\mathcal{H}}_M^j(-)$  defines a functor  $\mathbf{O}(G)^{op} \rightarrow R\text{-Mod}$ , and the  $R$ -module isomorphisms given in (5.1) define an isomorphism of  $\mathbf{RO}(G)$ -modules  $\tilde{M}^j \cong \overline{\mathcal{H}}_M^j$ .  $\square$

In our applications we use the  $\mathbf{O}(G)$ -module  $\overline{\mathcal{H}}_M^j$  as coefficients for the ordinary Bredon cohomology of a  $G$ -CW-complex. This is done by considering  $\overline{\mathcal{H}}_M^j$  as a  $R\mathcal{O}(G)$ -module via the restriction functor induced by the projection functor  $pr : \mathcal{O}(G) \rightarrow \mathbf{O}(G)$ .

**Definition 5.2.** For every  $\mathbf{RO}_C(G)$ -module  $M$ , and for every integer  $j \geq 0$ , we denote by  $\mathcal{H}_M^j$  the  $R\mathcal{O}(G)$ -module  $\text{Res}_{pr} \overline{\mathcal{H}}_M^j$ . Note that for each  $H \leq G$ , we have

$$\mathcal{H}_M^j(H) = H^j(\mathbf{O}_C(H); \text{Res}_{\mathbf{O}_C(H)}^{\mathbf{O}_C(G)} M).$$

**5.2. Fusion Bredon cohomology.** Let  $X$  be a  $G$ -CW-complex (with a left  $G$ -action). For each subgroup  $K \in \mathcal{C}$ , the fixed point set  $X^K$  can be identified with the set of  $G$ -maps from the transitive  $G$ -set  $G/K$  to  $X$ . This gives a contravariant functor

$$\mathbf{O}_C(G) \rightarrow \text{CW-complexes}$$

defined by  $K \rightarrow X^K$  when  $\mathbf{O}_C(G) = \mathcal{O}_C(G)$  and by  $K \rightarrow C_G(K) \backslash X^K$  when  $\mathbf{O}_C(G) = \overline{\mathcal{F}}_C(G)$ . Composing these functors with the functor which takes a CW-complex to its cellular chain complex with coefficients in  $R$ , we obtain a chain complex  $C_*(X^?; R)$  of  $\mathbf{RO}_C(G)$ -modules,

For each  $n \geq 0$ , we have

$$C_n(X^?; R) \cong \bigoplus_{i \in I_n} R[G/H_i^?] \cong \bigoplus_{i \in I_n} \text{RMor}_{\mathbf{O}_C(G)}(?, H_i)$$

where  $I_n$  is the set of  $G$ -orbits of  $n$ -dimensional cells in  $X$ , and  $H_i$  denotes the stabilizer of the  $i$ -th cell in  $X$ . Similarly,

$$C_n(C_G(?) \backslash X^?; R) \cong \bigoplus_{i \in I_n} R[C_G(?) \backslash (G/H_i)^?] \cong \bigoplus_{i \in I_n} \text{RMor}_{\overline{\mathcal{F}}_C(G)}(?, H_i)$$

as  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. Using the Yoneda functors introduced above, we can write this as

$$C_n(X^?; R) \cong \bigoplus_{i \in I_n} Y_{\mathbf{O}_{\mathcal{C}}(G)}^{\mathbf{O}(G)}(H_i)$$

as an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module.

**Definition 5.3.** Let  $X$  be a CW-complex. The *ordinary (fusion) Bredon cohomology* of  $X$  with coefficients in an  $R\mathbf{O}(G)$ -module  $M$  is defined by

$$H_{\mathbf{O}(G)}^*(X^?; M) := H^*(\mathrm{Hom}_{R\mathbf{O}(G)}(C_*(X^?; R); M).$$

For an arbitrary collection  $\mathcal{C}$ , the *(fusion) Bredon cohomology* of  $X$  with coefficients in an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module  $M$  is defined by

$$H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; M) := H^*(\mathrm{Hom}_{R\mathbf{O}_{\mathcal{C}}(G)}(\mathrm{Tot}^{\oplus}(P_{*,*}), M),$$

where  $P_{*,*}$  is a Cartan-Eilenberg resolution of  $C_*(X^?; R)$ .

When  $\mathbf{O}(G) = \mathcal{O}(G)$ , this gives the ordinary Bredon cohomology  $H_{\mathcal{O}(G)}^*(X^?; M)$  as defined in Definition 1.4. When  $\mathbf{O}_{\mathcal{C}}(G) = \overline{\mathcal{F}}_{\mathcal{C}}(G)$ , then  $H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; M)$  is the *fusion Bredon cohomology* of  $X$  as defined in Definition 1.5.

**Lemma 5.4.** *If the isotropy subgroups of  $X$  lie in the collection  $\mathcal{C}$ , then  $C_*(X^?; R)$  is a chain complex of projective  $R\mathbf{O}_{\mathcal{C}}(G)$ -modules. In this case, for every  $R\mathbf{O}_{\mathcal{C}}(G)$ -module  $M$  we have*

$$H_{\mathbf{O}(G)}^*(X^?; M) \cong H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; M).$$

*Proof.* The first sentence follows from the descriptions of  $C_n(X^?; R)$  given above. The second part follows from the fact that the Cartan-Eilenberg resolution of a chain complex of projective modules is itself.  $\square$

**Proposition 5.5.** *Let  $X$  be a  $G$ -CW-complex and  $\mathcal{C}$  be a collection of subgroups of  $G$  such that the isotropy subgroups of  $X$  lie in  $\mathcal{C}$ . Then for every  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ , there is an isomorphism*

$$H_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(X^?; M) \cong H_{\mathcal{O}_{\mathcal{C}}(G)}^*(X^?; \mathrm{Res}_{pr} M).$$

*Proof.* In this case  $C_*(X^?; R)$  is a chain complex of projective  $R\mathbf{O}_{\mathcal{C}}(G)$ -modules, hence we have

$$H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; M) \cong H^*(\mathrm{Hom}_{R\mathbf{O}_{\mathcal{C}}(G)}(C_*(X^?; R), M).$$

By Lemma 4.12, for every  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ , there is an isomorphism

$$\mathrm{Hom}_{R\mathbf{O}_{\mathcal{C}}(G)}(C_*(X^?; R), \mathrm{Res}_{pr} M) \cong \mathrm{Hom}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}(C_*(C_G(?) \backslash X^?; R), M).$$

Hence we have

$$\begin{aligned} H_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^*(X^?; M) &\cong H^*(\mathrm{Hom}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}(C_*(X^?; R), M) \\ &\cong H^*(\mathrm{Hom}_{R\mathbf{O}_{\mathcal{C}}(G)}(C_*(X^?; R), \mathrm{Res}_{pr} M) \cong H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; \mathrm{Res}_{pr} M). \end{aligned}$$

$\square$

In general when  $\mathcal{C}$  does not include all the isotropy subgroups of  $X$ , then the isomorphism in Proposition 5.5 does not hold.

**Example 5.6.** Let  $G$  be a discrete group. If  $\mathcal{C} = \{1\}$ ,  $X = pt$ , and  $A$  is an abelian group with trivial  $G$ -action, then  $H_{\mathcal{O}_{\mathcal{C}}(G)}^*(X; A)$  is isomorphic to the group cohomology  $H^*(G; A)$  whereas  $H_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^i(X^?; A) = 0$  for  $i \geq 1$  because  $\text{Aut}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}(1) = 1$ . So if  $G$  and  $A$  are taken such that  $H^i(G, A) \neq 0$  for  $i \geq 1$ , then in this case the fusion Bredon cohomology and Bredon cohomology are not isomorphic.

**5.3. Hypercohomology spectral sequence.** There are two hypercohomology spectral sequences converging to the Bredon cohomology of a  $G$ -space introduced by Symonds [34, p. 279]. We show in this section that these spectral sequences exist also for the fusion Bredon cohomology. Throughout this section,  $\mathbf{O}_{\mathcal{C}}(G)$  denotes either  $\mathcal{O}_{\mathcal{C}}(G)$  or  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ . For a  $G$ -CW-complex  $X$ ,  $X^?$  denotes either  $X^?$  or  $C_G(?) \backslash X^?$  depending on the category  $\mathbf{O}_{\mathcal{C}}(G)$ .

**Proposition 5.7.** *Let  $X$  be a  $G$ -CW-complex and  $M$  be an  $\mathbf{RO}_{\mathcal{C}}(G)$ -module. For every integer  $j \geq 0$ , let  $\mathcal{H}_M^j$  denote the  $\mathbf{RO}(G)$ -module defined in Definition 5.2. Then there are two spectral sequences*

$${}^I E_2^{s,t} \cong H_{\mathcal{O}(G)}^s(X^?; \mathcal{H}_M^t) \quad \text{and} \quad {}^{II} E_2^{s,t} \cong \text{Ext}_{\mathbf{RO}_{\mathcal{C}}(G)}^s(H_t(X^?; R), M)$$

converging to the (fusion) Bredon cohomology  $H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; M)$ .

*Proof.* Let  $C_*(X^?; R)$  denote the chain complex of  $\mathbf{RO}_{\mathcal{C}}(G)$ -modules for  $X$ , and  $P_{*,*}$  denote the Cartan-Eilenberg projective resolution of  $C_*(X^?; R)$  (see [36, Def 5.7.1]). Applying the hom-functor  $\text{Hom}_{\mathbf{RO}_{\mathcal{C}}(-; M)}$  to  $P_{*,*}$ , we obtain a double (cochain) complex where

$$C^{s,t} = \text{Hom}_{\mathbf{RO}_{\mathcal{C}}(G)}(P_{s,t}, M)$$

for every  $s, t \geq 0$ . The differentials are in two directions

$$d_v^{s,t} : C^{s,t} \rightarrow C^{s,t+1} \quad \text{and} \quad d_h^{s,t} : C^{s,t} \rightarrow C^{s+1,t}$$

satisfying  $d_h d_v + d_v d_h = 0$ . The total complex of  $C^{*,*}$  is defined by  $X^n = \text{Tot}^n(C^{*,*}) = \bigoplus_{s+t=n} C^{s,t}$  with differential  $d_v + d_h$ . Note that this is a first quadrant double complex so we have  $\text{Tot}^{\oplus} = \text{Tot}^{\pi}$  in this case.

We can filter the total complex  $X^*$  by taking

$$F^s(X^{s+t}) = \bigoplus_{\substack{i+j=s+t, \\ i \geq s}} C^{i,j}.$$

So the the layers of the filtration are the columns of  $C^{*,*}$ . This gives a spectral sequence  $\{{}^I E_r^{s,t}\}_{r \geq 0}$  with

$${}^I E_2^{s,t} = H_h^s(H_v^t(C^{*,*}))$$

(see [3, Thm 3.4.2] and [36, Def 5.6.1]). For each  $s \geq 0$ , let  $I_s$  denote the set of  $G$ -orbits of  $s$ -dimensional cells in  $X$ , and let  $H_i$  denote the stabilizer of the  $i$ -th cell in  $G$ . Then

$$C_s(X^?; R) \cong \bigoplus_{i \in I_s} Y_{\mathbf{O}_{\mathcal{C}}(G)}^{\mathbf{O}(G)}(H_i)$$

as an  $\mathbf{RO}_{\mathcal{C}}(G)$ -module. By Lemma 5.1, we obtain

$$H_v^t(C^{s,*}) \cong \prod_{i \in I_s} \text{Ext}_{\mathbf{RO}_{\mathcal{C}}(G)}^t(Y_{\mathbf{O}_{\mathcal{C}}(G)}^{\mathbf{O}(G)}(H_i), M) \cong \prod_{i \in I_s} \overline{\mathcal{H}}_M^t(H_i) \cong \text{Hom}_{\mathbf{RO}(G)}(C_s(X^?; R), \overline{\mathcal{H}}_M^t)$$

for every  $s, t \geq 0$ . The horizontal differential  $d_h$  is induced by the differentials of the chain complex  $C_*(X^?; R)$ . Hence we have

$${}^I E_2^{s,t} = H_h^s(H_v^t(C^{*,*})) \cong H^s(\mathrm{Hom}_{\mathbf{RO}(G)}(C_*(X^?; R), \overline{\mathcal{H}}_M^t)) \cong H_{\mathbf{O}(G)}^s(X^?; \overline{\mathcal{H}}_M^t).$$

When  $\mathbf{O}(G) = \overline{\mathcal{F}}(G)$ , by Proposition 5.5, we have

$$H_{\overline{\mathcal{F}}(G)}^*(X^?; \overline{\mathcal{H}}_M^t) \cong H_{\mathcal{O}(G)}^*(X^?; \mathrm{Res}_{pr} \overline{\mathcal{H}}_M^t) = H_{\mathcal{O}(G)}^*(X^?; \mathcal{H}_M^t).$$

So for both categories  $\mathcal{O}(G)$  and  $\overline{\mathcal{F}}(G)$ , we have

$${}^I E_2^{s,t} \cong H_{\mathcal{O}(G)}^s(X^?; \mathcal{H}_M^t).$$

The second spectral sequence comes from filtering the total complex  $X^*$  horizontally by taking

$$F^s(X^{s+t}) = \bigoplus_{\substack{i+j=s+t, \\ j \geq s}} C^{i,j}.$$

In this case the layers of the filtration are the rows of  $C^{*,*}$ . We take  ${}^{II} E_0^{s,t} = C^{t,s}$  and obtain a spectral sequence  $\{{}^{II} E_r^{s,t}\}$  with

$${}^{II} E_2^{s,t} = H_v^s(H_h^t(C^{*,*}))$$

(see [36, Def 5.6.2]). By the properties of the Cartan-Eilenberg resolutions, the horizontal cohomology group  $H_h^t(C^{*,s}) \cong \mathrm{Hom}_{\mathbf{RO}_{\mathcal{C}}(G)}(Q_s, M)$  where  $Q_*$  is a projective resolution of  $H_t(X^?; R)$  (see [36, Def 5.7.1]). Hence

$${}^{II} E_2^{s,t} \cong H^s(\mathrm{Hom}_{\mathbf{RO}_{\mathcal{C}}(G)}(Q_*, M)) \cong \mathrm{Ext}_{\mathbf{RO}_{\mathcal{C}}(G)}^s(H_t(X^?; R), M)$$

for all  $s, t \geq 0$ . □

One special case of the above hypercohomology spectral sequence is the case where  $C_*(X^H)$  (resp.  $C_*(C_G(H) \backslash X^H)$ ) has a homology of a point for every  $H \in \mathcal{C}$ . In this case we say  $X$  is  $\mathbf{RO}_{\mathcal{C}}(G)$ -acyclic (resp.  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -acyclic).

**Theorem 5.8.** *Let  $X$  be an  $\mathbf{RO}_{\mathcal{C}}(G)$ -acyclic  $G$ -CW-complex and  $M$  be an  $\mathbf{RO}_{\mathcal{C}}(G)$ -module. For every integer  $j \geq 0$ , let  $\mathcal{H}_M^j$  denote the  $\mathbf{RO}(G)$ -module defined in Definition 5.2. Then there is a spectral sequence*

$$E_2^{s,t} \cong H_{\mathcal{O}(G)}^s(X^?; \mathcal{H}_M^t) \Rightarrow H^{s+t}(\mathbf{O}_{\mathcal{C}}(G); M).$$

*Proof.* By Proposition 5.7, there are two spectral sequences which converge to the Bredon cohomology  $H_{\mathbf{O}_{\mathcal{C}}(G)}^*(X^?; M)$ . Since  $X$  is  $\mathbf{RO}_{\mathcal{C}}(G)$ -acyclic, we have

$$H_t(X^?; R) \cong \begin{cases} R & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

as an  $\mathbf{RO}_{\mathcal{C}}(G)$ -module. This gives that the spectral sequence  ${}^{II} E_2^{*,*}$  collapses at the  $E_2$ -page to the horizontal line at  $t = 0$ , hence for each  $n \geq 0$ , there is an isomorphism

$$H_{\mathbf{O}_{\mathcal{C}}(G)}^n(X^?; M) \cong H^n(\mathbf{O}_{\mathcal{C}}(G); M).$$

Now the first spectral sequence  ${}^I E_2^{s,t}$  in Proposition 5.7 becomes a spectral sequence that converges to  $H^*(\mathbf{O}_{\mathcal{C}}(G); M)$ . □

## 6. DWYER SPACES FOR HOMOLOGY DECOMPOSITIONS

The Dwyer spaces  $X_{\mathbf{C}}^{\alpha}, X_{\mathbf{C}}^{\beta}, X_{\mathbf{C}}^{\delta}$  for centralizer, subgroup, and normalizer decompositions were introduced by Dwyer to give a unified treatment for the homology decompositions for classifying spaces of discrete groups. In this section we give the definitions of Dwyer spaces and show that the Dwyer spaces for the centralizer and normalizer decompositions satisfy the conditions of Theorem 5.8. For more details on the homology decompositions of classifying spaces, we refer the reader to [12, §7], [4, Chp 5], and [13].

**6.1.  $G$ -Categories.** We first recall some definitions on categories with a group action. We follow the terminology and notation introduced by Grodal in [13] and [14]. Let  $G$  be a discrete group. A  $G$ -category is a category  $\mathbf{C}$  together with a set of functors  $F_g : \mathbf{C} \rightarrow \mathbf{C}$  for all  $g \in G$  which satisfy  $F_g \circ F_h = F_{gh}$  for all  $g, h \in G$ , and  $F_1 = \text{id}_{\mathbf{C}}$ . We denote the image of the object  $x \in \text{Ob}(\mathbf{C})$  under  $F_g$  by  $gx$ . Similarly for a morphism  $\alpha : x \rightarrow y$ , we write  $F_g(\alpha) = g\alpha$ . The nerve  $\mathcal{N}(\mathbf{C})$  of a small category  $\mathbf{C}$  is a simplicial set whose  $n$ -simplices are given by a chain of composable maps  $x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n$ . When  $\mathbf{C}$  is a  $G$ -category, the nerve  $\mathcal{N}(\mathbf{C})$  is a  $G$ -simplicial set where the  $G$ -action on the simplices is given by

$$g(x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n) = (gx_0 \xrightarrow{g\alpha_1} gx_1 \xrightarrow{g\alpha_2} \dots \xrightarrow{g\alpha_n} gx_n)$$

for every  $g \in G$ .

The geometric realization  $|\mathcal{N}(\mathbf{C})|$  of the simplicial set  $\mathcal{N}(\mathbf{C})$  is called the geometric realization of the category  $\mathbf{C}$ , and it is denoted by  $|\mathbf{C}|$ . When  $\mathbf{C}$  is a  $G$ -category, then the geometric realization  $|\mathbf{C}|$  has a  $G$ -CW-complex structure. For every subgroup  $H \leq G$ , let  $\mathbf{C}^H$  denote the subcategory of  $\mathbf{C}$  whose objects and morphisms are the objects and morphisms of  $\mathbf{C}$  that are fixed under the  $H$ -action. We have

$$|\mathbf{C}|^H = |\mathcal{N}(\mathbf{C})|^H = |\mathcal{N}(\mathbf{C})^H| = |\mathcal{N}(\mathbf{C}^H)| = |\mathbf{C}^H|.$$

The orbit space  $|\mathbf{C}|/G$  is homeomorphic to the topological realization of the simplicial set  $\mathcal{N}(\mathbf{C})/G$ .

A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  between two  $G$ -categories is called a  $G$ -functor if for every  $g \in G$ , (i)  $F(gx) = gF(x)$  for every  $x \in \text{Ob}(\mathbf{C})$ , and (ii)  $F(g\alpha) = gF(\alpha)$  for every morphism  $\alpha$  in  $\mathbf{C}$ . If  $F : \mathbf{C} \rightarrow \mathbf{D}$  is a  $G$ -functor, then the induced map on the realizations  $|F| : |\mathbf{C}| \rightarrow |\mathbf{D}|$  is a  $G$ -map. A natural transformation  $\mu : F \rightarrow F'$  between two  $G$ -functors  $F, F' : \mathbf{C} \rightarrow \mathbf{D}$  is called a *natural  $G$ -transformation* if  $\mu_{gx} : F(gx) \rightarrow F'(gx)$  is equal to  $g\mu_x : gF(x) \rightarrow gF'(x)$  for every  $g \in G$  and  $x \in \text{Ob}(\mathbf{C})$ . If  $\mu : F \rightarrow F'$  is a natural  $G$ -transformation, then the realizations  $|F|, |F'| : |\mathbf{C}| \rightarrow |\mathbf{D}|$  are  $G$ -homotopic. These give the following:

**Proposition 6.1** ([12, Prop 5.6]). *Let  $\mathbf{C}$  be a  $G$ -category and  $x_0 \in \text{Ob}(\mathbf{C}^G)$ . If the identity functor of  $\mathbf{C}$  is connected to the constant functor  $c_{x_0} : \mathbf{C} \rightarrow \mathbf{C}$  by a zigzag of natural  $G$ -transformations, then  $|\mathbf{C}|$  is  $G$ -equivariantly contractible. In particular, the orbit space  $|\mathbf{C}|/G$  is contractible.*

Our main example of a  $G$ -category is a category obtained by applying Grothendieck construction to a functor  $F : \mathbf{D} \rightarrow G\text{-sets}$ . In this case the Grothendieck construction is a category  $\mathbf{D} \wr F$  whose objects are pairs  $(d, x)$  where  $d \in \text{Ob}(\mathbf{D})$  and  $x \in F(d)$ . A morphism  $(d, x) \rightarrow (d', x')$  is a pair  $(\alpha, x)$  such that  $\alpha : d \rightarrow d'$  is a morphism in  $\mathbf{D}$  such that  $F(\alpha)(x) = x'$ . The composition of morphisms is defined by  $(\alpha', x') \circ (\alpha, x) = (\alpha' \circ \alpha, x)$ . The  $G$ -action on  $\mathbf{D} \wr F$  is given by  $g(d, x) = (d, gx)$  and  $g(\alpha, x) = (\alpha, gx)$  for every  $g \in G$ . There is a natural isomorphism of



simplicial sets

$$\operatorname{hocolim}_{d \in \mathbf{D}} F \xrightarrow{\cong} \mathcal{N}(\mathbf{D} \wr F)$$

(see [35, Thm 1.2] and [31, p. 3] for details). It is easy to see that this isomorphism is an isomorphism of  $G$ -simplicial sets, so the fixed point subspaces and the orbit space of  $X = |\operatorname{hocolim}_{d \in \mathbf{D}} F|$  can be computed using the nerve  $\mathcal{N}(\mathbf{D} \wr F)$ .

**Remark 6.2.** Throughout the paper whenever it makes sense we will work in the category of simplicial sets and call a simplicial set a *space* and a  $G$ -simplicial set a  $G$ -*space*. Note that the geometric realization of a  $G$ -space is a  $G$ -CW-complex, so we can associate to a  $G$ -space  $X$  a chain complex  $C_*(X^?; R)$  over (fusion) orbit category using its geometric realization. This allows us to apply the theorems proved for  $G$ -CW-complexes to  $G$ -spaces.

In the following subsections,  $G$  is a discrete group and  $\mathcal{C}$  denotes an arbitrary collection of subgroups in  $G$ .

**6.2. The Dwyer space for the subgroup decomposition.** Let  $\mathcal{O}_{\mathcal{C}}(G)$  denote the orbit category of  $G$  over the collection  $\mathcal{C}$ . The collection  $\mathcal{C}$  can be considered as a poset with the order relation given by the inclusion of subgroups. There is a  $G$ -action on this poset by conjugation. Let  $K_{\mathcal{C}}$  denote the order complex of the poset  $\mathcal{C}$ . The simplices of  $K_{\mathcal{C}}$  are the chains of subgroups  $\sigma = (H_0 < H_1 < \dots < H_n)$  in  $\mathcal{C}$ , and an element  $g \in G$  acts on a chain by the action defined by

$$g\sigma = (gH_0g^{-1} < gH_1g^{-1} < \dots < gH_ng^{-1}).$$

There is a simplicial set associated to the simplicial complex  $K_{\mathcal{C}}$ . We denote this simplicial set by  $\mathcal{N}\mathcal{C}$  and its geometric realization by  $|\mathcal{C}|$ .

**Definition 6.3.** Consider the Borel construction  $EG \times_G \mathcal{N}\mathcal{C} := (EG \times \mathcal{N}\mathcal{C})/G$ . A collection  $\mathcal{C}$  is called *p-ample* if the map induced by projection to the first coordinate  $EG \times_G \mathcal{N}\mathcal{C} \rightarrow BG$  is a mod- $p$  homology isomorphism.

The Dwyer space for the subgroup decomposition is defined as follows.

**Definition 6.4.** Let  $\tilde{\beta} : \mathcal{O}_{\mathcal{C}}(G) \rightarrow G\text{-Sets}$  denote the functor that sends  $H \in \mathcal{C}$  to the transitive  $G$ -set  $G/H$ . The  $G$ -space

$$X_{\mathcal{C}}^{\beta} := \operatorname{hocolim}_{\mathcal{O}_{\mathcal{C}}(G)} \tilde{\beta}$$

is called the *Dwyer space for the subgroup decomposition*.

The space  $X_{\mathcal{C}}^{\beta}$  is a nerve of a category  $\mathbf{X}_{\mathcal{C}}^{\beta}$  whose objects are pairs  $(H, gH)$  with  $H \in \mathcal{C}$  and  $gH \in G/H$ . A morphism  $(H, xH) \rightarrow (K, yK)$  is a  $G$ -map  $f : G/H \rightarrow G/K$  such that  $f(xH) = yK$ . There is a  $G$ -map  $X_{\mathcal{C}}^{\beta} \rightarrow \mathcal{N}\mathcal{C}$  which is a weak equivalence (non-equivariantly). This gives a homotopy equivalence  $EG \times_G X_{\mathcal{C}}^{\beta} \rightarrow EG \times_G \mathcal{N}\mathcal{C}$ . If  $\beta : \mathcal{O}_{\mathcal{C}}(G) \rightarrow \text{Spaces}$  is the functor defined by  $\beta = EG \times_G \tilde{\beta}$ , then there is a natural isomorphism

$$EG \times_G X_{\mathcal{C}}^{\beta} \cong \operatorname{hocolim}_{\mathcal{O}_{\mathcal{C}}(G)} \beta$$

(see [4, Prop 4.7.6]). Hence if  $\mathcal{C}$  is a  $p$ -ample collection, then the composition

$$\operatorname{hocolim}_{\mathcal{O}_{\mathcal{C}}(G)} \beta \cong EG \times_G X_{\mathcal{C}}^{\beta} \rightarrow EG \times_G \mathcal{N}\mathcal{C} \rightarrow BG$$

is a mod- $p$  homology isomorphism (see [12, Prop 7.14] or [4, Thm 5.5.4] for details). This mod- $p$  homology isomorphism is called the *mod- $p$  subgroup decomposition for  $BG$*

The Dwyer space  $X = X_{\mathcal{C}}^{\beta}$  is also denoted by  $E_{\mathcal{C}}G$  and called the universal space for the collection  $\mathcal{C}$ . For every  $H \in \mathcal{C}$ , the fixed point subspace  $X^H$  is homotopy equivalent to the realization of the subposet  $\mathcal{C}_{\geq H} = \{K \in \mathcal{C} \mid K \geq H\}$  of  $\mathcal{C}$ , hence it is contractible (see [15]). However, in general the orbit space  $C_G(H) \backslash X^H$  is not  $\mathbb{Z}_{(p)}$ -acyclic (see Example 11.6). So the Dwyer space  $X = X_{\mathcal{C}}^{\delta}$  does not satisfy the conditions of Theorem 5.8 for the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ .

**6.3. The Dwyer space for the centralizer decomposition.** The  $\mathcal{C}$ -conjugacy category  $\mathcal{A}_{\mathcal{C}}(G)$  is the category whose objects are pairs  $(H, [i])$  where  $H$  is a group and  $[i]$  is a  $G$ -conjugacy class of monomorphisms  $i : H \rightarrow G$  such that  $i(H) \in \mathcal{C}$ . The  $G$ -action on a monomorphism  $i : H \rightarrow G$  is defined by  $g \cdot i := c_g \circ i$  for  $g \in G$  where  $c_g : G \rightarrow G$  is conjugation map  $x \rightarrow gxg^{-1}$ . A morphism  $(H, [i]) \rightarrow (H', [i'])$  is given by a group homomorphism  $f : H \rightarrow H'$  such that  $[i] = [i' \circ f]$ .

**Remark 6.5.** The category  $\mathcal{A}_{\mathcal{C}}(G)$  is not small. To take homotopy colimits over  $\mathcal{A}_{\mathcal{C}}(G)$  we replace it with an equivalent category which is small. For example we can use the subcategory of  $\mathcal{A}_{\mathcal{C}}(G)$  where the objects are pairs  $(H, [i])$  where  $H \in \mathcal{C}$  and  $[i]$  is a conjugacy class of monomorphisms  $i : H \rightarrow G$  with  $i(H) \in \mathcal{C}$  as above.

**Definition 6.6.** Let  $\tilde{\alpha}_{\mathcal{C}} : \mathcal{A}_{\mathcal{C}}(G)^{op} \rightarrow G\text{-Sets}$  denote the functor which takes  $(H, [i])$  to the  $G$ -conjugacy class  $[i]$ . The *Dwyer space for the centralizer decomposition* over the collection  $\mathcal{C}$  is the  $G$ -space defined by

$$X_{\mathcal{C}}^{\alpha} := \operatorname{hocolim}_{\mathcal{A}_{\mathcal{C}}(G)^{op}} \tilde{\alpha}_{\mathcal{C}}.$$

The space  $X_{\mathcal{C}}^{\alpha}$  is also denoted by  $EA_{\mathcal{C}}$ . For every  $(H, [i])$  in  $\mathcal{A}_{\mathcal{C}}(G)$ , we have

$$\tilde{\alpha}_{\mathcal{C}}(H, [i]) \cong G/C_G(i(H))$$

as  $G$ -sets. The Dwyer space  $X_{\mathcal{C}}^{\alpha}$  is isomorphic to the nerve of the category  $\mathbf{Z}_{\mathcal{C}}^{\alpha}$  whose objects are the pairs  $(H, i)$  where  $H$  is a group and  $i : H \rightarrow G$  is a monomorphism such that  $i(H) \in \mathcal{C}$ . There is a unique morphisms from  $(H', i')$  to  $(H, i)$  if there is a group homomorphisms  $f : H' \rightarrow H$  such that  $i = i' \circ f$ . As it often done in the literature, we will work with the opposite category  $\mathbf{X}_{\mathcal{C}}^{\alpha} := (\mathbf{Z}_{\mathcal{C}}^{\alpha})^{op}$  where there is a unique morphisms from  $(H, i)$  to  $(H', i')$  if there is a group homomorphisms  $f : H \rightarrow H'$  such that  $i = i' \circ f$  (see [12, Prop 7.12] and [4, Thm 5.44]). Note that the realization of the category  $\mathbf{X}_{\mathcal{C}}^{\alpha}$  is  $G$ -homeomorphic to the realization of the Dwyer space  $X_{\mathcal{C}}^{\alpha}$ , so it does not affect the proofs to work with either category.

If  $\alpha_{\mathcal{C}} : \mathcal{A}_{\mathcal{C}}(G)^{op} \rightarrow Spaces$  denote the functor  $EG \times_G \tilde{\alpha}_{\mathcal{C}}$ , then there is a natural isomorphism

$$\operatorname{hocolim}_{\mathcal{A}_{\mathcal{C}}(G)^{op}} \alpha_{\mathcal{C}} \cong EG \times_G X_{\mathcal{C}}^{\alpha}.$$

There is also a  $G$ -map  $X_{\mathcal{C}}^{\alpha} \rightarrow \mathcal{N}\mathcal{C}$  which is a weak equivalence (see proof of [12, Prop 7.12]). If  $\mathcal{C}$  is a  $p$ -ample collection, then the composition

$$\operatorname{hocolim}_{\mathcal{A}_{\mathcal{C}}(G)^{op}} \alpha_{\mathcal{C}} \cong EG \times_G X_{\mathcal{C}}^{\alpha} \rightarrow EG \times_G \mathcal{N}\mathcal{C} \rightarrow BG$$

is a mod- $p$  homology isomorphism. This isomorphism is called the *centralizer decomposition* for  $G$  (see [12, Prop 7.12]). For the Dwyer space  $X_{\mathcal{C}}^{\alpha}$  we prove the following:

**Proposition 6.7.** *Let  $G$  be a discrete group and  $\mathcal{E}$  denote the collection of all nontrivial elementary abelian  $p$ -subgroups in  $G$ . If  $X = X_{\mathcal{E}}^{\alpha}$  is the Dwyer space for the centralizer decomposition for  $G$ , then for every nontrivial finite  $p$ -subgroup  $P \leq G$ , the fixed point subspace  $X^P$  and the orbit space  $C_G(P) \backslash X^P$  are contractible.*

*Proof.* A proof for the contractibility of  $X^P$  can be found in [12, §13.3]. The argument given there for the contractibility of  $X^P$  does not give a  $C_G(P)$ -equivariant contraction. We modify Dwyer's argument to obtain a  $C_G(P)$ -equivariant contraction.

Let  $X = X_{\mathcal{E}}^{\alpha}$  and  $P$  be a fixed nontrivial finite  $p$ -subgroup of  $G$ . The fixed point subspace  $|X|^P = |X^P|$  is  $G$ -homeomorphic to the realization of the category  $\mathbf{C} := (\mathbf{X}_{\mathcal{E}}^{\alpha})^P$  whose objects are the pairs  $(E, i)$  that satisfy  $i(E) \leq C_G(P)$ . Let  $Z$  be a central subgroup of order  $p$  in  $P$ . Then  $Z$  is a subgroup of  $C_G(P)$ , and furthermore it is a central subgroup of  $C_G(P)$  because  $Z \leq P$ . Let  $j : Z \rightarrow G$  denote the inclusion map and let  $c_{(Z,j)} : \mathbf{C} \rightarrow \mathbf{C}$  denote the constant functor that takes every object  $(E, i)$  in  $\mathbf{C}$  to  $(Z, j)$ . We will show that there is a zigzag of natural  $C_G(P)$ -transformations between  $\text{id}_{\mathbf{C}}$  and the constant functor  $c_{(Z,j)}$ . By Proposition 6.1, this will imply that  $C_G(P) \backslash X^P$  is contractible.

For every  $(E, i) \in \mathbf{C}$ , the elementary abelian  $p$ -subgroups  $i(E)$  and  $Z$  commute with each other. In particular the product  $i(E) \cdot Z$  is an elementary abelian  $p$ -subgroup of  $C_G(P)$ . Let  $(E', i')$  be the pair such that

$$E' := \begin{cases} E & \text{if } Z \leq i(E) \\ E \times Z & \text{if } Z \not\leq i(E) \end{cases} \quad i' := \begin{cases} i & \text{if } E' = E, \\ (e, z) \rightarrow i(e)z & \text{if } E' \neq E. \end{cases}$$

Let  $F : \mathbf{C} \rightarrow \mathbf{C}$  be the functor that sends the pair  $(E, i)$  to  $(E', i')$ , and sends a morphism  $f : (E_1, i_1) \rightarrow (E_2, i_2)$  to the morphism  $f' : (E'_1, i'_1) \rightarrow (E'_2, i'_2)$  where

- (1)  $f' = f$  if  $E'_1 = E_1$  and  $E'_2 = E_2$ ,
- (2)  $f' : E_1 \times Z \rightarrow E_2$  is defined by  $(e, z) \rightarrow f(e)i_2^{-1}(z)$  if  $E'_1 \neq E_1$  and  $E'_2 = E_2$ ,
- (3)  $f' = f \times \text{id}_Z$  if  $E'_1 \neq E_1$  and  $E'_2 \neq E_2$ .

Note that the case  $E'_1 = E_1$  and  $E'_2 \neq E_2$  can not happen because the morphism  $f : (E_1, i_1) \xrightarrow{f} (E_2, i_2)$  is a group homomorphism  $f : E_1 \rightarrow E_2$  such that  $i_2 \circ f = i_1$ , hence  $i_1(E_1) = i_2(f(E_1)) \leq i_2(E_2)$ , so  $Z \leq i_1(E_1)$  implies  $Z \leq i_2(E_2)$ .

There is a zigzag of natural transformations

$$\text{id}_{\mathbf{C}} \xrightarrow{\mu} F \xleftarrow{\eta} c_{(Z,j)}$$

between the identity functor and the constant functor with value  $(Z, j)$ . The morphism  $\mu_{(E,i)} : (E, i) \rightarrow (E', i')$  is defined by the group homomorphism

$$(\mu_{(E,i)} : E \rightarrow E') = \begin{cases} \text{id}_E : E \rightarrow E & \text{if } E' = E, \\ E \rightarrow E \times Z & \text{defined by } e \rightarrow (e, 1) & \text{if } E' \neq E. \end{cases}$$

It is easy to check that  $\mu$  is a natural transformation. The only nontrivial case to check is when  $E'_1 \neq E_1$  and  $E'_2 = E_2$ . In this case, for every morphism  $f : (E_1, i_1) \rightarrow (E_2, i_2)$ , and for every  $e \in E_1$ , we have

$$f'(\mu_{(E_1, i_1)}(e)) = f'(e, 1) = f(e)i_2^{-1}(1) = f(e) = \mu_{(E_2, i_2)}(f(e)).$$

Hence, the equality  $f' \circ \mu_{(E_1, i_1)} = \mu_{(E_2, i_2)} \circ f$  holds in this case.

For each pair  $(E, i)$ , the morphism  $\eta_{(E,i)}$  is defined by the group homomorphism

$$(\eta_{(E,i)} : Z \rightarrow E') = \begin{cases} i^{-1}|_Z & \text{if } E' = E, \\ Z \rightarrow E \times Z & \text{defined by } z \rightarrow (1, z) & \text{if } E' \neq E. \end{cases}$$

In the case  $E'_1 = E_1$  and  $E'_2 = E_2$ , for each morphism  $f : (E_1, i_1) \rightarrow (E_2, i_2)$ , and for each  $z \in Z$ , we have  $f(i_1^{-1}(z)) = i_2^{-1}(z)$ , hence  $f' \circ \eta_{(E_1, i_1)} = \eta_{(E_2, i_2)} \circ \text{id}_{(Z, j)}$  holds. It is easy to check that

this equality holds in the remaining cases and conclude that  $\eta$  is a natural transformation. Hence  $X^P$  is contractible.

It remains to show that  $F$  is a  $C_G(P)$ -functor, and  $\mu$  and  $\eta$  are  $C_G(P)$ -equivariant transformations. Since  $Z$  is central in  $C_G(P)$ , for every  $g \in C_G(P)$ , we have  $Z \leq i(E)$  if and only if  $Z \leq (c_g \circ i)(E)$ . So  $E'$  for  $(E, i)$  and  $E'$  for  $(E, c_g \circ i)$  are equal. We also have  $(c_g \circ i)' = c_g \circ i'$ . This gives

$$F(g \cdot (E, i)) = F(E, c_g \circ i) = (E', (c_g \circ i)') = (E', c_g \circ i') = g \cdot F(E, i)$$

for every  $g \in C_G(P)$ . This shows that the functor  $F$  is  $C_G(P)$ -invariant on the objects. If the morphism  $f : (E_1, i_1) \rightarrow (E_2, i_2)$  is defined by the group homomorphism  $f : E_1 \rightarrow E_2$ , then  $g \cdot f : (E_1, c_g \circ i_1) \rightarrow (E_2, c_g \circ i_2)$  is also defined by  $f : E_1 \rightarrow E_2$ . Then the morphism  $(g \cdot f)' : (E_1', c_g \circ i_1') \rightarrow (E_2', c_g \circ i_2')$  is given by  $f'$ . This gives that  $(g \cdot f)' = g \cdot f'$  for every  $g \in C_G(P)$ . Hence  $F$  is a  $C_G(P)$ -functor.

For every  $g \in C_G(P)$ , the morphism

$$\mu_{g \cdot (E, i)} = \mu_{(E, c_g \circ i)} : (E, c_g \circ i) \rightarrow (E', (c_g \circ i)') = (E', c_g \circ i')$$

is equal to the morphism

$$g \cdot \mu_{(E, i)} : g \cdot (E, i) \rightarrow g \cdot (E', i').$$

Hence  $\mu$  is a  $C_G(P)$ -transformation. For the transformation  $\eta$ , note that for every  $g \in C_G(P)$ , the morphism

$$\eta_{g \cdot (E, i)} = \eta_{(E, c_g \circ i)} : (Z, j) \rightarrow (E', (c_g \circ i)') = (E', c_g \circ i')$$

is defined by the group homomorphism  $(c_g \circ i)^{-1}|_Z : Z \rightarrow E'$  if  $E' = E$ . Since  $Z$  is central in  $C_G(P)$ , we have

$$(c_g \circ i)^{-1}|_Z = i^{-1} \circ c_{g^{-1}}|_Z = i^{-1}|_Z,$$

hence  $\eta_{g \cdot (E, i)}$  is equal to

$$g \cdot \eta_{(E, i)} : (Z, j) \rightarrow g \cdot (E', i') = (E', c_g \circ i').$$

In the case  $E' \neq E$ , both  $\eta_{g \cdot (E, i)}$  and  $g \cdot \eta_{(E, i)}$  are given by the group homomorphism  $Z \rightarrow E \times Z$  defined by  $z \rightarrow (1, z)$ . Hence  $\eta$  is  $C_G(P)$ -equivariant.

By Proposition 6.1, we conclude that the orbit space  $C_G(P) \backslash X^P$  is contractible.  $\square$

**6.4. Dwyer space for the normalizer decomposition.** Let  $K_{\mathcal{C}}$  denote the order complex of the poset  $\mathcal{C}$ . There is a  $G$ -action on the poset  $\mathcal{C}$  by conjugation, which makes  $K_{\mathcal{C}}$  a  $G$ -simplicial complex. The simplices of  $K_{\mathcal{C}}$  forms a poset with order relation given by  $\tau \leq \sigma$  if  $\tau$  is a face of  $\sigma$ . We denote this poset by  $sd(K_{\mathcal{C}})$ , and its opposite poset by  $Sd_{\mathcal{C}}(G)$ . There is a  $G$ -action on  $Sd_{\mathcal{C}}(G)$ , and the stabilizer of the simplex  $\sigma := (H_0 < H_1 < \dots < H_n)$  is the subgroup  $N_G(\sigma) = \bigcap_{i=0}^n N_G(H_i)$ .

**Definition 6.8.** The category of *orbit simplices*, denoted by  $Sd_{\mathcal{C}}(G)/G$ , is the category whose objects are the  $G$ -orbits  $[\sigma]$  of the simplices of  $K_{\mathcal{C}}$ . Between two objects there is a unique morphism  $[\sigma] \rightarrow [\tau]$  if there is an element  $g \in G$  such that  $\tau$  is a face of  $g\sigma$ .

Note that  $Sd_{\mathcal{C}}(G)/G$  is a poset category. There is a functor  $\tilde{\delta}_{\mathcal{C}} : Sd_{\mathcal{C}}(G)/G \rightarrow G\text{-Sets}$  which takes a  $G$ -orbit  $[\sigma]$  to itself as a  $G$ -set. We can describe the effect of  $\tilde{\delta}_{\mathcal{C}}$  on morphisms as follows: Choose a representative in each  $G$ -orbit of simplices and write  $[\sigma]$  for the  $G$ -orbit whose representative is  $\sigma$ . The stabilizer of  $\sigma = (H_0 < H_1 < \dots < H_n)$  under the  $G$ -action is  $N_G(\sigma) = \bigcap_{i=0}^n N_G(H_i)$ . Given a morphism  $[\sigma] \rightarrow [\tau]$ , let  $g \in G$  be such that  $\tau$  is a face of  $g\sigma$ . This gives that  $gN_G(\sigma)g^{-1} = N_G(g\sigma) \leq N_G(\tau)$ . Hence there is a  $G$ -map  $f : [\sigma] \rightarrow [\tau]$  defined

by  $f(g'\sigma) = g'g^{-1}\tau$  for all  $g' \in G$ . Note that the  $G$ -map  $f$  does not depend on the group element  $g \in G$ . If  $g_1, g_2 \in G$  are such that  $\tau$  is a face of  $g_1\sigma$  and  $g_2\sigma$ , then we must have  $g_1^{-1}\tau = g_2^{-1}\tau$  since they are both subchains of  $\sigma$  with subgroups in both chains having the same order. Hence the  $G$ -map  $f : [\sigma] \rightarrow [\tau]$  is well-defined. It is clear that  $\tilde{\delta}_{\mathcal{C}} : Sd_{\mathcal{C}}(G)/G \rightarrow G\text{-Sets}$  with these assignments defines a functor.

**Definition 6.9.** The *Dwyer space for the normalizer decomposition* is the  $G$ -space

$$X_{\mathcal{C}}^{\delta} := \operatorname{hocolim}_{Sd_{\mathcal{C}}(G)/G} \tilde{\delta}.$$

The space  $X_{\mathcal{C}}^{\delta}$  is isomorphic to the nerve of the poset category  $\mathbf{X}_{\mathcal{C}}^{\delta}$  whose objects are simplices of  $K_{\mathcal{C}}$ , and there is one morphism  $\sigma \rightarrow \tau$  if  $\tau$  is a subchain of  $\sigma$ . The category  $\mathbf{X}_{\mathcal{C}}^{\delta}$  is the opposite category of the simplex category of  $K_{\mathcal{C}}$ . This gives that  $|X_{\mathcal{C}}^{\delta}|$  is  $G$ -homeomorphic to  $|sd(K_{\mathcal{C}})|$ . By a standard result on barycentric subdivisions,  $|K_{\mathcal{C}}|$  is  $G$ -homeomorphic to  $|sd(K_{\mathcal{C}})|$  (see [33, Lemma 1.6.4]). Hence  $|X_{\mathcal{C}}^{\delta}|$  is  $G$ -homeomorphic to  $|K_{\mathcal{C}}|$ .

**Lemma 6.10.** *Let  $|\mathcal{C}|$  denote the realization of the simplicial set  $\mathcal{NC}$  of the poset  $\mathcal{C}$ . Then  $|X_{\mathcal{C}}^{\delta}|$  is  $G$ -homeomorphic to  $|\mathcal{C}|$ .*

*Proof.*  $\mathcal{NC}$  is the simplicial set for the simplicial complex  $K_{\mathcal{C}}$ , so we have a  $G$ -homeomorphism  $|\mathcal{C}| \cong |K_{\mathcal{C}}|$ . Hence by the remarks above  $|X_{\mathcal{C}}^{\delta}|$  and  $|\mathcal{C}|$  are  $G$ -homeomorphic.  $\square$

If  $\delta_{\mathcal{C}} : Sd_{\mathcal{C}}(G)/G \rightarrow Spaces$  denote the functor  $EG \times_G \tilde{\delta}_{\mathcal{C}}$ , then there is a natural isomorphism

$$\operatorname{hocolim}_{Sd_{\mathcal{C}}(G)/G} \delta_{\mathcal{C}} \cong EG \times_G X_{\mathcal{C}}^{\delta}.$$

If  $\mathcal{C}$  is a  $p$ -ample collection, then the composition

$$|\operatorname{hocolim}_{Sd_{\mathcal{C}}(G)/G} \delta_{\mathcal{C}}| \cong EG \times_G |X_{\mathcal{C}}^{\delta}| \rightarrow EG \times_G |\mathcal{C}| \rightarrow BG$$

is a mod- $p$  homology isomorphism (see [12, Prop 7.17]). This mod- $p$  homology isomorphism is called the *normalizer decomposition* for  $G$ .

We are now going to prove an important property of the Dwyer space  $X_{\mathcal{C}}^{\delta}$ . We first give a definition.

**Definition 6.11.** We say a collection  $\mathcal{C}$  of subgroups in  $G$  is *closed under taking products* if it satisfies the following condition: if  $P, Q \in \mathcal{C}$  such that  $PQ$  is a subgroup in  $G$ , then  $PQ \in \mathcal{C}$ .

We prove the following:

**Proposition 6.12.** *Let  $\mathcal{C}$  be a collection of subgroups of  $G$  closed under taking products, and let  $X = X_{\mathcal{C}}^{\delta}$  be the Dwyer space for the normalizer decomposition for  $G$ . Then for every  $P \in \mathcal{C}$ , the fixed point set  $X^P$  and the orbit space  $C_G(P) \backslash X^P$  are contractible.*

*Proof.* By Lemma 6.10, the realization  $|X|$  is  $G$ -homeomorphic to  $|\mathcal{C}|$ , and hence it is enough to prove the statements for the  $G$ -space  $|\mathcal{C}|$ . For each  $P \in \mathcal{C}$ , we have  $|\mathcal{C}|^P = |\mathcal{C}^P|$ , so we need to show that  $\mathcal{C}^P$  is a contractible poset. If  $Q \in \mathcal{C}^P$ , then  $P \leq N_G(Q)$ , hence  $PQ$  is a subgroup in  $G$ . Since  $\mathcal{C}$  is closed under taking products,  $PQ \in \mathcal{C}$ . Since  $P$  normalizes  $Q$ , it normalizes  $PQ$ . Hence  $PQ \in \mathcal{C}^P$ . There is a zigzag of poset contractions  $P \leq PQ \geq Q$  in  $\mathcal{C}^P$ , hence the poset  $\mathcal{C}^P$  is canonically contractible. The poset maps in the above contractions are  $C_G(P)$ -equivariant maps, hence  $|\mathcal{C}|^P$  is  $C_G(P)$ -equivariantly contractible. We conclude that the orbit space  $C_G(P) \backslash X^P$  is contractible.  $\square$

## 7. HYPERCOHOMOLOGY SPECTRAL SEQUENCES FOR DWYER SPACES

As an immediate consequence of Theorem 5.8 and Propositions 6.7 and 6.12, we obtain spectral sequences which converge to the cohomology of the orbit category and the fusion orbit category. We can state this conclusion as a corollary as follows:

**Corollary 7.1.** *Let  $\mathbf{O}_{\mathcal{C}}(G) = \mathcal{O}_{\mathcal{C}}(G)$  or  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ , and let  $M$  be an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module.*

- (1) *Let  $X = X_{\mathcal{E}}^{\alpha}$  be the Dwyer space for the centralizer decomposition over the collection  $\mathcal{E}$  of nontrivial elementary abelian  $p$ -subgroups, and  $\mathcal{C}$  be any collection of nontrivial  $p$ -subgroups of  $G$ , or*
- (2) *let  $\mathcal{C}$  be a collection of  $p$ -subgroups of  $G$  that is closed under taking products, and  $X = X_{\mathcal{C}}^{\delta}$  be the Dwyer space for normalizer decomposition over  $\mathcal{C}$ .*

For  $j \geq 0$ , let  $\mathcal{H}_M^j$  denote the  $R\mathcal{O}(G)$ -module defined in Definition 5.2. Then there is a spectral sequence

$$E_2^{s,t} = H_{\mathcal{O}(G)}^s(X^?; \mathcal{H}_M^t) \Rightarrow H^{s+t}(\mathbf{O}_{\mathcal{C}}(G); M).$$

Our aim in this section is to show that the Bredon cohomology groups appearing in the  $E_2$ -term of the above spectral sequences can be expressed as higher limits over the fusion category  $\mathcal{F}_{\mathcal{E}}(G)$  in the first case, and over the category of orbit simplices  $Sd_{\mathcal{C}}(G)/G$  in the second case. To show these we need to introduce more definitions on equivariant local coefficient systems. We follow the terminology introduced by Grodal in [13] and [14, §2.4].

**7.1. Local coefficient systems.** Given a simplicial set  $X$ , let  $\Delta X$  denote the simplex category whose objects are simplices of  $X$  and morphisms are compositions of face maps  $d_i : X_n \rightarrow X_{n-1}$  and degeneracy maps  $s_i : X_n \rightarrow X_{n+1}$ . If  $X$  is a  $G$ -simplicial set, the simplex category  $\Delta X$  is a  $G$ -category. Let  $(\Delta X)_G$  denote the transporter category of the  $G$ -category  $\Delta X$ . This is the category whose objects are simplices of  $X$  and morphisms from  $\sigma$  to  $\tau$  are given by pairs  $(g, \varphi : g\sigma \rightarrow \tau)$  where  $g \in G$  and  $\varphi$  is a morphism in  $\Delta X$ . For a commutative ring  $R$ , a general (contravariant)  $G$ -local coefficient system on  $X$  is a functor  $\mathcal{M} : ((\Delta X)_G)^{op} \rightarrow R\text{-mod}$ . The cochain complex  $(C^*(X, \mathcal{M}), \delta)$  is defined by

$$C^n(X; \mathcal{M}) = \prod_{\sigma \in X_n} \mathcal{M}(\sigma) \text{ and } (\delta^n f)(\sigma) = \sum_i (-1)^i \mathcal{M}(1, d_i)(f(d_i\sigma))$$

for every  $f \in C^n(X; \mathcal{M})$ , where  $d_i$  denotes the  $i$ -th face map  $d_i : X_n \rightarrow X_{n-1}$  and  $(1, d_i)$  denotes the morphism  $(1, d_i : \sigma \rightarrow d_i\sigma)$  in  $(\Delta X)_G$ . Note that in the above formula, elements of  $C^n(X; \mathcal{M})$  are considered as a function  $f : X_n \rightarrow \prod_{\sigma \in X_n} \mathcal{M}(\sigma)$  such that  $f(\sigma) \in \mathcal{M}(\sigma)$ . The (right)  $G$ -action on  $f \in C^n(X; \mathcal{M})$  is defined by

$$(fg)(\sigma) = \mathcal{M}(g, \text{id}_{g\sigma})(f(g\sigma))$$

for every  $g \in G$ , where  $\mathcal{M}(g, \text{id}_{g\sigma}) : \mathcal{M}(g\sigma) \rightarrow \mathcal{M}(\sigma)$  is the  $R$ -module homomorphism induced by the morphism  $(g, \text{id}_{g\sigma}) : \sigma \rightarrow g\sigma$  in  $(\Delta X)_G$ . We can convert this action to a left action by taking  $gf = fg^{-1}$ . With this definition of  $G$ -action, the chain complex  $C^*(X; \mathcal{M})$  becomes a chain complex of left  $RG$ -modules. The  $G$ -equivariant cohomology  $H_G^*(X; \mathcal{M})$  of a  $G$ -space  $X$  with coefficients in  $\mathcal{M}$  is defined to be the cohomology of the cochain complex  $(C^*(X; \mathcal{M})^G, \delta^*)$ .

Given an  $R\mathcal{O}(G)$ -module  $M$ , we can define a coefficient system  $\mathcal{M} : ((\Delta X)_G)^{op} \rightarrow R\text{-Mod}$  associated to  $M$  as follows: Let  $F : (\Delta X)_G \rightarrow \mathcal{O}(G)$  be the functor that sends the simplex  $\sigma$  to its stabilizer  $G_{\sigma}$ , and sends a morphism  $(g, \varphi : g\sigma \rightarrow \tau) : \sigma \rightarrow \tau$  to the  $G$ -map

$$G/G_{\sigma} \xrightarrow{g^{-1}} G/G_{g\sigma} \xrightarrow{\varphi} G/G_{\tau}$$

(see [14, §2.4]). We define  $\mathcal{M}$  to be the composition  $((\Delta X)_G)^{op} \xrightarrow{F^{op}} \mathcal{O}(G)^{op} \xrightarrow{M} R\text{-Mod}$ . The coefficient system obtained this way is called the  $G$ -isotropy coefficient system associated to  $M$ . The following is proved by Bredon [8, §I.9].

**Lemma 7.2.** *Let  $M$  be an  $RO(G)$ -module and  $\mathcal{M}$  be the coefficient system associated to  $M$ . Then for every  $G$ -simplicial set  $X$ , the  $G$ -equivariant cohomology  $H_G^*(X; \mathcal{M})$  is isomorphic to the Bredon cohomology  $H_{\mathcal{O}(G)}^*(X; M)$  defined in Definition 1.4.*

Let  $\mathbf{C}$  be a  $G$ -category. The transporter category of  $\mathbf{C}$  is a category  $\mathbf{C}_G$  whose objects are the same as the objects of  $\mathbf{C}$  and morphisms from  $x \rightarrow y$  are given by pairs  $(g, \varphi : gx \rightarrow y)$  where  $g \in G$ , and  $\varphi$  is a morphism in  $\mathbf{C}$ . If  $X = \mathcal{N}(\mathbf{C})$  is the nerve of the  $G$ -category  $\mathbf{C}$ , and  $M : (\mathbf{C}_G)^{op} \rightarrow R\text{-mod}$  is a functor, then there is a  $G$ -local coefficient system  $\mathcal{M}$  on  $X$  defined via the functor  $(\Delta X)_G \rightarrow \mathbf{C}_G$  which takes  $x_0 \rightarrow \cdots \rightarrow x_n$  to  $x_0$ . Similarly for a functor  $M : \mathbf{C}_G \rightarrow R\text{-mod}$ , we can define a coefficient system via the functor  $((\Delta X)_G)^{op} \rightarrow \mathbf{C}_G$  which takes  $x_0 \rightarrow \cdots \rightarrow x_n$  to  $x_n$ .

**7.2. Spectral sequence for the centralizer decomposition.** Let  $\mathcal{A}_{\mathcal{C}}(G)$  denote the conjugacy category of  $G$  over the collection  $\mathcal{C}$  as defined in Section 6.3. Recall that the fusion category  $\mathcal{F}_{\mathcal{C}}(G)$  of  $G$  over  $\mathcal{C}$  is the category whose objects are subgroups  $H \in \mathcal{C}$  where morphisms  $H \rightarrow K$  in  $\mathcal{F}_{\mathcal{C}}(G)$  are given by conjugation maps  $c_g : H \rightarrow K$  defined by  $c_g(h) = ghg^{-1}$  for all  $h \in H$ . The following is well-known.

**Lemma 7.3.** *The conjugacy category  $\mathcal{A}_{\mathcal{C}}(G)$  is equivalent to  $\mathcal{F}_{\mathcal{C}}(G)$  as categories.*

*Proof.* For each  $G$ -conjugacy class  $[i]$  with  $i(H) \in \mathcal{C}$ , choose a representative monomorphism  $i : H \rightarrow G$ . Let  $T : \mathcal{A}_{\mathcal{C}}(G) \rightarrow \mathcal{F}_{\mathcal{C}}(G)$  be the functor defined by  $(H, [i]) \rightarrow i(H)$  on objects. For every morphism  $f : (H_1, [i_1]) \rightarrow (H_2, [i_2])$ , we define  $T(f) : i_1(H_1) \rightarrow i_2(H_2)$  to be the composition

$$i_1(H_1) \xrightarrow{c_g} i_2(f(H_1)) \rightarrow i_2(H_2)$$

where the first conjugation map  $c_g$  exists because  $[i_2 \circ f] = [i_1]$ , and the second map is the inclusion map. There is a functor  $S : \mathcal{F}_{\mathcal{C}}(G) \rightarrow \mathcal{A}_{\mathcal{C}}(G)$  in the other direction defined by  $S(H) = (H, [inc_H])$  for every  $H \in \mathcal{C}$ . It is clear that  $T \circ S = \text{id}_{\mathcal{F}_{\mathcal{C}}(G)}$ . There is a natural transformation  $\eta : \text{id}_{\mathcal{A}_{\mathcal{C}}(G)} \rightarrow S \circ T$  such that the morphism

$$\eta_{(H, [i])} : (H, [i]) \rightarrow (i(H), [inc_H])$$

is given by the group homomorphism  $i : H \rightarrow i(H)$ . It is straightforward to check that  $\eta$  is a natural isomorphism. Hence these two categories are equivalent.  $\square$

The Dwyer space  $X_{\mathcal{C}}^{\alpha}$  is the realization of the  $G$ -category  $\mathbf{X}_{\mathcal{C}}^{\alpha}$  whose objects are the pairs  $(H, i)$  where  $i$  is a monomorphism  $i : H \rightarrow G$  with  $i(H) \in \mathcal{C}$ . Let  $X$  denote the nerve of the category  $\mathbf{X}_{\mathcal{C}}^{\alpha}$ . The simplex category  $\Delta X$  of  $X$  is the category whose objects are chains of morphisms  $(H_0, i_0) \xrightarrow{\alpha_1} (H_1, i_1) \rightarrow \cdots \xrightarrow{\alpha_n} (H_n, i_n)$  in  $\mathbf{X}_{\mathcal{C}}^{\alpha}$  and whose morphisms are given by face and degeneracy maps of the nerve construction. There is a functor  $F : ((\Delta X)_G)^{op} \rightarrow \mathcal{F}_{\mathcal{C}}(G)$  defined by the composition of functors

$$((\Delta X)_G)^{op} \rightarrow ((\Delta \mathcal{N}\mathcal{C})_G)^{op} \rightarrow \mathcal{C}_G = \mathcal{T}_{\mathcal{C}}(G) \rightarrow \mathcal{F}_{\mathcal{C}}(G)$$

where the first functor takes a simplex  $(H_0, i_0) \xrightarrow{\alpha_1} (H_1, i_1) \rightarrow \cdots \xrightarrow{\alpha_n} (H_n, i_n)$  in  $X$  to the simplex  $i_0(H_0) \leq \cdots \leq i_n(H_n)$  in  $\mathcal{N}\mathcal{C}$ , and the second functor takes this chain of subgroups to  $i_n(H_n)$  in  $\mathcal{C}$  (see [13, p. 416] for more details). The equality  $\mathcal{C}_G = \mathcal{T}_{\mathcal{C}}(G)$  follows from definitions, and

the last functor  $\mathcal{T}_{\mathcal{C}}(G) \rightarrow \mathcal{F}_{\mathcal{C}}(G)$  is the quotient functor defined in Section 2.3. The following is proved by Grodal in [13].

**Proposition 7.4** ([13, Prop 2.10]). *Let  $N : \mathcal{F}_{\mathcal{C}}(G) \rightarrow R\text{-Mod}$  be a (covariant) functor, and let  $\mathcal{M}$  denote the local coefficient system defined on  $X$  by the composition*

$$((\Delta X)_G)^{op} \xrightarrow{F} \mathcal{F}_{\mathcal{C}}(G) \xrightarrow{N} R\text{-Mod}.$$

*Then there is an isomorphism*

$$H_G^*(X_{\mathcal{C}}^{\alpha}; \mathcal{M}) \cong \lim_{\mathcal{F}_{\mathcal{C}}(G)}^* N.$$

There is a functor  $\xi_{\mathcal{C}} : \mathcal{F}_{\mathcal{C}}(G) \rightarrow \mathcal{O}(G)^{op}$  which sends  $H \in \mathcal{C}$  to its centralizer  $C_G(H) \leq G$ . Given an  $R\mathcal{O}(G)$ -module  $M$ , precomposing  $M$  with  $\xi_{\mathcal{C}}$  gives a functor  $M \circ \xi_{\mathcal{C}} : \mathcal{F}_{\mathcal{C}}(G) \rightarrow R\text{-Mod}$ . Combining Proposition 7.4 with our earlier observations, we obtain the following:

**Proposition 7.5.** *Let  $M$  be an  $R\mathcal{O}(G)$ -module and let  $X = X_{\mathcal{C}}^{\alpha}$ . Then there is an isomorphism*

$$H_{\mathcal{O}(G)}^*(X^?; M) \cong \lim_{\mathcal{F}_{\mathcal{C}}(G)}^*(M \circ \xi_{\mathcal{C}}).$$

*Proof.* Let  $\mathcal{M}$  be the coefficient system on  $X$  by defined by the composition

$$M \circ \xi_{\mathcal{C}} \circ F : ((\Delta X)_G)^{op} \rightarrow ((\Delta \mathcal{N}\mathcal{C})_G)^{op} \rightarrow \mathcal{C}_G \rightarrow \mathcal{F}_{\mathcal{C}}(G) \xrightarrow{\xi_{\mathcal{C}}} \mathcal{O}(G)^{op} \xrightarrow{M} R\text{-Mod}.$$

By Proposition 7.4, there is an isomorphism

$$H_G^*(X; \mathcal{M}) \cong \lim_{\mathcal{F}_{\mathcal{C}}(G)}^*(M \circ \xi_{\mathcal{C}}). \quad (7.1)$$

The composition  $\xi_{\mathcal{C}} \circ F$  takes the simplex

$$\sigma := (H_0, i_0) \xrightarrow{\alpha_1} (H_1, i_1) \rightarrow \dots \xrightarrow{\alpha_n} (H_n, i_n)$$

in  $X$  to the subgroup  $C_G(i_n(H_n))$  in  $\mathcal{O}(G)$ , which is the stabilizer of  $\sigma$  under the  $G$ -action on  $\sigma$ . This implies that  $\mathcal{M} = M \circ \xi_{\mathcal{C}} \circ F$  is the  $G$ -isotropy coefficient system for  $X$  associated to  $M$ . Hence by Lemma 7.2, there is an isomorphism

$$H_{\mathcal{O}(G)}^*(X^?; M) \cong H_G^*(X; \mathcal{M}).$$

Combining this isomorphism with the isomorphism in (7.1) gives the desired isomorphism.  $\square$

Let  $\mathcal{C}$  and  $\mathcal{D}$  be two arbitrary collections in  $G$ . Let  $\mathbf{O}_{\mathcal{C}}(G)$  denote either  $\mathcal{O}_{\mathcal{C}}(G)$  or  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$  and  $M$  denote an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module. For every integer  $j \geq 0$ , let  $\mathcal{H}_M^j$  denote the  $R\mathcal{O}(G)$ -module defined in Definition 5.2. Let  $\xi_{\mathcal{D}} : \mathcal{F}_{\mathcal{D}}(G) \rightarrow \mathcal{O}(G)^{op}$  be the functor defined above which sends  $D \in \mathcal{D}$  to the centralizer  $C_G(D) \leq G$ . Precomposing  $\mathcal{H}_M^j$  with  $\xi_{\mathcal{D}}$  gives a functor  $\mathcal{F}_{\mathcal{D}}(G) \rightarrow R\text{-Mod}$ .

**Definition 7.6.** For an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module  $M$  and for  $j \geq 0$ , the functor

$$\mathcal{H}_{M, C_G}^j : \mathcal{F}_{\mathcal{D}}(G) \rightarrow R\text{-Mod}$$

is defined to be the composition  $\mathcal{H}_M^j \circ \xi_{\mathcal{D}}$ . Note that for every  $D \in \mathcal{D}$ ,

$$\mathcal{H}_{M, C_G}^j(D) = H^*(\mathbf{O}_{\mathcal{C}}(C_G(D)); \text{Res}_{\mathbf{O}_{\mathcal{C}}(C_G(D))}^{\mathbf{O}_{\mathcal{C}}(G)} M).$$

We have the following result.



**Proposition 7.7.** *Let  $\mathcal{C}$  be any collection of all nontrivial  $p$ -subgroups in  $G$ , and  $\mathcal{E}$  be the collection of all nontrivial elementary abelian  $p$ -subgroups in  $G$ . Let  $\mathbf{O}_{\mathcal{C}}(G) = \mathcal{O}_{\mathcal{C}}(G)$  or  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ , and  $M$  be an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module. For each integer  $j \geq 0$ , let  $\mathcal{H}_{M, \mathcal{C}_G}^j$  denote the functor defined in Definition 7.6. Then, there is a spectral sequence*

$$E_2^{s,t} = \lim_{\mathcal{F}_{\mathcal{E}}(G)}^s \mathcal{H}_{M, \mathcal{C}_G}^t \Rightarrow H^{s+t}(\mathbf{O}_{\mathcal{C}}(G); M).$$

*Proof.* Since  $\mathcal{H}_{M, \mathcal{C}_G}^t = \mathcal{H}_M^t \circ \xi_{\mathcal{E}}$ , this follows from Corollary 7.1 and Proposition 7.5.  $\square$

**7.3. Spectral sequence for the normalizer decomposition.** Let  $G$  be a discrete group and  $\mathcal{C}$  be a collection of subgroups in  $G$  such that  $\mathcal{C}$  is closed taking products. Let  $\mathbf{O}_{\mathcal{C}}(G) = \mathcal{O}_{\mathcal{C}}(G)$  or  $\mathcal{F}_{\mathcal{C}}(G)$ , and  $X := X_{\mathcal{C}}^{\delta}$  denote the Dwyer space for the normalizer decomposition over  $\mathcal{C}$ . By Corollary 7.1 there is a spectral sequence

$$E_2^{s,t} = H_{\mathcal{O}(G)}^s(X^?; \mathcal{H}_M^t) \Rightarrow H^{s+t}(\mathbf{O}_{\mathcal{C}}(G); M)$$

where  $\mathcal{H}_M^t$  is the  $R\mathcal{O}(G)$ -module defined in Definition 5.2. By Lemma 6.10,  $X$  is  $G$ -homeomorphic to the geometric realization  $|\mathcal{C}|$  of  $\mathcal{C}$ , hence we can replace  $X^?$  with  $|\mathcal{C}|^?$  in the above spectral sequence. Note that  $|\mathcal{C}|$  is the realization of the simplicial set  $\mathcal{N}\mathcal{C}$  and there is a functor

$$\eta_{\mathcal{C}} : (\Delta\mathcal{N}\mathcal{C})_G \rightarrow Sd_{\mathcal{C}}(G)/G$$

which takes each simplex  $\sigma$  in  $\mathcal{N}\mathcal{C}$  to its  $G$ -orbit  $[\sigma]$ , and each morphism

$$(g, \varphi : g\sigma \rightarrow \tau) : \sigma \rightarrow \tau$$

in  $(\Delta\mathcal{N}\mathcal{C})_G$  to the unique morphism  $[\sigma] \rightarrow [\tau]$  in  $Sd_{\mathcal{C}}(G)/G$ . For higher limits over the subdivision category we have the following result which is attributed to Slomińska [32] by Grodal (see [13, Prop 7.1] for a proof).

**Proposition 7.8.** *Let  $N : (Sd_{\mathcal{C}}(G)/G)^{op} \rightarrow R\text{-Mod}$  be an arbitrary functor. Then*

$$\lim_{Sd_{\mathcal{C}}(G)/G}^* N \cong H_G^*(|\mathcal{C}|; \mathcal{M})$$

where  $\mathcal{M}$  is the  $G$ -local coefficient system given via  $\eta_{\mathcal{C}}^{op} : ((\Delta\mathcal{N}\mathcal{C})_G)^{op} \rightarrow (Sd_{\mathcal{C}}(G)/G)^{op}$ .

The definition of the functor  $\tilde{\delta}_{\mathcal{C}} : Sd_{\mathcal{C}}(G)/G \rightarrow G\text{-Sets}$  can be adjusted to define a functor

$$\zeta_{\mathcal{C}} : Sd_{\mathcal{C}}(G)/G \rightarrow \mathcal{O}(G)$$

which takes a  $G$ -orbit  $[\sigma]$  to the normalizer  $N_G(\sigma)$ . Given a morphism  $[\sigma] \rightarrow [\tau]$ , there is a  $g \in G$  be such that  $\tau$  is a face of  $g\sigma$ . This gives that  $gN_G(\sigma)g^{-1} = N_G(g\sigma) \leq N_G(\tau)$ . Hence there is a  $G$ -map  $f : G/N_G(\sigma) \rightarrow G/N_G(\tau)$  defined by  $f(g'N_G(\sigma)) = g'g^{-1}N_G(\tau)$  for all  $g' \in G$ . The  $G$ -map  $f : [\sigma] \rightarrow [\tau]$  does not depend on the group element  $g \in G$ , hence  $\zeta_{\mathcal{C}}$  with these assignments defines a functor.

**Proposition 7.9.** *Let  $M$  be an  $R\mathcal{O}(G)$ -module. Then there is an isomorphism*

$$H_{\mathcal{O}(G)}^*(|\mathcal{C}|^?; M) \cong \lim_{Sd_{\mathcal{C}}(G)/G}^* (M \circ \zeta_{\mathcal{C}}^{op}).$$

*Proof.* By Proposition 7.8, we have

$$\lim_{Sd_{\mathcal{C}}(G)/G}^* (M \circ \zeta_{\mathcal{C}}^{op}) \cong H_G^*(|\mathcal{C}|; \mathcal{M}),$$

where the coefficient system  $\mathcal{M}$  on  $|\mathcal{C}|$  is defined by the composition

$$((\Delta\mathcal{N}\mathcal{C})_G)^{op} \xrightarrow{\eta_{\mathcal{C}}^{op}} (Sd_{\mathcal{C}}(G)/G)^{op} \xrightarrow{\zeta_{\mathcal{C}}^{op}} \mathcal{O}(G)^{op} \xrightarrow{M} R\text{-Mod}.$$

Note that  $\zeta_{\mathcal{C}} \circ \eta_{\mathcal{C}}$  sends a simplex  $\sigma$  to  $N_G(\sigma)$  and a morphism  $(g, \varphi : g\sigma \rightarrow \tau) : \sigma \rightarrow \tau$  in  $((\Delta\mathcal{N}\mathcal{C})_G)^{op}$  to the  $G$ -map

$$G/N_G(\sigma) \xrightarrow{g^{-1}} G/N_G(g\sigma) \xrightarrow{\varphi} G/N_G(\tau).$$

Hence  $\mathcal{M}$  defines the  $G$ -isotropy coefficient system for  $|\mathcal{C}|$ . Applying Lemma 7.2, we obtain the desired isomorphism.  $\square$

Let  $M$  be an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module  $M$ . For every integer  $j \geq 0$ , let  $\mathcal{H}_M^j : \mathcal{O}(G)^{op} \rightarrow R\text{-Mod}$  denote the functor defined in Definition 5.2.

**Definition 7.10.** Precomposing  $\mathcal{H}_M^j$  with the functor  $\zeta_{\mathcal{C}}^{op} : (Sd_{\mathcal{C}}(G)/G)^{op} \rightarrow \mathcal{O}(G)^{op}$ , we obtain a functor

$$\mathcal{H}_{M, N_G}^j : (Sd_{\mathcal{C}}(G)/G)^{op} \rightarrow R\text{-Mod}$$

such that for every  $[\sigma] \in Sd_{\mathcal{C}}(G)/G$ , we have

$$\mathcal{H}_{M, N_G}^j([\sigma]) = H^*(\mathbf{O}_{\mathcal{C}}(N_G(\sigma)); \text{Res}_{\mathbf{O}_{\mathcal{C}}(N_G(\sigma))}^{\mathbf{O}_{\mathcal{C}}(G)} M).$$

As a consequence of the results above, we obtain the following spectral sequence.

**Proposition 7.11.** *Let  $G$  be a discrete group and  $\mathcal{C}$  be a collection subgroups of  $G$  closed under taking products. Let  $\mathbf{O}_{\mathcal{C}}(G) = \mathcal{O}_{\mathcal{C}}(G)$  or  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ , and  $M$  be an  $R\mathbf{O}_{\mathcal{C}}(G)$ -module. For each integer  $j \geq 0$ , let  $\mathcal{H}_{M, N_G}^j$  denote the functor defined in Definition 7.10. Then there is a spectral sequence*

$$E_2^{s,t} = \lim_{Sd_{\mathcal{C}}(G)/G}^s \mathcal{H}_{M, N_G}^t \Rightarrow H^{s+t}(\mathbf{O}_{\mathcal{C}}(G); M).$$

*Proof.* This follows from Corollary 7.1 and Proposition 7.9.  $\square$

In the next section we consider centralizer and normalizer fusion systems. The results we prove will allow us to express the spectral sequences we obtained in Propositions 7.7 and 7.11 in terms of the cohomology of the centric orbit category of the centralizer and normalizer fusion systems.

## 8. THE CENTRALIZER AND NORMALIZER FUSION SYSTEMS

If  $G$  is an infinite group with a Sylow  $p$ -subgroup  $S$ , the subgroups of  $G$  may not have Sylow  $p$ -subgroups. This makes it difficult to work with centralizers and normalizers of  $p$ -subgroups of  $G$  when  $G$  is an infinite group. We overcome this difficulty by using a result due to Libman [22, Prop 3.8] (see also Parker [27, Lemma 2.14]) which states that if  $G$  is a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F}_S(G)$  is a saturated fusion system then the normalizer  $N_G(P)$  of a  $p$ -subgroup  $P$  always has a Sylow  $p$ -subgroup. In this section we prove a generalization of Libman's theorem to  $K$ -normalizer subgroups so that we can also apply it to centralizer fusion systems and to the normalizers of chains of subgroups in  $S$ .

Let  $G$  be a discrete group and  $S$  be a Sylow  $p$ -subgroup of  $G$ . For every subgroup  $Q \leq S$  and every subgroup  $K \leq \text{Aut}(Q)$ , the  $K$ -normalizer of  $Q$  in  $G$  is the subgroup

$$N_G^K(Q) := \{g \in N_G(Q) \mid (c_g)|_Q \in K\}.$$

Let  $N_S^K(Q) := S \cap N_G^K(Q)$ . If  $K = \text{Aut}(Q)$ , then  $N_G^K(Q) = N_G(Q)$  and if  $K = 1$ , then  $N_G^K(Q) = C_G(Q)$ . Another interesting case is where  $K$  is the subgroup of automorphisms of  $Q$  which stabilizes a chain  $\sigma := (Q_0 < Q_1 < \cdots < Q_n)$  of subgroups of  $Q$  such that  $Q_n = Q$ . In this case we write  $N_G(\sigma)$  for  $N_G^K(Q)$ .

Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $K \leq \text{Aut}(Q)$ . We say  $Q$  is *fully  $K$ -normalized* if for any morphism  $\varphi : Q \rightarrow S$  in  $\mathcal{F}$ ,

$$|N_S^K(Q)| \geq |N_S^{\varphi K}(\varphi Q)|$$

where  $\varphi K := \{\varphi k \varphi^{-1} \mid k \in K\} \leq \text{Aut}(\varphi Q)$ .

**Definition 8.1.** The  *$K$ -normalizer fusion system*  $N_{\mathcal{F}}^K(Q)$  is the fusion system over  $N_S^K(Q)$  whose morphisms  $P \rightarrow P'$  are the morphisms  $\varphi \in \text{Mor}_{\mathcal{F}}(P, P')$  which extend to a morphism  $\tilde{\varphi} : QP \rightarrow QP'$  in  $\mathcal{F}$  in such a way that  $\tilde{\varphi}|_Q \in K$ .

If  $K = \text{Aut}(Q)$ , then  $N_{\mathcal{F}}^K(Q)$  is denoted by  $N_{\mathcal{F}}(Q)$  and called the *normalizer fusion system*. If  $K = 1$ , then  $N_{\mathcal{F}}^K(Q)$  is denoted by  $C_{\mathcal{F}}(Q)$  and called the *centralizer fusion system*.

**Theorem 8.2** (Puig [28]). *Let  $\mathcal{F}$  be a saturated fusion system over a finite  $p$ -group  $S$ , and let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . If  $Q$  is fully  $K$ -normalized then  $N_{\mathcal{F}}^K(Q)$  is a saturated fusion system.*

As special cases of Puig's theorem, we obtain that for every fully  $\mathcal{F}$ -normalized subgroup  $Q \leq S$ , the normalizer fusion system  $N_{\mathcal{F}}(Q)$  is saturated, and for every fully  $\mathcal{F}$ -centralized subgroup  $Q$ , the centralizer fusion system  $C_{\mathcal{F}}(Q)$  is saturated. We will need the following lemma in our proofs below.

**Lemma 8.3** ([10, Lemma 4.36]). *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ . Let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . If  $\phi : Q \rightarrow S$  is a morphism in  $\mathcal{F}$  such that  $\phi Q$  is fully  ${}^{\phi}K$ -normalized, then there is a morphism  $\psi : QN_S^K(Q) \rightarrow S$  in  $\mathcal{F}$  and  $\alpha \in K$  such that  $\psi|_Q = \phi \circ \alpha$ , and  $\beta \in {}^{\phi}K$  such that  $\beta \circ \psi|_Q = \phi$ .*

The following proposition is a generalization of Libman's result [22, Prop 3.8] (see also [27, Lemma 2.14]).

**Proposition 8.4.** *Let  $G$  be a discrete group and  $S$  be a Sylow  $p$ -subgroup of  $G$ . Suppose that  $\mathcal{F} = \mathcal{F}_S(G)$  is a saturated fusion system. Let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . Then,  $Q$  is fully  $K$ -normalized if and only if  $N_S^K(Q)$  is a Sylow  $p$ -subgroup of  $N_G^K(Q)$ .*

*Proof.* Assume that  $Q$  is fully  $K$ -normalized. We will show that for every finite  $p$ -subgroup  $P$  of  $N_G^K(Q)$ , there is an  $x \in N_G^K(Q)$  such that  ${}^xP \leq N_S^K(Q)$ . Let  $P$  be a finite  $p$ -subgroup of  $N_G^K(Q)$ , and let  $R = QP$ . Since  $P \leq N_G^K(Q) \leq N_G(Q)$ , we have  $R \leq N_G(Q)$ . Let  $g \in G$  such that  ${}^gR \leq S$ . Then we have  ${}^gQ \leq {}^gR \leq S$ . Let  $\phi : {}^gQ \rightarrow S$  be the conjugation map  $c_{g^{-1}} : {}^gQ \rightarrow S$  defined by  $c_{g^{-1}}(x) = g^{-1}xg$ . Note that  $\phi$  is a morphism in  $\mathcal{F}$ . Let  $L \leq \text{Aut}({}^gQ)$  be the subgroup defined by  $L := \phi^{-1}K$ . Since  $\phi({}^gQ) = Q$ ,  $\phi L = K$ , and  $Q$  is fully  $K$ -normalized, by Lemma 8.3, there is a morphism

$$\psi : {}^gQN_S^L({}^gQ) \rightarrow S$$

in  $\mathcal{F}$  and an automorphism  $\beta \in K$  such that  $\beta \circ \psi|_Q = \phi$ .

Since  ${}^gP \leq {}^gR \leq S$  and  ${}^gP \leq {}^g(N_G^K(Q)) = N_G^L({}^gQ)$ , we have  ${}^gP \leq N_S^L({}^gQ)$ . This gives that  ${}^gR = {}^gQ{}^gP \leq {}^gQN_S^L({}^gQ)$ , hence  $\psi$  is defined on  ${}^gR$ . Let  $u \in G$  be such that  $\psi = c_u$ . Let  $x = ug$ . Then

$${}^xR = {}^u({}^gR) = \psi({}^gR) \leq S.$$

We also have

$$c_x|_Q = c_u \circ c_g|_Q = \psi \circ \phi^{-1}|_Q = \psi \circ (\beta \circ \psi|_{gQ})^{-1} = \beta^{-1} \in K.$$

This gives that  $x \in N_G^K(Q)$ . Hence  ${}^xP \leq N_G^K(Q)$ . Since  ${}^xP \leq {}^xR \leq S$ , we have

$${}^xP \leq S \cap N_G^K(Q) = N_S^K(Q).$$

This completes the proof that  $N_S^K(Q)$  is a Sylow  $p$ -subgroup of  $N_G^K(Q)$ .

For the converse, suppose that  $N_S^K(Q)$  is a Sylow  $p$ -subgroup of  $N_G^K(Q)$ . Let  $\varphi : Q \rightarrow S$  be a morphism in  $\mathcal{F}$ . Let  $g \in G$  such that  $\varphi = c_g$ . Then we have

$$N_S^{\varphi K}(\varphi Q) = N_S^{c_g K}(gQ) = g(N_S^K(Q)).$$

Note that  $N_{Sg}^K(Q)$  is a  $p$ -subgroup of  $N_G^K(Q)$ . Since  $N_S^K(G)$  is a Sylow  $p$ -subgroup of  $N_G^K(Q)$ , we have  $|N_S^K(Q)| \geq |N_{Sg}^K(Q)|$ . This gives  $|N_S^K(Q)| \geq |N_S^{\varphi K}(\varphi Q)|$ , hence  $Q$  is fully  $K$ -normalized.  $\square$

The next proposition easily follows from Proposition 8.4.

**Proposition 8.5.** *Let  $G$  be a discrete group and  $S$  be a Sylow  $p$ -subgroup of  $G$ . Let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . If  $\mathcal{F} = \mathcal{F}_S(G)$  is a saturated fusion system, then the  $K$ -normalizer subgroup  $N_G^K(Q)$  has a Sylow  $p$ -subgroup.*

*Proof.* Assume that  $\mathcal{F}$  is saturated. Let  $\phi : Q \rightarrow S$  be a morphism in  $\mathcal{F}$  such that  $N_S^{\phi K}(\phi Q)$  has the maximum order among all such normalizers. Let  $L = \phi K$  and  $R = \phi Q$ . Then for every morphism  $\varphi : R \rightarrow S$  in  $\mathcal{F}$ , we have

$$|N_S^L(R)| = |N_S^{\phi K}(\phi Q)| \geq |N_S^{\varphi \phi K}(\varphi \phi Q)| = |N_S^{\varphi L}(\varphi L)|,$$

hence  $R$  is fully  $L$ -normalized. By Proposition 8.4, we conclude that  $N_S^L(R)$  is a Sylow  $p$ -subgroup of  $N_G^L(R)$ . If  $g \in G$  such that  $\phi = c_g$ , then

$$N_G^L(R) = N_G^{c_g K}(gQ) = g(N_G^K(Q)).$$

Since  $N_G^L(R)$  has a Sylow  $p$ -subgroup, its conjugate  $N_G^K(Q)$  also has a Sylow  $p$ -subgroup.  $\square$

Another consequence of Proposition 8.4 is the following proposition which is a generalization of [22, Prop 3.8].

**Proposition 8.6.** *Let  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} = \mathcal{F}_S(G)$  is a saturated fusion system. Let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . If  $Q$  is fully  $K$ -normalized, then  $N_{\mathcal{F}}^K(Q) = \mathcal{F}_{N_S^K(Q)}(N_G^K(Q))$ .*

*Proof.* The argument given in [10, Thm 4.27] for finite groups also holds here. By Proposition 8.4,  $N_S^K(Q)$  is a Sylow  $p$ -subgroup of  $N_G^K(Q)$ . Let  $\varphi : P \rightarrow P'$  be a morphism in  $N_{\mathcal{F}}^K(Q)$ . Then it extends to  $\tilde{\varphi} : QP \rightarrow QP'$  in  $\mathcal{F}$  such that  $\tilde{\varphi}|_Q \in K$ . Let  $x \in G$  such that  $\tilde{\varphi} = c_x$ . Since  $\tilde{\varphi}|_Q \in K$ , we have  $x \in N_G^K(Q)$ . This implies that  $\varphi = c_x|_Q$  is a morphism in  $\mathcal{F}_{N_S^K(Q)}(N_G^K(Q))$ . Conversely, if  $c_x : P \rightarrow P'$  is a morphism in  $\mathcal{F}_{N_S^K(Q)}(N_G^K(Q))$ , then  $c_x$  extends to a homomorphism  $QP \rightarrow QP'$  defined also by conjugation with  $x$ , hence  $c_x$  lies in  $N_{\mathcal{F}}^K(Q)$ .  $\square$

A subgroup  $P$  of  $S$  is called  $\mathcal{F}$ -centric if  $C_S(P') \leq P'$  for every  $P' \sim_{\mathcal{F}} P$ . Note that if  $P$  is  $\mathcal{F}$ -centric, then  $C_S(P') = Z(P')$  has the same order for every  $P' \sim_{\mathcal{F}} P$ . Thus if  $P$  is an  $\mathcal{F}$ -centric subgroup then  $P$  and all its  $\mathcal{F}$ -conjugates are fully  $\mathcal{F}$ -centralized.

**Lemma 8.7.** *Let  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} = \mathcal{F}_S(G)$  is a saturated fusion system. Let  $P \leq S$ . Then  $P$  is  $\mathcal{F}$ -centric if and only if  $Z(P)$  is a Sylow  $p$ -subgroup of  $C_G(P)$ .*

*Proof.* If  $P$  is  $\mathcal{F}$ -centric, then by the argument above,  $P$  is fully  $\mathcal{F}$ -centralized. Then by Proposition 8.4,  $C_S(P)$  is a Sylow  $p$ -subgroup of  $C_G(P)$ . Since  $P$  is  $\mathcal{F}$ -centric, we have  $C_S(P) = Z(P)$ . Hence  $Z(P)$  is a Sylow  $p$ -subgroup of  $C_G(P)$ . For the converse, assume that  $Z(P)$  is a Sylow  $p$ -subgroup of  $C_G(P)$ . For every  $g \in G$ , we have  $C_G({}^gP) = {}^gC_G(P)$  and  $Z({}^gP) = {}^gZ(P)$ , hence  $Z({}^gP)$  is a Sylow  $p$ -subgroup of  $C_G({}^gP)$ . If  $g \in G$  such that  ${}^gP \leq S$ , we have  $Z({}^gP) \leq C_S({}^gP) \leq C_G({}^gP)$ . Since  $Z({}^gP)$  is a Sylow  $p$ -subgroup of  $C_G({}^gP)$ , we obtain that  $C_S({}^gP) = Z({}^gP)$ . This implies that  $P$  is  $\mathcal{F}$ -centric.  $\square$

Note that the argument above also proves that if  $P$  is an  $\mathcal{F}$ -centric subgroup, then for every  $g \in G$ , the center  $Z({}^gP)$  is a Sylow  $p$ -subgroup of  $C_G({}^gP)$ . This suggests that we extend the definition of  $p$ -centric subgroups for discrete groups in the following way.

**Definition 8.8.** Let  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$ . A  $p$ -subgroup in  $G$  is called  *$p$ -centric* if  $Z(P)$  is a Sylow  $p$ -subgroup of  $C_G(P)$ .

We have the following observation.

**Lemma 8.9.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} \cong \mathcal{F}_S(G)$ . A  $p$ -subgroup  $P \leq G$  is conjugate to an  $\mathcal{F}$ -centric subgroup in  $S$  if and only if it is a  $p$ -centric subgroup of  $G$ . Hence the collection of all  $p$ -subgroups of  $G$  conjugate to an  $\mathcal{F}$ -centric subgroup in  $S$  is equal to the collection of  $p$ -centric subgroups in  $G$ .*

*Proof.* This follows from Lemma 8.7 and from the fact that for every  $g \in G$ , a subgroup  $P \leq G$  is  $p$ -centric if and only if  ${}^gP$  is  $p$ -centric.  $\square$

In Sections 9 and 10, we consider  $\mathcal{F}$ -centric subgroups in centralizer and normalizer subgroups. In our proofs we will need following lemma.

**Lemma 8.10.** *Let  $\mathcal{F}$  be a saturated fusion system over a finite  $p$ -group  $S$ . Let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . Assume that  $Q$  is a fully  $K$ -normalized subgroup of  $S$ . Then for every  $P \leq N_S^K(Q)$ , the following hold:*

- (i) *If  $P$  is  $\mathcal{F}$ -centric, then  $P$  is  $N_{\mathcal{F}}^K(Q)$ -centric.*
- (ii) *If  $P$  is  $N_{\mathcal{F}}^K(Q)$ -centric, then  $QP$  is  $\mathcal{F}$ -centric.*

*Proof.* Assume that  $P$  is  $\mathcal{F}$ -centric. Let  $P' \leq N_S^K(Q)$  be such that  $P'$  is isomorphic to  $P$  in  $N_{\mathcal{F}}^K(Q)$ . Then  $P' \sim_{\mathcal{F}} P$ , hence  $C_S(P') \leq P'$  since  $P$  is  $\mathcal{F}$ -centric. From this we obtain that

$$C_{N_S^K(Q)}(P') \leq C_S(P') \leq P',$$

hence  $P$  is  $N_{\mathcal{F}}^K(Q)$ -centric. This proves (i).

To prove (ii), we use an argument similar to the argument given in the proof of [6, Lemma 6.2]. Assume that  $P \leq N_S^K(Q)$  is  $N_{\mathcal{F}}^K(Q)$ -centric. Let  $\varphi : QP \rightarrow S$  be a morphism in  $\mathcal{F}$ . We need to show that  $C_S(\varphi(QP)) \leq \varphi(QP)$ . Since  $Q$  is fully  $K$ -normalized, we can apply Lemma 8.3 to  $\varphi^{-1} : \varphi Q \rightarrow Q$  to obtain that there is a morphism

$$\psi : \varphi Q \cdot N_S^{\varphi K}(\varphi Q) \rightarrow S$$

in  $\mathcal{F}$  and an automorphism  $\beta \in K$  such that  $\beta \circ (\psi|_{\varphi Q}) = \varphi^{-1}$ . We have

$$\varphi(QP) = \varphi Q \cdot \varphi P \leq \varphi Q \cdot N_S^{\varphi K}(\varphi Q)$$

hence the composition  $\psi \circ \varphi : QP \rightarrow S$  is defined. Moreover  $(\psi \circ \varphi)|_Q = \beta^{-1} \in K$ .

We claim that  $(\psi \circ \varphi)|_P : P \rightarrow S$  is a morphism in  $N_{\mathcal{F}}^K(Q)$ . We will first show that  $(\psi \circ \varphi)(P) \leq N_S^K(Q)$ . We write  $\psi\varphi$  for  $\psi \circ \varphi$  to simplify the notation. Let  $x \in P$  and  $y = \psi\varphi(x)$ . For every  $q \in Q$ , we have

$$c_y(q) = yqy^{-1} = \psi\varphi(x) \cdot q \cdot \psi\varphi(x^{-1}) = \psi\varphi(x \cdot \beta(q) \cdot x^{-1}) = \beta^{-1}(c_x(\beta(q))) = (\beta^{-1}c_x\beta)(q).$$

Hence  $c_y = \beta^{-1}c_x\beta$ . Since  $x \in P \leq N_S^K(Q)$ , we have  $c_x \in K$ , hence  $c_y \in K$ . Note that  $y = \psi\varphi(x) \in S$ , hence we can conclude that  $y \in N_S^K(Q)$ . This gives  $\psi\varphi(P) \leq N_S^K(Q)$ . Since  $\psi\varphi|_Q : Q \rightarrow Q$  is equal to  $\beta^{-1} \in K$ , we conclude that  $\psi\varphi|_P : P \rightarrow S$  is a morphism in  $N_{\mathcal{F}}^K(Q)$ .

Note that we have

$$C_S(\varphi(QP)) \leq C_S(\varphi Q) \leq N_S^{\varphi K}(\varphi Q),$$

so we can apply  $\psi$  to obtain  $\psi(C_S(\varphi(QP))) \leq N_S^K(Q)$ . We also have

$$\psi(C_S(\varphi(QP))) \leq C_S(\psi\varphi(QP)) = C_S(Q \cdot \psi\varphi(P)) \leq C_S(\psi\varphi(P)).$$

These two inclusions give

$$\psi(C_S(\varphi(QP))) \leq C_S(\psi\varphi(P)) \cap N_S^K(Q) = C_{N_S^K(Q)}(\psi\varphi(P)) \leq \psi\varphi(P)$$

where the last inclusion follows from the fact that  $P$  is  $N_{\mathcal{F}}^K(Q)$ -centric and  $\psi\varphi|_P$  is a morphism in  $N_{\mathcal{F}}^K(Q)$ . We conclude that

$$C_S(\varphi(QP)) \leq \varphi(P) \leq \varphi(QP),$$

hence  $QP$  is  $\mathcal{F}$ -centric. □

As an easy corollary of Lemma 8.10, we obtain the following.

**Lemma 8.11** ([6, Prop 2.4]). *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and  $Q$  be a fully  $\mathcal{F}$ -centralized subgroup of  $S$ . Suppose that  $Q$  is abelian. Then a subgroup  $P \leq C_S(Q)$  is  $C_{\mathcal{F}}(Q)$ -centric if and only if it is  $\mathcal{F}$ -centric. If one of these equivalent conditions hold, then  $Q \leq P$ .*

*Proof.* Take  $K = 1$  in Lemma 8.10. Then  $N_S^K(Q) = C_S(Q)$  and  $N_{\mathcal{F}}^K(Q) = C_{\mathcal{F}}(Q)$ . Let  $P \leq C_S(Q)$ . By Lemma 8.10, if  $P$  is  $\mathcal{F}$ -centric then it is  $C_{\mathcal{F}}(Q)$ -centric. For the converse, assume that  $P$  is  $C_{\mathcal{F}}(Q)$ -centric. Then by Lemma 8.10,  $QP$  is  $\mathcal{F}$ -centric. Since  $Q$  is abelian, we have  $Q \leq C_S(Q)$ . We also have  $Q \leq C_S(P)$ , hence  $Q \leq C_S(P) \cap C_S(Q) = C_{C_S(Q)}(P)$ . Since  $P$  is  $C_{\mathcal{F}}(Q)$ -centric, we have  $C_{C_S(Q)}(P) \leq P$ , hence  $Q \leq P$ . Thus  $P = PQ$  is  $\mathcal{F}$ -centric.

If one of these equivalent conditions holds, then  $P$  is  $\mathcal{F}$ -centric. Since  $P \leq C_S(Q)$ , we obtain  $Q \leq C_S(P) \leq P$ . □

A subgroup  $Q \leq S$  is called *normal* in  $\mathcal{F}$  if  $N_{\mathcal{F}}(Q) = \mathcal{F}$ . In this case we write  $Q \triangleleft \mathcal{F}$ . We say  $Q \leq S$  is *central* in  $\mathcal{F}$  if  $C_{\mathcal{F}}(Q) = \mathcal{F}$ . If a subgroup  $Q$  is central in  $\mathcal{F}$ , then it is also normal in  $\mathcal{F}$ . The product of all central subgroups in  $\mathcal{F}$  is called the *center* of  $\mathcal{F}$  and denoted by  $Z(\mathcal{F})$ . The product of all normal subgroups of  $\mathcal{F}$  is denoted by  $O_p(\mathcal{F})$ .

A subgroup  $P \leq S$  is  $\mathcal{F}$ -*radical* if  $\text{Out}_{\mathcal{F}}(P) := \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$  has no normal  $p$ -subgroups. A subgroup  $P \leq S$  is  $\mathcal{F}$ -*centric-radical* if it is both  $\mathcal{F}$ -radical and  $\mathcal{F}$ -centric. The full subcategories of  $\mathcal{F}$  generated by subgroups which are  $\mathcal{F}$ -centric,  $\mathcal{F}$ -radical, and  $\mathcal{F}$ -centric-radical are denoted by  $\mathcal{F}^c$ ,  $\mathcal{F}^r$ , and  $\mathcal{F}^{cr}$ , respectively. We recall the following well-known fact on normal subgroups of a fusion system.

**Proposition 8.12.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $Q$  be a subgroup of  $S$ . If  $Q$  is normal in  $\mathcal{F}$ , then it is included in every  $\mathcal{F}$ -centric-radical subgroup of  $S$ .*

*Proof.* See Proposition 4.46 in [10].  $\square$

As a consequence we obtain the following.

**Lemma 8.13.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ . Let  $Q \leq S$  and  $K \leq \text{Aut}(Q)$ . Suppose that  $Q$  is fully  $K$ -normalized, and assume that  $N_S^K(Q)$  contains some  $Q_0$  such that  $Q_0$  is normal in  $N_{\mathcal{F}}^K(Q)$  and  $Q_0 \in \mathcal{F}^c$ . Then, if  $P \leq N_S^K(Q)$  is an  $N_{\mathcal{F}}^K(Q)$ -centric-radical subgroup, then  $P$  is  $\mathcal{F}$ -centric.*

*Proof.* Suppose that  $P \leq N_S^K(Q)$  is  $N_{\mathcal{F}}^K(Q)$ -centric-radical. Since  $Q_0$  is normal in  $N_{\mathcal{F}}^K(Q)$ , by Proposition 8.12 applied to the fusion system  $N_{\mathcal{F}}^K(Q)$ , we obtain  $Q_0 \leq P$ . Since  $Q_0$  is  $\mathcal{F}$ -centric,  $P$  is  $\mathcal{F}$ -centric.  $\square$

## 9. THE CENTRALIZER DECOMPOSITION FOR $\mathcal{O}^c(\mathcal{F})$

Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $\mathcal{F}^e$  denote the full subcategory of  $\mathcal{F}$  whose objects are the nontrivial elementary abelian  $p$ -subgroups of  $S$  which are fully  $\mathcal{F}$ -centralized. By Theorem 2.4, there is a discrete group  $G$  with a Sylow subgroup isomorphic to  $S$  such that  $\mathcal{F}_S(G) \cong \mathcal{F}$ . By Lemmas 2.5 and 8.9, if we take  $\mathcal{C}$  to be the collection of all  $p$ -centric subgroups in  $G$ , then there is an equivalence of categories  $\overline{\mathcal{F}}_{\mathcal{C}}(G) \simeq \mathcal{O}^c(\mathcal{F})$ . Let  $\mathcal{E}$  denote the collection of all finite nontrivial elementary abelian  $p$ -subgroups of  $G$ .

**Lemma 9.1.** *Let  $G$  be a discrete group that realizes the fusion system  $\mathcal{F}$ , and  $\mathcal{C}$  be the collection of  $p$ -centric subgroups in  $G$ . For every  $E \in \mathcal{F}^e$ , the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E))$  over the collection  $\mathcal{C}|_{C_G(E)}$  is equivalent to the centric orbit category  $\mathcal{O}^c(C_{\mathcal{F}}(E))$ .*

*Proof.* By Proposition 8.6, we have  $\mathcal{F}_{C_S(E)}(C_G(E)) = C_{\mathcal{F}}(E)$ . This gives that  $\mathcal{F}_{\mathcal{C}}(C_G(E))$  is equivalent to the full subcategory of the fusion system  $C_{\mathcal{F}}(E)$  whose objects are subgroups of  $C_S(E)$  which are  $\mathcal{F}$ -centric. By Lemma 8.11, a subgroup  $P \leq C_S(E)$  is  $\mathcal{F}$ -centric if and only if it is  $C_{\mathcal{F}}(E)$ -centric. Hence  $\mathcal{F}_{\mathcal{C}}(C_G(E))$  is equivalent to the fusion system  $C_{\mathcal{F}}(E)^c$  as categories. We conclude that  $\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E))$  is equivalent to the centric orbit category  $\mathcal{O}^c(C_{\mathcal{F}}(E))$ .  $\square$

We introduce the following notation.

**Definition 9.2.** Let  $\theta : \overline{\mathcal{F}}_{\mathcal{C}}(G) \rightarrow \mathcal{O}^c(\mathcal{F})$  denote the functor that gives the equivalence of categories proved in Lemma 2.5. For every  $R\mathcal{O}^c(\mathcal{F})$ -module  $M$ , we denote the  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M \circ \theta$  by  $\overline{M}$ .

We have the following.

**Lemma 9.3.** *Let  $M$  be an  $R\mathcal{O}^c(\mathcal{F})$ -module. For every integer  $j \geq 0$ , there is a functor*

$$\mathcal{H}_{M, C_{\mathcal{F}}}^j : \mathcal{F}^e \rightarrow R\text{-Mod}$$

such that for every  $E \in \mathcal{F}^e$ ,

$$\mathcal{H}_{M, C_{\mathcal{F}}}^j(E) = H^j(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M).$$

*Proof.* Let  $G$  be a discrete group with Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} \cong \mathcal{F}_S(G)$ , and let  $\mathcal{C}$  be the collection of all  $p$ -centric subgroups in  $G$ . Consider the functor  $\mathcal{H}_{M, C_G}^j : \mathcal{F}_{\mathcal{D}}(G) \rightarrow R\text{-Mod}$  defined in Definition 7.6 for an arbitrary collection  $\mathcal{D}$ . Take  $\mathcal{D} = \mathcal{E}$  and  $\mathbf{O}_{\mathcal{C}}(C_G(E)) = \overline{\mathcal{F}}_{\mathcal{C}}(C_G(E))$ . Then for every  $j \geq 0$ , we obtain a functor

$$\mathcal{H}_{M, C_G}^j : \mathcal{F}_{\mathcal{E}}(G) \rightarrow R\text{-Mod}$$

such that for every  $E \in \mathcal{E}$ ,

$$\mathcal{H}_{M, C_G}^j(E) = H^*(\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E)); \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E))}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} \overline{M})$$

where  $\overline{M}$  is the  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module associated to  $M$  as defined in Definition 9.2. By Lemma 9.1, for every  $E \in \mathcal{F}^e$ , the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E))$  over the collection  $\mathcal{C}|_{C_G(E)}$  is equivalent to the centric orbit category  $\mathcal{O}^c(C_{\mathcal{F}}(E))$ . Hence by Proposition 3.9, there is an isomorphism

$$H^*(\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E)); \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E))}^{\overline{\mathcal{F}}_{\mathcal{C}}(G)} \overline{M}) \cong H^*(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M)$$

induced by the equivalence of categories. Using these isomorphisms, as it is done in the proof of Lemma 5.1, we obtain a functor  $\mathcal{H}_{M, C_{\mathcal{F}}}^j : \mathcal{F}^e \rightarrow R\text{-Mod}$  such that

$$\mathcal{H}_{M, C_{\mathcal{F}}}^j(E) = H^*(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M).$$

□

The following is stated as Theorem 1.6 in the introduction.

**Theorem 9.4.** *Let  $R$  be a commutative ring with unity,  $M$  be an  $R\mathcal{O}^c(\mathcal{F})$ -module, and  $\mathcal{H}_{M, C_{\mathcal{F}}}^j$  denote the functor defined in Lemma 9.3. Then there is a spectral sequence*

$$E_2^{s,t} = \lim_{\mathcal{F}^e}^s \mathcal{H}_{M, C_{\mathcal{F}}}^t \Rightarrow H^*(\mathcal{O}^c(\mathcal{F}); M).$$

*Proof.* Let  $G$  be a discrete group that realizes  $\mathcal{F}$ , and  $\mathcal{C}$  be the collection of all  $p$ -centric subgroups in  $G$ . Let  $\overline{M}$  denote the  $\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module that corresponds to  $M$  as defined in Definition 9.2. By Proposition 7.7, there is a spectral sequence

$$E_2^{s,t} = \lim_{\mathcal{F}_{\mathcal{E}}(G)}^s \mathcal{H}_{\overline{M}, C_G}^t \Rightarrow H^{s+t}(\overline{\mathcal{F}}_{\mathcal{C}}(G); \overline{M}).$$

By Lemmas 2.5 and 9.1, we have the equivalence of categories  $\overline{\mathcal{F}}_{\mathcal{C}}(G) \cong \mathcal{O}^c(\mathcal{F})$  and  $\overline{\mathcal{F}}_{\mathcal{C}}(C_G(E)) \cong \mathcal{O}^c(C_{\mathcal{F}}(E))$  for every  $E \in \mathcal{F}^e$ . Since every elementary abelian  $p$ -subgroup in  $\mathcal{E}$  is  $G$ -conjugate to a subgroup  $E \in \mathcal{F}^e$ , we also have  $\mathcal{F}_{\mathcal{E}}(G) \cong \mathcal{F}^e$ . Hence by Proposition 3.9, we can replace  $\mathcal{F}_{\mathcal{E}}(G)$  with  $\mathcal{F}^e$  and  $\mathcal{H}_{\overline{M}, C_G}^t$  with  $\mathcal{H}_{M, C_{\mathcal{F}}}^t$ . □

In [6], Broto, Levi, and Oliver introduced the centralizer decomposition for  $p$ -local finite groups. Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. This means that  $\mathcal{F}$  is a saturated fusion system over  $S$ , and  $\mathcal{L}$  is a centric linking system associated to  $\mathcal{F}$  (see Definition 10.9).

**Definition 9.5** ([6, Def 2.4]). For each  $E \in \mathcal{F}^e$ , the centralizer linking system  $C_{\mathcal{L}}(E)$  is the category whose objects are the  $C_{\mathcal{F}}(E)$ -centric subgroups  $P \leq C_S(E)$ , and whose morphisms  $\text{Mor}_{C_{\mathcal{L}}(E)}(P, P')$  are the set of morphisms  $\varphi \in \text{Mor}_{\mathcal{L}}(PE, P'E)$  whose underlying homomorphisms are the identity on  $E$  and send  $P$  into  $P'$ .

By [6, Prop 2.5], the category  $C_{\mathcal{L}}(E)$  is the centric linking system associated to  $C_{\mathcal{F}}(E)$ . For every  $E \in \mathcal{F}^e$ , let  $\tilde{C}_{\mathcal{L}}(E)$  denote the category whose objects are the pairs  $(P, \alpha)$  with  $P \in \mathcal{F}^c$  and  $\alpha \in \text{Mor}_{\mathcal{F}}(E, Z(P))$ . The morphisms in  $\tilde{C}_{\mathcal{L}}(E)$  are defined by

$$\text{Mor}_{\tilde{C}_{\mathcal{L}}(E)}((P, \alpha), (Q, \beta)) := \{\varphi \in \text{Mor}_{\mathcal{L}}(P, Q) \mid \pi(\varphi) \circ \alpha = \beta\}$$

where  $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$  denotes the canonical projection functor. There is a natural map

$$f : \text{hocolim}_{E \in (\mathcal{F}^e)^{op}} |\tilde{C}_{\mathcal{L}}(E)| \longrightarrow |\mathcal{L}|$$



induced by the forgetful functors  $(P, \alpha) \rightarrow P$ . It is proved by Broto, Levi, and Oliver [6, Thm 2.6] that  $f$  is a homotopy equivalence, and that there is a homotopy equivalence  $|C_{\mathcal{L}}(E)| \rightarrow |\tilde{C}_{\mathcal{L}}(E)|$  induced by the functor  $P \rightarrow (P, \text{incl})$ . This homotopy equivalence given by  $f$  is called *the centralizer decomposition* for  $(S, \mathcal{F}, \mathcal{L})$  over the collection of all nontrivial elementary abelian  $p$ -subgroups of  $S$ .

Associated to the centralizer decomposition, there is a Bousfield-Kan spectral sequence

$$E_2^{s,t} = \lim_{\mathcal{F}^e}^s H^t(|C_{\mathcal{L}}(-)|; \mathbb{F}_p) \Rightarrow H^{s+t}(|\mathcal{L}|; \mathbb{F}_p).$$

We say the centralizer decomposition is *sharp* if  $E_2^{s,t} = 0$  for  $s \geq 1$ . When the centralizer decomposition is sharp, then there is an isomorphism

$$H^*(|\mathcal{L}|; \mathbb{F}_p) \cong \lim_{E \in \mathcal{F}^e} H^*(|C_{\mathcal{L}}(E)|; \mathbb{F}_p).$$

Broto, Levi, and Oliver [6] proved that the centralizer decomposition for every  $p$ -local finite group is sharp. We can also state their result in terms of the cohomology of the centralizer fusion systems.

**Theorem 9.6** (Broto-Levi-Oliver [6], p. 821). *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ . Then, for every  $n \geq 0$ , and every  $i \geq 1$ ,*

$$\lim_{\mathcal{F}^e}^i H^n(C_{\mathcal{F}}(-); \mathbb{F}_p) = 0.$$

**Remark 9.7.** In the above theorem  $H^n(C_{\mathcal{F}}(-); \mathbb{F}_p)$  denotes the functor which sends  $E \in \mathcal{F}^e$  to

$$H^n(C_{\mathcal{F}}(E); \mathbb{F}_p) := \lim_{P \in \mathcal{O}^c(C_{\mathcal{F}}(E))} H^n(P; \mathbb{F}_p).$$

Note that  $H^n(C_{\mathcal{F}}(E); \mathbb{F}_p)$  is the subring of  $C_{\mathcal{F}}(E)$ -stable elements in  $H^n(C_S(E); \mathbb{F}_p)$ .

We now recall the definition of the sharpness of the subgroup decomposition. For every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , there is a homotopy equivalence

$$|\mathcal{L}| \simeq \text{hocolim}_{\mathcal{O}^c(\mathcal{F})} \tilde{B}$$

where  $\tilde{B} : \mathcal{O}^c(\mathcal{F}) \rightarrow \text{Top}$  is a functor such that  $\tilde{B}(P)$  is homotopy equivalent to the classifying space  $BP$  for every  $P \in \mathcal{F}^c$  (see [6, Prop 2.2]). This homotopy equivalence is called the subgroup decomposition for the  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  over  $\mathcal{F}$ -centric subgroups. The subgroup decomposition for  $(S, \mathcal{F}, \mathcal{L})$  is sharp if for every  $n \geq 0$  and every  $i \geq 1$ ,

$$\lim_{\mathcal{O}^c(\mathcal{F})}^i H^n(-; \mathbb{F}_p) = 0.$$

Now we are ready to prove the main theorem of this section (stated as Theorem 1.8 in the introduction). Recall that a subgroup  $Q \leq S$  is *central* in  $\mathcal{F}$  if  $C_{\mathcal{F}}(Q) = \mathcal{F}$ . The product of all central subgroups in  $\mathcal{F}$  is called the center of  $\mathcal{F}$  and denoted by  $Z(\mathcal{F})$ .

**Theorem 9.8.** *If the subgroup decomposition is sharp for every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  with  $Z(\mathcal{F}) \neq 1$ , then it is sharp for every  $p$ -local finite group.*

*Proof.* Let  $n \geq 0$  be a fixed integer, and let  $M := H^n(-; \mathbb{F}_p)$  denote the group cohomology functor considered as a module over  $\mathcal{O}^c(\mathcal{F})$ . By Theorem 9.4, there is a spectral sequence

$$E_2^{s,t} = \lim_{\mathcal{F}^e}^s \mathcal{H}_{M, C_{\mathcal{F}}}^t \Rightarrow H^{s+t}(\mathcal{O}^c(\mathcal{F}); M)$$

where for each  $E \in \mathcal{F}^e$ , we have

$$\mathcal{H}_{M, C_{\mathcal{F}}}^t(E) = H^t(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M).$$

For every  $E \in \mathcal{F}^e$ , the centralizer fusion system  $C_{\mathcal{F}}(E)$  has a nontrivial center, hence by assumption, the subgroup decomposition for  $(C_S(E), C_{\mathcal{F}}(E), C_{\mathcal{L}}(E))$  is sharp. This gives that for every  $E \in \mathcal{F}^e$ ,

$$\mathcal{H}_{M, C_{\mathcal{F}}}^t(E) = H^t(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M) = H^t(\mathcal{O}^c(C_{\mathcal{F}}(E)); H^n(-; \mathbb{F}_p)) = 0$$

for  $t \geq 1$ . Therefore  $E_2^{s,t} = 0$  for all  $t \geq 1$ . By Remark 9.7, we have

$$\mathcal{H}_{M, C_{\mathcal{F}}}^0(E) = H^0(\mathcal{O}^c(C_{\mathcal{F}}(E)); \text{Res}_{\mathcal{O}^c(C_{\mathcal{F}}(E))}^{\mathcal{O}^c(\mathcal{F})} M) \cong \lim_{\mathcal{O}^c(C_{\mathcal{F}}(E))} H^n(-; \mathbb{F}_p) = H^n(C_{\mathcal{F}}(E); \mathbb{F}_p).$$

Hence we have

$$E_2^{s,0} = \lim_{\mathcal{F}^e}^s \mathcal{H}_{M, C_{\mathcal{F}}}^0 \cong \lim_{\mathcal{F}^e}^s H^n(C_{\mathcal{F}}(-); \mathbb{F}_p).$$

By Theorem 9.6, these higher limits vanish for  $s \geq 1$ , hence  $H^i(\mathcal{O}^c(\mathcal{F}); M) = 0$  for all  $i \geq 1$ .  $\square$

Another consequence of the spectral sequence argument given above is the following: Given a  $p$ -local group  $(S, \mathcal{F}, \mathcal{L})$ , if we know that

- (1) the subgroup decomposition for  $(S, \mathcal{F}, \mathcal{L})$  is sharp, and
- (2) for every  $E \in \mathcal{F}^e$ , the subgroup decomposition for  $(C_S(E), C_{\mathcal{F}}(E), C_{\mathcal{L}}(L))$  is sharp,

then we can conclude that the centralizer decomposition for  $(S, \mathcal{F}, \mathcal{L})$  over the nontrivial elementary abelian  $p$ -subgroups is sharp. We can consider this as a propagation result between two different types of homology decompositions in the sense given by Grodal and Smith in [15].

## 10. THE NORMALIZER DECOMPOSITION FOR $\mathcal{O}^c(\mathcal{F})$

Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $sd(\mathcal{F}^c)$  denote the poset category of all chains  $\sigma = (P_0 < P_1 < \cdots < P_n)$  with  $P_i \in \mathcal{F}^c$  where there is a unique morphism  $\sigma \rightarrow \tau$  in  $sd(\mathcal{F}^c)$  if  $\tau$  is a subchain of  $\sigma$ . Two chains  $\sigma = (P_0 < P_1 < \cdots < P_n)$  and  $\tau = (Q_0 < Q_1 < \cdots < Q_n)$  are  $\mathcal{F}$ -conjugate if there is an isomorphism  $\varphi : P_n \rightarrow Q_n$  in  $\mathcal{F}$  such that  $\varphi(P_i) = Q_i$  for all  $i$ . In this case we write  $\sigma \sim_{\mathcal{F}} \tau$ . The  $\mathcal{F}$ -conjugacy class of a chain  $\sigma$  in  $sd(\mathcal{F}^c)$  is denoted by  $[\sigma]$ .

**Definition 10.1.** The category of  $\mathcal{F}$ -conjugacy classes of chains in  $\mathcal{F}^c$ , denoted by  $\bar{sd}(\mathcal{F}^c)$ , is the poset category whose objects are  $\mathcal{F}$ -conjugacy classes  $[\sigma]$  of chains of subgroups in  $\mathcal{F}^c$  where there is a unique morphism  $[\sigma] \rightarrow [\tau]$  in  $\bar{sd}(\mathcal{F}^c)$  if  $\tau$  is  $\mathcal{F}$ -conjugate to a subchain of  $\sigma$ .

Let  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} \cong \mathcal{F}_S(G)$ . If  $\mathcal{C}$  is the collection of all  $p$ -centric subgroups in  $G$ , then by Lemma 8.9 we have  $\mathcal{F}^c \cong \mathcal{F}_{\mathcal{C}}(G)$ . We defined the category  $Sd_{\mathcal{C}}(G)/G$  to be the category whose objects are the  $G$ -orbits  $[\sigma]$  of the chains of subgroups in  $\mathcal{C}$  and such that there is a unique morphism  $[\sigma] \rightarrow [\tau]$  in  $Sd_{\mathcal{C}}(G)/G$  if there is an element  $g \in G$  such that  $\tau$  is a face of  $g\sigma$  (see Definition 6.8). It is easy to see that since  $\mathcal{F}^c \cong \mathcal{F}_{\mathcal{C}}(G)$ , the poset categories  $Sd_{\mathcal{C}}(G)/G$  and  $\bar{sd}(\mathcal{F}^c)$  are isomorphic.

If  $\sigma = (P_0 < \cdots < P_n)$  is a chain of subgroups of a discrete group  $G$ , then the stabilizer of  $\sigma$  under the  $G$ -action defined by conjugation is the subgroup  $N_G(\sigma) = \bigcap_i N_G(P_i)$ . Note that  $C_G(P_n) \leq N_G(\sigma)$  is a normal subgroup and the quotient group  $N_G(\sigma)/C_G(P_n)$  is a subgroup of  $N_G(P_n)/C_G(P_n) = \text{Aut}_G(P_n) \leq \text{Aut}(P_n)$ . We define the  $G$ -automorphism group of  $\sigma$  to be the subgroup

$$\text{Aut}_G(\sigma) := N_G(\sigma)/C_G(P_n) \leq \text{Aut}(P_n).$$

For a chain of  $p$ -groups  $\sigma = (P_0 < \cdots < P_n)$  in  $S$ , the automorphism group of  $\sigma$  is defined by

$$\text{Aut}(\sigma) := \{\alpha \in \text{Aut}(P_n) \mid \alpha(P_i) = P_i \text{ for all } i\} \leq \text{Aut}(P_n).$$

In Section 8, for every subgroup  $K \leq \text{Aut}(Q)$ , we defined the  $K$ -normalizer group  $N_G^K(Q)$  and the  $K$ -normalizer fusion system  $N_{\mathcal{F}}^K(Q)$  (see Definition 8.1). It is easy to see that if we take  $K = \text{Aut}(\sigma) \leq \text{Aut}(P_n)$ , then  $N_G^K(P_n) = N_G(\sigma)$ . For the  $K$ -normalizer fusion system  $N_{\mathcal{F}}^K(P_n)$ , we have the following alternative description.

**Definition 10.2.** Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $\sigma = (P_0 < \cdots < P_n)$  be a chain of subgroups in  $S$ . The *normalizer fusion system*  $N_{\mathcal{F}}(\sigma)$  is the fusion system over  $N_S(\sigma)$  whose morphisms  $\text{Mor}_{N_{\mathcal{F}}(\sigma)}(P, R)$  are all the morphisms  $\varphi \in \text{Mor}_{\mathcal{F}}(P, R)$  which extend to a morphism  $\tilde{\varphi} : P_n P \rightarrow P_n R$  in  $\mathcal{F}$  in such a way that  $\tilde{\varphi}(P_i) = P_i$  for all  $i \in \{1, \dots, n\}$ .

We say a chain  $\sigma = (P_0 < \cdots < P_n)$  in  $S$  is *fully  $\mathcal{F}$ -normalized* if  $|N_S(\sigma)| \geq |N_S(\tau)|$  for every  $\tau \sim_{\mathcal{F}} \sigma$ . Note that this is equivalent to saying that  $P_n$  is fully  $K$ -normalized with  $K = \text{Aut}(\sigma) \leq \text{Aut}(P_n)$ . Using the results of Section 8 we conclude the following proposition which was also proved in [22, Prop 3.9] using a different approach.

**Proposition 10.3.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and let  $\sigma = (P_0 < \cdots < P_n)$  be a chain of subgroups in  $S$  which is fully  $\mathcal{F}$ -normalized. Then*

- (1) *the normalizer fusion system  $N_{\mathcal{F}}(\sigma)$  is saturated, and*
- (2) *if  $\mathcal{F} = \mathcal{F}_S(G)$  for a discrete group  $G$  with a Sylow  $p$ -subgroup  $S$ , then  $N_S(\sigma)$  is a Sylow  $p$ -subgroup of  $N_G(\sigma)$  and  $\mathcal{F}_{N_S(\sigma)}(N_G(\sigma)) = N_{\mathcal{F}}(\sigma)$ .*

*Proof.* The first statement follows from Theorem 8.2 and the second part follows from Propositions 8.4 and 8.6.  $\square$

Let  $\sigma$  be a chain of subgroups in  $\mathcal{F}^c$  which is fully  $\mathcal{F}$ -normalized. Let  $\mathcal{N} := N_{\mathcal{F}}(\sigma)$ , and let  $\mathcal{N}^c$  denote the full subcategory of  $\mathcal{N}$  generated by  $\mathcal{N}$ -centric subgroups in  $N_S(\sigma)$ . We denote by  $\mathcal{N}^{cn}$  the full subcategory of  $\mathcal{N}$  generated by subgroups of  $N_S(\sigma)$  which are  $\mathcal{F}$ -centric. Let  $\mathcal{N}^{cr}$  denote the full subcategory of  $\mathcal{N}$  generated by  $\mathcal{N}$ -centric-radical subgroups of  $N_S(\sigma)$ .

**Lemma 10.4.** *Let  $\sigma = (P_0 < \cdots < P_n)$  be a chain of subgroups in  $\mathcal{F}^c$ . Suppose that  $\sigma$  is fully  $\mathcal{F}$ -normalized. Let  $\mathcal{N} := N_{\mathcal{F}}(\sigma)$ , and  $\mathcal{N}^{cr}, \mathcal{N}^{cn}, \mathcal{N}^c$  be the subcategories defined above. Then*

$$\mathcal{N}^{cr} \subseteq \mathcal{N}^{cn} \subseteq \mathcal{N}^c.$$

*Proof.* Note that  $N_S(\sigma)$  contains  $P_0 \in \mathcal{F}^c$  as a subgroup and  $P_0$  is normal in  $N_{\mathcal{F}}(\sigma)$ . Applying Lemma 8.13, we obtain that if  $P \leq N_S(\sigma)$  is an  $\mathcal{N}$ -centric-radical subgroup, then  $P$  is  $\mathcal{F}$ -centric. Hence  $\mathcal{N}^{cr} \subseteq \mathcal{N}^{cn}$ . For the second inequality, note that by Lemma 8.10(i), if  $P \leq N_S(\sigma)$  is  $\mathcal{F}$ -centric then it is  $\mathcal{N}$ -centric, hence  $\mathcal{N}^{cn}$  is a full subcategory of  $\mathcal{N}^c$ .  $\square$

For higher limits over different collections of subgroups, we will now prove a proposition. Although it is not stated in the following way, we think this statement is essentially what was proved in [6, Cor 3.6].

**Proposition 10.5.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ , and  $\mathcal{C}$  be the collection of subgroups of  $S$  closed under taking overgroups. Assume that  $\text{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{C} \subseteq \text{Ob}(\mathcal{F}^c)$ , and let  $\nu : \mathcal{O}(\mathcal{F}_{\mathcal{C}}) \rightarrow \mathcal{O}^c(\mathcal{F})$  denote the inclusion functor for the corresponding orbit categories. Then for every  $\mathbb{Z}_{(p)}\mathcal{O}^c(\mathcal{F})$ -module  $M$ , there is an isomorphism*

$$\lim_{\mathcal{O}(\mathcal{F}_{\mathcal{C}})}^*(M \circ \nu) \cong \lim_{\mathcal{O}^c(\mathcal{F})}^* M.$$

*Proof.* We can filter the module  $M$  with atomic functors, i.e. with functors that vanish except on a single  $\mathcal{F}$ -conjugacy class, and prove the result by induction using long exact sequences. Hence it is enough to prove the above isomorphism when  $M$  is an atomic functor.

Let  $Q \in \mathcal{F}^c$  be such that  $M$  vanishes on subgroups outside of the  $\mathcal{F}$ -conjugacy class of  $Q$ . If  $Q \in \mathcal{C}$ , then by [26, Lemma 1.5(a)], the isomorphism in the proposition holds. So assume that  $Q \notin \mathcal{C}$ . In this case the restriction of  $M$  to the subcategory  $\mathcal{O}(\mathcal{F}_\mathcal{C})$  is zero. Therefore in this case we need to show that  $\lim_{\mathcal{O}^c(\mathcal{F})}^* M = 0$ .

Let  $\Gamma$  be a finite group and  $N$  be a  $\mathbb{Z}_{(p)}G$ -module. Let  $F_N$  denote the  $\mathcal{O}_p(\Gamma)$ -module such that  $F_N(1) = N$  and  $F(P) = 0$  for all  $p$ -subgroups  $1 \neq P \leq \Gamma$ . We define

$$\Lambda^*(\Gamma; N) := \lim_{\mathcal{O}_p(\Gamma)}^* F_N.$$

By [6, Prop 3.2], there is an isomorphism

$$\lim_{\mathcal{O}^c(\mathcal{F})}^* M \cong \Lambda^*(\text{Out}_{\mathcal{F}}(Q); M(Q)).$$

Since  $Q$  is not  $\mathcal{F}$ -radical, the group  $\text{Out}_{\mathcal{F}}(Q)$  has a normal  $p$ -subgroup. Then by [19, Prop 6.1(ii)] we have  $\Lambda^*(\text{Out}_{\mathcal{F}}(Q); M(Q)) = 0$ . Hence the proof is complete.  $\square$

Now, we are ready to prove the following.

**Proposition 10.6.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ ,  $G$  be a discrete group realizing  $\mathcal{F}$ , and  $\mathcal{C}$  be the collection of all  $p$ -centric subgroups in  $G$ . Let  $M$  denote an  $\mathbb{Z}_{(p)}\mathcal{O}(\mathcal{F})$ -module and  $\overline{M}$  denote the corresponding  $\mathbb{Z}_{(p)}\overline{\mathcal{F}}(G)$ -module. For every fully  $\mathcal{F}$ -normalized chain of subgroups  $\sigma = (P_0 < \dots < P_n)$  with  $P_i \in \mathcal{F}^c$ , there is an isomorphism*

$$H^*(\overline{\mathcal{F}}_{\mathcal{C}}(N_G(\sigma)); \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(N_G(\sigma))}^{\overline{\mathcal{F}}(G)} \overline{M}) \cong H^*(\mathcal{O}^c(N_{\mathcal{F}}(\sigma)); \text{Res}_{\mathcal{O}^c(N_{\mathcal{F}}(\sigma))}^{\mathcal{O}(\mathcal{F})} M).$$

*Proof.* Let  $\mathcal{N} = \mathcal{N}_{\mathcal{F}}(\sigma)$ . By Proposition 10.3,  $\mathcal{F}_{N_S(\sigma)}(N_G(\sigma)) = \mathcal{N}$ , hence the fusion orbit category  $\overline{\mathcal{F}}_{\mathcal{C}}(N_G(\sigma))$  is equivalent to the orbit category  $\mathcal{O}^{cn}(\mathcal{N})$ . This gives an isomorphism

$$H^*(\overline{\mathcal{F}}_{\mathcal{C}}(N_G(\sigma)); \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(N_G(\sigma))}^{\overline{\mathcal{F}}(G)} \overline{M}) \cong H^*(\mathcal{O}(\mathcal{N}^{cn}); \text{Res}_{\mathcal{O}(\mathcal{N}^{cn})}^{\mathcal{O}(\mathcal{F})} M).$$

By Lemma 10.4, we have  $\mathcal{N}^{cr} \subseteq \mathcal{N}^{cn} \subseteq \mathcal{N}^c$ . Applying Proposition 10.5 to the fusion system  $\mathcal{N}$ , we obtain the desired isomorphism.  $\square$

Using the isomorphism in Proposition 10.6, we define the following functor.

**Lemma 10.7.** *Let  $M$  be an  $\mathbb{Z}_{(p)}\mathcal{O}(\mathcal{F})$ -module. For every integer  $j \geq 0$ , there is a functor*

$$\mathcal{H}_{M, N_{\mathcal{F}}}^j : \overline{sd}(\mathcal{F}^c) \rightarrow \mathbb{Z}_{(p)}\text{-Mod}$$

such that for every  $[\sigma] \in \overline{sd}(\mathcal{F}^c)$ ,

$$\mathcal{H}_{M, N_{\mathcal{F}}}^j([\sigma]) = H^j(\mathcal{O}^c(N_{\mathcal{F}}(\sigma)); \text{Res}_{\mathcal{O}^c(N_{\mathcal{F}}(\sigma))}^{\mathcal{O}(\mathcal{F})} M)$$

where  $\sigma$  is a representative for  $[\sigma]$  such that  $\sigma$  is a fully  $\mathcal{F}$ -normalized chain of subgroups of  $S$ .

*Proof.* The proof follows from the same argument given in the proof of Lemma 9.3. In this case we use the isomorphisms proved in Proposition 10.6.  $\square$

The following is stated as Theorem 1.7 in the introduction.

**Theorem 10.8.** *Let  $M$  be an  $\mathbb{Z}_{(p)}\mathcal{O}(\mathcal{F})$ -module, and  $\mathcal{H}_{M, N_{\mathcal{F}}}^j$  denote the functor defined in Lemma 10.7. Then there is a spectral sequence*

$$E_2^{s,t} = \lim_{\overline{sd}(\mathcal{F}^c)}^s \mathcal{H}_{M, N_{\mathcal{F}}}^t \Rightarrow H^*(\mathcal{O}^c(\mathcal{F}); \text{Res}_{\mathcal{O}^c(\mathcal{F})}^{\mathcal{O}(\mathcal{F})} M).$$

*Proof.* Let  $G$  be a discrete group with a Sylow  $p$ -subgroup  $S$  such that  $\mathcal{F} \cong \mathcal{F}_S(G)$ . Let  $\mathcal{C}$  be the collection of all  $p$ -centric subgroups in  $G$ . The collection  $\mathcal{C}$  is closed under taking overgroups, hence it is closed under taking products. Let  $\overline{M}$  denote the  $\overline{\mathcal{F}}(G)$ -module corresponding to  $M$ . By Proposition 7.11, there is a spectral sequence

$$E_2^{s,t} = \lim_{Sd_{\mathcal{C}}(G)/G}^s \mathcal{H}_{\overline{M}, N_G}^t \Rightarrow H^{s+t}(\overline{\mathcal{F}}_{\mathcal{C}}(G); \text{Res}_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^{\overline{\mathcal{F}}(G)} \overline{M})$$

where  $\mathcal{H}_{\overline{M}, N_G}^j$  is the functor defined in Definition 7.10. Since  $\mathcal{F}^c \cong \mathcal{F}_{\mathcal{C}}(G)$ , the poset categories  $Sd_{\mathcal{C}}(G)/G$  and  $\overline{sd}(\mathcal{F}^c)$  are isomorphic. In each  $\mathcal{F}$ -conjugacy class  $[\sigma] \in \overline{sd}(\mathcal{F}^c)$ , we choose the chain  $\sigma$  which is fully  $\mathcal{F}$ -normalized. By Proposition 10.6, there is an isomorphism of functors

$$\mathcal{H}_{\overline{M}, N_G}^t \cong \mathcal{H}_{M, N_{\mathcal{F}}}^t$$

once we identify  $Sd_{\mathcal{C}}(G)/G$  with  $\overline{sd}(\mathcal{F})$ . We also have  $\overline{\mathcal{F}}_{\mathcal{C}}(G) \cong \mathcal{O}^c(\mathcal{F})$ . Hence applying Proposition 3.9, we obtain the desired spectral sequence.  $\square$

In the rest of the section we consider the normalizer decomposition for  $p$ -local finite groups introduced by Libman [21]. A  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is a triple where  $S$  is a finite  $p$ -group,  $\mathcal{F}$  is a saturated fusion system over  $S$ , and  $\mathcal{L}$  is a centric linking system associated to  $\mathcal{F}$ . We recall the definition of a centric linking system.

**Definition 10.9** ([2, Def 3.2]). Let  $\mathcal{F}$  be a fusion system over the  $p$ -group  $S$ . A centric linking system associated to  $\mathcal{F}$  is the category  $\mathcal{L}$  whose objects are the  $\mathcal{F}$ -centric subgroups of  $S$ , together with a pair of functors

$$\mathcal{T}_S^{\mathcal{F}^c}(S) \xrightarrow{\epsilon} \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c$$

satisfying the following conditions:

- (1) The functor  $\epsilon$  is the identity on objects and injective on morphisms, while  $\pi$  is the inclusion on objects and is surjective on each morphism set.
- (2) For each  $P, Q \in \text{Ob}(\mathcal{L})$ ,  $\epsilon_P(P) \leq \text{Aut}_{\mathcal{L}}(P)$  acts freely on the set  $\text{Mor}_{\mathcal{L}}(P, Q)$  by precomposition and  $\pi_{P, Q}$  induces a bijection from  $\text{Mor}_{\mathcal{L}}(P, Q)/\epsilon_P(Z(P))$  onto  $\text{Mor}_{\mathcal{F}}(P, Q)$ .
- (3) For each  $P, Q \in \text{Ob}(\mathcal{L})$  and  $g \in \mathcal{T}_S(P, Q)$ , the composite functor  $\pi \circ \epsilon$  sends  $g \in \text{Mor}_{\mathcal{T}}(P, Q)$  to  $c_g \in \text{Mor}_{\mathcal{F}}(P, Q)$ .
- (4) For  $P, Q \in \text{Ob}(\mathcal{L})$ ,  $\psi \in \text{Mor}_{\mathcal{L}}(P, Q)$ , and  $g \in P$ , the following square commutes in  $\mathcal{L}$ .

$$\begin{array}{ccc} P & \xrightarrow{\psi} & Q \\ \epsilon_P(g) \downarrow & & \downarrow \epsilon_Q(\pi(\psi)(g)) \\ P & \xrightarrow{\psi} & Q \end{array}$$

For every  $P, Q \in \mathcal{F}^c$  such that  $P \leq Q$ , let  $i_P^Q : P \rightarrow Q$  in  $\mathcal{L}$  denote the image of the inclusion map  $P \rightarrow Q$  under  $\epsilon_P$ . For each chain  $\sigma = (P_0 < \cdots < P_n)$  of  $\mathcal{F}$ -centric subgroups in  $S$ , let

$\text{Aut}_{\mathcal{L}}(\sigma)$  denote the subgroup of  $\prod_{i=1}^n \text{Aut}_{\mathcal{L}}(P_i)$  formed by tuples  $(\alpha_0, \dots, \alpha_n)$  of automorphisms  $\alpha_i \in \text{Aut}_{\mathcal{L}}(P_i)$  such that the following diagram commutes

$$\begin{array}{ccccccc} P_0 & \xrightarrow{i_{P_0}^{P_1}} & P_1 & \xrightarrow{i_{P_1}^{P_2}} & \cdots & \longrightarrow & P_{n-1} & \xrightarrow{i_{P_{n-1}}^{P_n}} & P_n \\ \downarrow \alpha_0 & & \downarrow \alpha_1 & & & & \downarrow \alpha_{n-1} & & \downarrow \alpha_n \\ P_0 & \xrightarrow{i_{P_0}^{P_1}} & P_1 & \xrightarrow{i_{P_1}^{P_2}} & \cdots & \longrightarrow & P_{n-1} & \xrightarrow{i_{P_{n-1}}^{P_n}} & P_n \end{array}$$

(see [21, Def 1.4] for details).

In  $\mathcal{L}$ , every morphism is a monomorphism and an epimorphism in the categorical sense (see [21, Remark 2.10 and Prop 2.11]). As a consequence, for each  $j = 0, \dots, n$ , the map

$$\pi_j : \text{Aut}_{\mathcal{L}}(\sigma) \rightarrow \text{Aut}_{\mathcal{L}}(P_j)$$

defined by  $(\alpha_0, \dots, \alpha_n) \rightarrow \alpha_j$  is a monomorphism, so the automorphism group  $\text{Aut}_{\mathcal{L}}(\sigma)$  can be considered as a subgroup of  $\text{Aut}_{\mathcal{L}}(P_j)$  via the projection map  $\pi_j$ . This allows us to identify  $\mathcal{B}(\text{Aut}_{\mathcal{L}}(\sigma))$  with a subcategory of  $\mathcal{L}$ . Note that if  $\tau$  is a face of  $\sigma$ , then there is group homomorphism  $\text{Aut}_{\mathcal{L}}(\sigma) \rightarrow \text{Aut}_{\mathcal{L}}(\tau)$  defined by restriction.

**Theorem 10.10** ([21, Thm A], [1, Thm 5.36]). *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group. Then there is a functor  $\delta : \overline{sd}(\mathcal{F}^c) \rightarrow \text{Top}$  such that the following hold:*

(1) *There is a homotopy equivalence*

$$\text{hocolim}_{[\sigma] \in \overline{sd}(\mathcal{F}^c)} \delta([\sigma]) \xrightarrow{\simeq} |\mathcal{L}|.$$

(2) *For each  $\sigma \in \overline{sd}(\mathcal{F}^c)$ , there is a natural homotopy equivalence*

$$B\text{Aut}_{\mathcal{L}}(\sigma) \xrightarrow{\simeq} \delta([\sigma]).$$

(3) *For each  $\sigma \in \overline{sd}(\mathcal{F}^c)$ , the map from  $B\text{Aut}_{\mathcal{L}}(\sigma)$  to  $|\mathcal{L}|$  induced by the equivalences in (1) and (2) is equal to the map induced by the inclusion of  $\mathcal{B}(\text{Aut}_{\mathcal{L}}(\sigma))$  into  $\mathcal{L}$ .*

Associated to the normalizer decomposition, there is a Bousfield-Kan spectral sequence

$$E_2^{s,t} = \lim_{\overline{sd}(\mathcal{F}^c)}^s H^t(\text{Aut}_{\mathcal{L}}(-); \mathbb{F}_p) \Rightarrow H^{s+t}(|\mathcal{L}|; \mathbb{F}_p).$$

**Definition 10.11.** We say the normalizer decomposition for  $(S, \mathcal{F}, \mathcal{L})$  is *sharp* if  $E_2^{s,t} = 0$  for all  $s > 0$  and  $t \geq 0$ .

We prove below (Theorem 10.16) that the sharpness of the normalizer decomposition is equivalent to the sharpness of the subgroup decomposition. For this, we need to introduce more definitions and prove some lemmas.

Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and  $\sigma = (P_0 < \cdots < P_n)$  be a fully  $\mathcal{F}$ -normalized chain of  $\mathcal{F}$ -centric subgroups in  $S$ . The normalizer fusion system  $N_{\mathcal{F}}(\sigma)$  is defined in Definition 10.2 as the  $K$ -normalizer fusion system  $N_{\mathcal{F}}^K(P_n)$  where  $K = \text{Aut}(\sigma) \leq \text{Aut}(P_n)$ . By Proposition 10.3,  $N_{\mathcal{F}}(\sigma)$  is saturated. By Lemma 8.10, for every  $N_{\mathcal{F}}(\sigma)$ -centric subgroup  $P$ , the subgroup  $P_n P$  is  $\mathcal{F}$ -centric.

**Definition 10.12.** Let  $\sigma$  be a fully  $\mathcal{F}$ -normalized chain of subgroups in  $\mathcal{F}^c$  and  $\mathcal{N} = N_{\mathcal{F}}(\sigma)$ . The *normalizer linking system*  $N_{\mathcal{L}}(\sigma)$  is the category whose objects are the  $\mathcal{N}$ -centric subgroups in  $N_S(\sigma)$ , and whose morphisms for  $P, P' \in \mathcal{N}^c$  are defined by

$$\text{Mor}_{N_{\mathcal{L}}(\sigma)}(P, P') = \{\varphi \in \text{Mor}_{\mathcal{L}}(P_n P, P_n P') \mid \pi(\varphi)(P) \leq P', \pi(\varphi)(P_i) \leq P_i \text{ for all } i\}.$$

We have the following lemma.

**Lemma 10.13.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and  $\sigma$  be a fully  $\mathcal{F}$ -normalized chain of subgroups in  $\mathcal{F}^c$ . Then the normalizer linking system  $N_{\mathcal{L}}(\sigma)$  is the linking system associated to the normalizer fusion system  $N_{\mathcal{F}}(\sigma)$ .*

*Proof.* When  $\sigma = (P_0)$  is a chain with one subgroup this statement is proved in [6, Lemma 6.2]. We modify the argument given there to a chain of subgroups. Let  $\mathcal{N} := N_{\mathcal{F}}(\sigma)$  and  $\mathcal{N}^c$  denote the full subcategory of  $\mathcal{N}$ -centric subgroups of  $N_S(\sigma)$ . The projection functor  $\pi^\sigma : N_{\mathcal{L}}(\sigma) \rightarrow \mathcal{N}$  is defined to be the inclusion map on objects and it sends a morphism  $\varphi \in \text{Mor}_{N_{\mathcal{L}}(\sigma)}(P, P')$  to  $\pi(\varphi)|_P$ . Let

$$\epsilon^\sigma : \mathcal{T}_{N_S(\sigma)}^{\mathcal{N}^c} \rightarrow N_{\mathcal{L}}(\sigma)$$

be the functor which is the identity map on objects and such that for  $P, P' \in \mathcal{N}^c$ , the map  $\epsilon_{P, P'}^\sigma : N_{N_S(\sigma)}(P, P') \rightarrow \text{Mor}_{N_{\mathcal{L}}(\sigma)}(P, P')$  is defined by restricting  $\epsilon_{PP_n, P'P_n} : N_S(PP_n, P'P_n) \rightarrow \text{Aut}_{\mathcal{L}}(PP_n, P'P_n)$  to the subset  $N_{N_S(\sigma)}(P, P')$ . It is clear that with these definitions the condition (1) of Definition 10.9 holds.

The subgroup  $PP_n$  acts freely on  $\text{Mor}_{\mathcal{L}}(PP_n, P'P_n)$  via  $\epsilon_{PP_n}$ . Hence  $Z(P)$  acts freely on  $\text{Mor}_{N_{\mathcal{L}}(\sigma)}(P, P')$ . If  $\varphi_1, \varphi_2$  are two morphisms in  $\text{Mor}_{N_{\mathcal{L}}(\sigma)}(P, P')$  such that  $\pi^\sigma(\varphi_1) = \pi^\sigma(\varphi_2)$ , then  $\pi(\varphi_1)|_P = \pi(\varphi_2)|_P$ . Since  $P$  is  $\mathcal{N}$ -centric, by [6, Prop A.8] applied to the fusion system  $\mathcal{N}$ , there is some  $g \in Z(P)$  such that  $\pi(\varphi_2) = \pi(\varphi_1) \circ c_g$ . By condition (2), there is an  $h \in Z(PP_n) \leq Z(P)$  such that

$$\varphi_2 = \varphi_1 \circ \epsilon_{PP_n}(g) \circ \epsilon_{PP_n}(h) = \varphi_1 \circ \epsilon_P^\sigma(gh)$$

holds in  $\text{Mor}_{\mathcal{L}}(PP_n, P'P_n)$ . Hence the condition (2) holds for the category  $N_{\mathcal{L}}(\sigma)$ . Conditions (3) and (4) hold for  $N_{\mathcal{L}}(\sigma)$  since they hold for  $\mathcal{L}$ . We conclude that  $N_{\mathcal{L}}(\sigma)$  is the linking system associated to the normalizer fusion system  $N_{\mathcal{F}}(\sigma)$ .  $\square$

Recall that the cohomology of the normalizer fusion system  $\mathcal{N} = N_{\mathcal{F}}(\sigma)$  is defined by

$$H^n(\mathcal{N}; \mathbb{F}_p) := \lim_{P \in \mathcal{O}^c(\mathcal{N})} H^n(P; \mathbb{F}_p)$$

which is equal to the subring of  $\mathcal{N}$ -stable elements in  $H^n(N_S(\sigma); \mathbb{F}_p)$ .

**Lemma 10.14.** *Let  $(S, \mathcal{F}, \mathcal{L})$  be a  $p$ -local finite group and let  $\sigma$  be a chain of  $\mathcal{F}$ -centric subgroups in  $S$  which is fully  $\mathcal{F}$ -normalized. Let  $\mathcal{N} := N_{\mathcal{F}}(\sigma)$  denote the normalizer fusion system of  $\sigma$ . Then for every  $n \geq 0$ , we have*

$$H^t(\mathcal{O}^c(\mathcal{N}); H^n(-; \mathbb{F}_p)) = 0$$

for  $t > 0$ , and for  $t = 0$ , we have

$$H^0(\mathcal{O}^c(\mathcal{N}); H^n(-; \mathbb{F}_p)) = H^n(\mathcal{N}; \mathbb{F}_p) \cong H^n(\text{Aut}_{\mathcal{L}}(\sigma); \mathbb{F}_p).$$

*Proof.* Let  $\sigma = (P_0 < \dots < P_n)$ . The subgroup  $P_0$  is  $\mathcal{F}$ -centric and normal in  $\mathcal{N} = N_{\mathcal{F}}(\sigma)$ . Hence by [5, Prop 4.3], there is a finite group  $H$  with a Sylow  $p$ -subgroup  $T$  such that  $\mathcal{N} \cong \mathcal{F}_T(H)$ . By Theorem [11, Thm B], the subgroup decomposition for a  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$  is sharp if the fusion system  $\mathcal{F}$  is realized by a finite group. Hence we have

$$H^t(\mathcal{O}^c(\mathcal{N}); H^n(-; \mathbb{F}_p)) = 0$$

for every  $t > 0$ , and for  $t = 0$ , we have

$$H^0(\mathcal{O}^c(\mathcal{N}); H^n(-; \mathbb{F}_p)) \cong \lim_{\mathcal{O}^c(\mathcal{N})} H^n(-; \mathbb{F}_p) = H^n(\mathcal{N}; \mathbb{F}_p).$$

For the last isomorphism in the statement of the lemma, we claim that  $\mathcal{N} \cong \mathcal{F}_{N_S(\sigma)}(\text{Aut}_{\mathcal{L}}(\sigma))$ . By Lemma 10.13,  $\mathcal{N}_{\mathcal{L}}(\sigma)$  is the linking system associated to  $\mathcal{N} = \mathcal{N}_{\mathcal{F}}(\sigma)$ . Then by [5, Prop 4.3 (a), (b)],  $H \cong \text{Aut}_{N_{\mathcal{L}}(\sigma)}(P_0)$  and  $N_{\mathcal{L}}(\sigma) \cong \mathcal{L}_T^c(H)$ . This gives

$$H \cong \text{Aut}_{N_{\mathcal{L}}(\sigma)}(P_0) = \{\varphi \in \text{Aut}_{\mathcal{L}}(P_n) \mid \pi(\varphi)(P_i) \leq P_i \text{ for all } i\} \cong \text{Aut}_{\mathcal{L}}(\sigma).$$

Hence  $\mathcal{N} \cong \mathcal{F}_{N_S(\sigma)}(\text{Aut}_{\mathcal{L}}(\sigma))$ . This gives

$$H^n(\mathcal{N}; \mathbb{F}_p) \cong H^*(\text{Aut}_{\mathcal{L}}(\sigma); \mathbb{F}_p)$$

by the Cartan-Eilenberg theorem for finite groups. Hence the proof is complete.  $\square$

Now we prove our main theorem in this section.

**Theorem 10.15.** *Let  $\mathcal{F}$  be a saturated fusion system over  $S$ . For every  $n \geq 0$ , and for every  $i \geq 0$ , there is an isomorphism*

$$\lim_{\mathcal{O}^c(\mathcal{F})}^i H^n(-; \mathbb{F}_p) \cong \lim_{\overline{sd}(\mathcal{F}^c)}^i H^n(\text{Aut}_{\mathcal{L}}(-); \mathbb{F}_p).$$

*Proof.* By Theorem 10.8, for every  $\mathbb{Z}_{(p)}\mathcal{O}(\mathcal{F})$ -module  $M$ , there is a spectral sequence

$$E_2^{s,t} = \lim_{\overline{sd}(\mathcal{F}^c)}^s \mathcal{H}_{M, N_{\mathcal{F}}}^t \Rightarrow H^*(\mathcal{O}^c(\mathcal{F}); \text{Res}_{\mathcal{O}^c(\mathcal{F})}^{\mathcal{O}(\mathcal{F})} M)$$

where  $\mathcal{H}_{M, N_{\mathcal{F}}}^j$  denotes the functor defined in Lemma 10.7. Fix  $n \geq 0$  and take  $M = H^n(-; \mathbb{F}_p)$ . By Lemma 10.14, for every  $[\sigma] \in \overline{sd}(\mathcal{F}^c)$  we have

$$\mathcal{H}_{M, N_{\mathcal{F}}}^t([\sigma]) = H^t(\mathcal{O}^c(N_{\mathcal{F}}(\sigma)); H^n(-; \mathbb{F}_p)) = 0$$

for every  $t > 0$ , and for  $t = 0$ , we have

$$\mathcal{H}_{M, N_{\mathcal{F}}}^0([\sigma]) = H^0(\mathcal{O}^c(N_{\mathcal{F}}(\sigma)); H^n(-; \mathbb{F}_p)) \cong H^n(\text{Aut}_{\mathcal{L}}(\sigma); \mathbb{F}_p).$$

This gives  $E_2^{s,t} = 0$  for  $t > 0$ , and for  $t = 0$  we obtain

$$E_2^{s,0} \cong \lim_{\overline{sd}(\mathcal{F}^c)}^s H^n(\text{Aut}_{\mathcal{L}}(-); \mathbb{F}_p).$$

Hence we conclude that

$$\lim_{\mathcal{O}^c(\mathcal{F})}^i H^n(-; \mathbb{F}_p) \cong E_2^{i,0} \cong \lim_{\overline{sd}(\mathcal{F}^c)}^i H^n(\text{Aut}_{\mathcal{L}}(-); \mathbb{F}_p)$$

for every  $i \geq 0$ .  $\square$

As an immediate corollary of Theorem 10.15, we obtain the following theorem which was stated as Theorem 1.9 in the introduction.

**Theorem 10.16.** *For every  $p$ -local finite group  $(S, \mathcal{F}, \mathcal{L})$ , the subgroup decomposition is sharp if and only if the normalizer decomposition is sharp.*

*Proof.* This follows from Theorem 10.15 and Definitions 1.1 and 10.11.  $\square$

This reduces the problem of showing the sharpness of subgroup decomposition to a problem of showing the sharpness of the normalizer decomposition. Of course this could be an equivalently difficult problem to solve.



## 11. FUSION SYSTEMS REALIZED BY FINITE GROUPS

Let  $G$  be a discrete group and  $\mathcal{C}$  be a collection of subgroups of  $G$ . Let  $X_{\mathcal{C}}^{\beta} = E_{\mathcal{C}}G$  denote the Dwyer space for the subgroup decomposition as defined in Section 6.2. For every  $H \in \mathcal{C}$ , the fixed point subspace  $(E_{\mathcal{C}}G)^H$  is homotopy equivalent to the realization of the subposet  $\mathcal{C}_{\geq H}$  hence it is contractible (see [15]). We show below that if  $G$  is a finite group and  $\mathcal{C}$  is a collection of  $p$ -subgroups in  $G$  closed under taking  $p$ -overgroups, then for every  $P \in \mathcal{C}$ , the orbit space  $C_G(P) \backslash (E_{\mathcal{C}}G)^P$  is  $\mathbb{Z}_{(p)}$ -acyclic. We start with a lemma.

**Lemma 11.1.** *Let  $G$  be a finite group and  $X$  be a  $G$ -CW-complex such that for every  $p$ -subgroup  $P \leq G$ , the fixed point subspace  $X^P$  is contractible. Then the orbit space  $X/G$  is  $\mathbb{Z}_{(p)}$ -acyclic.*

*Proof.* Let  $S \leq G$  be a Sylow  $p$ -subgroup of  $G$ . By a transfer argument it is enough to show that  $X/S$  is  $\mathbb{Z}_{(p)}$ -acyclic (see [7, §III.2]). Consider the fixed point map  $f : X \rightarrow pt$ . Since for every  $P \leq S$ , the map  $X^P \rightarrow pt$  is a homotopy equivalence, we can conclude that  $f$  is an  $S$ -homotopy equivalence (see [4, Prop 2.5.8]). This gives that the orbit space  $X/S$  is contractible, hence it is  $\mathbb{Z}_{(p)}$ -acyclic.  $\square$

As a consequence of the above lemma we obtain the following:

**Proposition 11.2.** *Let  $G$  be a finite group and  $\mathcal{C}$  be a collection of  $p$ -subgroups in  $G$  closed under taking  $p$ -overgroups. Then for every  $P \in \mathcal{C}$ , the orbit space  $C_G(P) \backslash (E_{\mathcal{C}}G)^P$  is  $\mathbb{Z}_{(p)}$ -acyclic.*

*Proof.* Let  $X = E_{\mathcal{C}}G$  and  $P \in \mathcal{C}$ . The fixed point set  $X^P$  has a natural  $N_G(P)$ -action on it, hence it has an action of  $C_G(P)$  via restriction. For every  $p$ -subgroup  $Q \leq C_G(P)$ , we have  $(X^P)^Q = X^{PQ}$ . Since  $\mathcal{C}$  is closed under taking  $p$ -overgroups, we have  $PQ \in \mathcal{C}$ , hence  $X^{PQ}$  is contractible. This allows us to apply Lemma 11.1 and conclude that  $C_G(P) \backslash X^P$  is  $\mathbb{Z}_{(p)}$ -acyclic.  $\square$

An interesting consequence of this proposition is the following:

**Theorem 11.3.** *Let  $G$  be a finite group and  $\mathcal{C}$  be a collection of  $p$ -subgroups of  $G$  closed under taking  $p$ -overgroups. Then for every  $\mathbb{Z}_{(p)}\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ , there is an isomorphism*

$$H^*(\overline{\mathcal{F}}_{\mathcal{C}}(G); M) \cong H^*(\mathcal{O}_{\mathcal{C}}(G); \text{Res}_{pr} M)$$

induced by the projection functor  $pr : \mathcal{O}_{\mathcal{C}}(G) \rightarrow \overline{\mathcal{F}}_{\mathcal{C}}(G)$ .

*Proof.* Let  $X = E_{\mathcal{C}}G$  and  $R = \mathbb{Z}_{(p)}$ . The isotropy subgroups of  $X$  are in  $\mathcal{C}$  and for every  $P \in \mathcal{C}$ , the fixed point set  $X^P$  is contractible. This implies that the chain complex  $C_*(X^?; R)$  is a projective resolution of the constant functor  $\underline{R}$  as an  $R\mathcal{O}_{\mathcal{C}}(G)$ -module. By Proposition 11.2, the chain complex of  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules  $C_*(C_G(?) \backslash X^?; R)$  is also  $R$ -acyclic, hence it gives a projective resolution of  $\underline{R}$  as an  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module. Combining these we obtain

$$\begin{aligned} H^*(\overline{\mathcal{F}}_{\mathcal{C}}(G); M) &\cong H^*(\text{Hom}_{R\overline{\mathcal{F}}_{\mathcal{C}}(G)}(C_*(C_G(?) \backslash X^?; R), M)) \\ &\cong H^*(\text{Hom}_{R\mathcal{O}_{\mathcal{C}}(G)}(C_*(X^?; R), \text{Res}_{pr} M)) \\ &\cong H^*(\mathcal{O}_{\mathcal{C}}(G); \text{Res}_{pr} M). \end{aligned}$$

The isomorphism in the middle follows from Lemma 4.12.  $\square$

For a finite group  $G$ , let  $O_p(G)$  denote the (unique) maximal normal subgroup in  $G$ . A  $p$ -subgroup  $P \leq G$  is called a *principal  $p$ -radical subgroup* if it is  $p$ -centric and  $O_p(N_G(P)/PC_G(P))$  is trivial. The collection of principal  $p$ -radical subgroups is denoted by  $\mathcal{D}_p(G)$ . Now as a corollary of Theorem 11.3, a theorem of Grodal [13, Thm 1.2] gives the following.

**Corollary 11.4.** *Let  $G$  be a finite group, and  $\mathcal{C}$  and  $\mathcal{C}'$  be two collections of  $p$ -subgroups of  $G$  closed under taking  $p$ -overgroups satisfying  $\mathcal{C}' \cap \mathcal{D}_p(G) \subseteq \mathcal{C} \subseteq \mathcal{C}'$ . Then for any  $\mathbb{Z}_{(p)}\overline{\mathcal{F}}_{\mathcal{C}'}(G)$ -module  $M$ , there is an isomorphism*

$$H^*(\overline{\mathcal{F}}_{\mathcal{C}'}(G); M) \cong H^*(\overline{\mathcal{F}}_{\mathcal{C}}(G); \text{Res}_{\mathcal{C}} M).$$

*Proof.* By Theorem 11.3, it is enough to prove that

$$H^*(\mathcal{O}_{\mathcal{C}'}(G); \text{Res}_{pr} M) \cong H^*(\mathcal{O}_{\mathcal{C}}(G); \text{Res}_{pr} \text{Res}_{\mathcal{C}} M).$$

Since  $C_G(P)$  acts trivially on  $(\text{Res}_{pr} M)(P)$  for every  $P \in \mathcal{C}$ , the result follows from [13, Thm 1.2].  $\square$

As a consequence we obtain a generalization of the vanishing result due to Diaz and Park [11].

**Theorem 11.5.** *Let  $\mathcal{F} = \mathcal{F}_S(G)$  for a finite group  $G$  with a Sylow  $p$ -subgroup  $S$ . Let  $\mathcal{C}$  be a collection of subgroups of  $S$  closed under taking  $p$ -overgroups such that  $\mathcal{C}$  includes all  $\mathcal{F}$ -centric-radical subgroups in  $S$ . Then for every  $n \geq 0$  and for every  $i \geq 1$ ,*

$$\lim_{\mathcal{O}(\mathcal{F}_{\mathcal{C}})}^i H^n(-; \mathbb{F}_p) = 0.$$

*Proof.* By Lemma 8.9, a subgroup  $P \leq S$  is  $\mathcal{F}$ -centric if and only if it is  $p$ -centric in  $G$ . For  $P \leq S$ , we have

$$\text{Out}_{\mathcal{F}}(P) := \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P) \cong N_G(P)/PC_G(P).$$

This means  $P \leq S$  is  $\mathcal{F}$ -radical if and only if  $O_p(N_G(P)/PC_G(P)) = 1$ . Hence  $P \leq S$  is  $\mathcal{F}$ -centric-radical if and only if  $P$  is a principal  $p$ -radical subgroup.

Let  $\mathcal{C}'$  denote the collection of all  $p$ -subgroups in  $G$ . We denote the orbit category over the collection  $\mathcal{C}'$  by  $\mathcal{O}_p(G)$ . By the assumption on  $\mathcal{C}$  we have

$$\mathcal{C}' \cap \mathcal{D}_p(G) = \mathcal{D}_p(G) \subseteq \mathcal{C} \subseteq \mathcal{C}'.$$

Applying Theorem 11.3 and Corollary 11.4 to these collections, we obtain

$$\lim_{\mathcal{O}(\mathcal{F}_{\mathcal{C}})}^i H^n(-; \mathbb{F}_p) \cong \lim_{\mathcal{O}(\mathcal{F}_{\mathcal{C}'})}^i H^n(-; \mathbb{F}_p) \cong \lim_{\mathcal{O}_p(G)}^i H^n(-; \mathbb{F}_p).$$

Since  $H^n(-; \mathbb{F}_p)$  is a cohomological Mackey functor, by [18, Prop 5.14],

$$\lim_{\mathcal{O}_p(G)}^i H^n(-; \mathbb{F}_p) = 0.$$

Hence the proof is complete.  $\square$

It is interesting to ask whether Proposition 11.2 and Theorem 11.3 still hold when  $G$  is an infinite group. The following example illustrates that these results do not hold for infinite groups in general.

**Example 11.6.** Let  $\mathcal{F}$  be the fusion system of the symmetric group  $S_3$  at prime 3. If we apply the Leary-Stancu construction to  $\mathcal{F}$ , we find the infinite group

$$G = \langle b, a \mid b^3 = 1, aba^{-1} = b^2 \rangle \cong C_3 \rtimes \mathbb{Z}$$

realizing  $\mathcal{F}$  (see [16, Ex 4.3]). The subgroup  $\langle a^2 \rangle \cong \mathbb{Z}$  is central in  $G$  and  $G/\langle a^2 \rangle$  is isomorphic to the symmetric group  $S_3$ . Hence we have an extension of groups  $1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow S_3 \rightarrow 1$  whose extension class in  $\mathbb{F}_3$ -coefficients is zero. This gives that for every  $n \geq 0$ ,

$$H^n(G; \mathbb{F}_3) \cong H^n(S_3; \mathbb{F}_3) \oplus H^{n-1}(S_3; \mathbb{F}_3).$$

Note that

$$H^*(S_3; \mathbb{F}_3) \cong H^*(C_3; \mathbb{F}_3)^{C_2} \cong (\Lambda(y) \otimes \mathbb{F}_3[x])^{C_2}$$

where  $|y| = 1$ ,  $|x| = 2$ , and  $C_2$  acts on  $x$  and  $y$  via  $x \rightarrow -x$  and  $y \rightarrow -y$ . Hence  $H^i(S_3; \mathbb{F}_3) \cong \mathbb{F}_3$  for  $i = 0, 3 \pmod 4$ , and zero otherwise. We have

$$H^*(\mathcal{F}; \mathbb{F}_3) = \lim_{P \in \mathcal{O}(\mathcal{F})} H^*(P; \mathbb{F}_3) \cong H^*(S_3; \mathbb{F}_3) \neq H^*(G; \mathbb{F}_3),$$

since  $H^{n-1}(S_3; \mathbb{F}_3) \neq 0$  for  $n = 0, 1 \pmod 4$ . Hence this is an example of the situation where the cohomology of the infinite group  $G$  realizing the fusion system  $\mathcal{F}$  is not isomorphic to the cohomology of  $\mathcal{F}$  (see [16]).

Let  $\mathcal{C}$  be the collection of all finite 3-subgroups in  $G$ . We can take  $X = E_{\mathcal{C}}G$  to be the real line with  $G$ -action via the homomorphism  $G \rightarrow \mathbb{Z}$ . For any commutative ring  $R$ , the chain complex  $C_*(X^?; R)$  gives a sequence

$$0 \rightarrow R[G/C_3^?] \xrightarrow{1-a} R[G/C_3^?] \rightarrow \underline{R} \rightarrow 0$$

which is a projective resolution of  $\underline{R}$  as an  $R\mathcal{O}_{\mathcal{C}}(G)$ -module. For every  $R\mathcal{O}_{\mathcal{C}}(G)$ -module  $M$ , the higher limits  $\lim_{\mathcal{O}_{\mathcal{C}}(G)}^* M$  are the cohomology modules of the cochain complex

$$0 \rightarrow M(C_3) \xrightarrow{1-a} M(C_3) \rightarrow 0.$$

Hence  $\lim_{\mathcal{O}_{\mathcal{C}}(G)}^0 M \cong \ker(1-a)$  and  $\lim_{\mathcal{O}_{\mathcal{C}}(G)}^1 M \cong \operatorname{coker}(1-a)$ . If we take  $M = H^n(-; \mathbb{F}_p)$  with  $n = 0, 3 \pmod 4$ , then the action of the element  $a \in G$  on  $M$  is trivial, hence we have

$$\lim_{\mathcal{O}_{\mathcal{C}}(G)}^i H^n(-; \mathbb{F}_p) \cong \mathbb{F}_3 \neq 0$$

for  $i = 0, 1$  and  $n = 0, 3 \pmod 4$ .

The orbit space  $C_G(C_3) \backslash X^{C_3} \cong \mathbb{Z} \backslash \mathbb{R} \cong S^1$  is not  $\mathbb{Z}_{(3)}$ -acyclic. This shows that Proposition 11.2 does not hold for this group. If we take the orbit spaces of  $C_G(P)$ -actions on  $X^P$  for all  $P \in \mathcal{C}$ , then we obtain a chain complex

$$0 \rightarrow H_1(?) \rightarrow \operatorname{RMor}_{\mathcal{F}_{\mathcal{C}}(G)}(?, C_3) \rightarrow \operatorname{RMor}_{\mathcal{F}_{\mathcal{C}}(G)}(?, C_3) \rightarrow \underline{R} \rightarrow 0$$

of  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -modules. Note that  $H_1(P) \cong R$  for  $P = 1$  and  $P = C_3$ , and the restriction map  $H_1(C_3) \rightarrow H_1(1)$  is given by multiplication by 2. Thus if we take  $R = \mathbb{Z}_{(3)}$ , then  $H_1(?) \cong \underline{R}$ , and we can splice the sequences to get a projective resolution for  $\underline{R}$  as an  $R\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module. Using this projective resolution, we obtain that for every  $\mathbb{Z}_{(3)}\overline{\mathcal{F}}_{\mathcal{C}}(G)$ -module  $M$ ,

$$\lim_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^i M \cong H^i(C_2; M(C_3)) = 0$$

for every  $i \geq 1$ . In particular,

$$\lim_{\overline{\mathcal{F}}_{\mathcal{C}}(G)}^i H^n(-; \mathbb{F}_3) = 0$$

for all  $i \geq 1$ . Hence Conjecture 1.2 holds for the fusion system  $\mathcal{F} = \mathcal{F}_{C_3}(S_3)$ .

To summarize, for the infinite group  $G = C_3 \rtimes \mathbb{Z}$  realizing the fusion system  $\mathcal{F} = \mathcal{F}_{C_3}(S_3)$ , we conclude that

- (1) the statements of Proposition 11.2 and Theorem 11.3 do not hold for  $G$  when we take  $\mathcal{C}$  as the collection of all 3-subgroups in  $G$ ,
- (2) the mod-3 subgroup decomposition for  $BG$  over the collection of all finite 3-subgroups is not sharp for  $G$ , and
- (3) Conjecture 1.2 holds for the fusion system  $\mathcal{F}$ .

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