FINITE GROUPS OF RANK TWO WHICH DO NOT INVOLVE Qd(p)

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Abstract. Let $p > 3$ be a prime. We show that if G is a finite group with p-rank equal to 2, then G involves $\mathrm{Qd}(p)$ if and only if G p'-involves $\mathrm{Qd}(p)$. This allows us to use a version of Glauberman's ZJ-theorem to give a more direct construction of finite group actions on mod-p homotopy spheres. We give an example to illustrate that the above conclusion does not hold for $p \leq 3$.

1. Introduction

Throughout the paper all groups are finite, and p denotes a prime number. A p-group is said to be of *rank* k if the largest possible order of an elementary abelian subgroup of the group is p^k. We say that a group G is of *p-rank k* if a Sylow p-subgroup of G is of rank k. We denote the p-rank of G by $\mathrm{rk}_p(G)$.

The group $\mathrm{Qd}(p)$ is defined to be the semidirect product

$$\mathrm{Qd}(p) := (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL(2,p)$$

where the action of the group $SL(2,p)$ on $\mathbb{Z}/p \times \mathbb{Z}/p$ is the usual action of 2×2 matrices on a two-dimensional vector space. A group G is said to *involve $Qd(p)$* if there exist subgroups $K \lhd H \leq G$ such that $H/K \cong \mathrm{Qd}(p)$. We say G *p'-involves $Qd(p)$* if there exist $K \lhd H \leq G$ such that K has order coprime to p and $H/K \cong \mathrm{Qd}(p)$. If a group p'-involves $\mathrm{Qd}(p)$, then obviously it involves $\mathrm{Qd}(p)$, but the converse does not hold in general. We show that for finite groups with p-rank equal to 2 these two conditions are equivalent when $p > 3$.

Theorem 1.1. *Let G be a finite group with $rk_p(G) = 2$, where $p > 3$ is a prime. Then G involves $Qd(p)$ if and only if G p'-involves $Qd(p)$.*

Theorem 1.1 is proved in Section 2. The key step is the case where G involves $\mathrm{Qd}(p)$ with subgroups $K \lhd H \leq G$ where K is a p-group. This case is handled by using the classification of p-groups of rank 2. Theorem 1.1 is no longer true when $p = 3$. We illustrate this by constructing a group extension of $\mathrm{Qd}(3)$ by a cyclic group of order 3 in a way that the extension group does not $3'$-involve $\mathrm{Qd}(3)$ (see Example 2.2). For $p = 2$, there is a similar example (see Example 2.3).

The condition that G does not involve $\mathrm{Qd}(p)$ appears in Glauberman's ZJ-theorem [5, Thm B]. Let G be a finite group, and S be a Sylow p-subgroup of G. The Thompson subgroup $J(S)$ of S is defined to be the subgroup generated by all abelian p-subgroups of S with maximal order. The center of $J(S)$ is denoted by

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$ZJ(S)$ and its normalizer in G by $N_G(ZJ(S))$. Given a subgroup H of G containing S, we say H *controls* G*-fusion in* S if for every $P \leq S$ and $g \in G$ such that $gPg^{-1} \leq S$, there exist $h \in H$ and $c \in C_G(P)$ such that $g = hc$. Glauberman's ZJ-theorem [5, Thm B] states that if G is a finite group that does not involve Qd(p), and S is a Sylow p-subgroup of G, where p is odd, then $N_G(ZJ(S))$ controls G-fusion in S. There is a version of Glauberman's ZJ-theorem due to Stellmacher which also works for $p = 2$. We find Stellmacher's version of Glauberman's theorem more useful for our purpose even for the p odd case (see Theorem 3.1).

The condition that G p'-involves Qd(p) appears in the construction of finite group actions on products of spheres, particularly in the construction of mod-p spherical fibrations over the classifying space BG of a finite group G. One requires these spherical fibrations to have a p-effective Euler class (see Section 3 for definitions). Jackson [9] showed that if G is a finite group of rank 2 that does not p'-involve Qd(p) for any odd prime p, then there is a spherical fibration over BG with an effective Euler class. Jackson proves this theorem using results from two papers, one on the homotopy theory of maps between classifying spaces [8] and the other on representations that respect fusion [9].

The motivation for proving Theorem 1.1 comes from the question of whether or not the constructions of mod-p spherical fibrations are related to Glauberman's ZJ-theorem. Having shown that these two theorems have equivalent conditions at least for $p > 3$, we consider whether Glauberman's ZJ-theorem can be used directly to obtain a mod-p spherical fibration with a p-effective Euler class, providing an alternative to Jackson's construction. We prove this for a finite group of arbitrary rank and for any prime p, using Stellmacher's version of Glauberman's ZJ-theorem.

Theorem 1.2. *Let G be a finite group that does not involve Qd(p), where p is a prime. Then there is a mod-p spherical fibration over BG with a p-effective Euler class.*

Theorem 1.2 is proved in Section 3. In the proof we use Stellmacher's theorem (Theorem 3.1) to obtain that there is a homotopy equivalence $BG^{\wedge}_p \simeq BN^{\wedge}_p$ between p-completions of classifying spaces of G and $N = N_G(W(S))$. We then show that over BN the desired spherical fibration can be constructed using the Borel construction $EN \times_N S(V) \to BN$ for a linear sphere $S(V)$.

Note that Theorem 1.1 and Theorem 1.2 together give a different proof for Jackson's theorem for $p > 3$. We believe Theorem 1.2 is interesting in its own right for constructing actions on products of spheres for finite groups with arbitrary rank.

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2. The proof of Theorem 1.1

We first consider the case where G involves Qd(p) with $U \lhd G$ where U is a p-group.

Lemma 2.1. *Let $U \lhd G$. Suppose that U is a p-subgroup of G such that $G/U \cong Qd(p)$. If $rk_p(G) = 2$ and $p > 3$ then $U = 1$, that is, $G \cong Qd(p)$.*

Proof. Let $P \in Syl_p(G)$. Note that P is of rank 2 and P/U is isomorphic to a Sylow p-subgroup of Qd(p) by the hypothesis, and so P/U is isomorphic to an extra-special p-group of order p^3 and of exponent p. Note that P belongs to one of the 3-possible family of p-groups by the classification of p-groups of rank 2 (see [4, Thm A.1]).

(i) P is a metacyclic group. It follows that P/U is also a metacyclic group of order p^3. However this is not possible as the exponent of P/U is p. Thus, P cannot be metacyclic.

(ii) $P = C(p, r) := \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = 1, [a, b] = c^{p^{r-3}}, [a, c] = [b, c] = 1\rangle$ where $r \geq 3$. First suppose that $r > 3$. Since P/U has exponent p, we have $[a, b] = c^{p^{r-3}} \in U$. It follows that P/U is abelian as $[a, c] = [b, c] = 1$. This contradiction shows that $r = 3$. In this case, one can easily see that $|P| = p^3 = |P/U|$, and so $U = 1$.

(iii) $P = G(p, r; \epsilon) := \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = [b, c] = 1, [a, b^{-1}] = c^{\epsilon p^{r-3}}, [a, c] = b\rangle$ where $r \geq 4$ and ϵ is either 1 or a quadratic non-residue at modulo p. Since $[a, c] = b$ and $[b, c] = 1$, we have $[a, c^p] = [a, c]^p = b^p = 1$. Thus, $c^p \in Z(P)$. Moreover, $c^p \in U$ since P/U is of exponent p. It is easy to see that $P/\langle c^p\rangle$ has order p^3 by the given presentation. This forces that $U = \langle c^p\rangle$, and so $U \leq Z(P)$. Clearly, we have an embedding of $G/C_G(U)$ into the $Aut(U)$. Note that $G/C_G(U) \cong (G/U)/(C_G(U)/U)$ and $Aut(U)$ is cyclic, and so $G/C_G(U)$ is isomorphic to a cyclic quotient of Qd(p). As a consequence, $C_G(U) = G$, that is, $U \leq Z(G)$.

Let A be a normal subgroup of G such that $G/A \cong SL(2, p)$ and $A/U \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Note that A is a maximal subgroup in P. Since the Frattini subgroup of P is $\langle b, c^p\rangle$, either $A = \langle b, c\rangle$, or $A = \langle b, c^p, ac^i\rangle$ for some $i = 0, \ldots, p-1$. If $A = \langle b, c\rangle$, then A is abelian, hence $A \cong C_p \times C_{p^{r-2}}$. If $A = \langle b, c^p, a\rangle$, then A is isomorphic to $C(p, r-1)$. If $i \neq 0$, then $(ac^i)^p = c^{ip}$ which gives that $A = \langle b, ac^i\rangle$ is a metacyclic group $M(p, r-1)$ isomorphic to a split extension $C_{p^{r-2}} \rtimes C_p$ (see [4, Lemma A.8]).

Let S be a subgroup of G such that $A \cap S = U$ and $S/U \cong SL_2(p)$. Note that S acts transitively (by conjugation) on the set of maximal subgroups of A containing U. If $A \cong C_p \times C_{p^{r-2}}$ or $A \cong C_{p^{r-2}} \rtimes C_p$, then $U = \Phi(A)$ and the maximal subgroups of A are not isomorphic, so they cannot be permuted transitively by S. Hence these cases are not possible. Now assume that $A = \langle b, c^p, a\rangle \cong C(p, r-1)$. In this case A can be expressed as a central product $E * U$ where $E = \Omega_1(A)$ is the subgroup generated by elements of order p, in fact generated by a and b, and $U = \langle c^p\rangle$. Note that E is an extra-speical p-group of order p^3 and U is a cyclic group of order p^{r-3}. Since $\Omega_1(A)$ is characteristic in A, E is normal in G.

We already observed above that $U \leq Z(G)$, hence U is a central subgroup of S. Since $S/U \cong SL_2(p)$ is perfect and has a trivial Schur multiplier (see for example [7, Chapter 5]), we have $S \cong U \times SL_2(p)$. So G has some subgroup $L \cong SL_2(p)$. Consider the semidirect product $LE \leq G$. Let z denote a central element in E. Observe that $E/\langle z\rangle$ is a natural module for L, and L centralizes $\langle z\rangle$. According to [14, (3F)] there is some $x \in L$ satisfying $a^x = ab$ and $b^x = b$. Then the order of x is

p and $\langle x, b, z\rangle \cong (\mathbb{Z}/p)^3$ which contradicts with the rank assumption. We conclude that the case $P = G(p, r; \epsilon)$ with $r \geq 4$ cannot occur. □

Proof of Theorem 1.1. Note that if G p'-involves $\mathrm{Qd}(p)$ then clearly G involves $\mathrm{Qd}(p)$. Thus, we only show that if G involves $\mathrm{Qd}(p)$ then G p'-involves $\mathrm{Qd}(p)$. Let G be a minimal counterexample to the claim. Suppose that $X/Y \cong \mathrm{Qd}(p)$ for $Y \lhd X < G$. Clearly, the Sylow p-subgroups of X are of rank 2. Then we obtain that X p'-involves $\mathrm{Qd}(p)$ by the minimality of G, and hence so does G. This contradiction shows that if G has a section X/Y isomorphic to $\mathrm{Qd}(p)$, then $X = G$. In particular, there exists $H \lhd G$ such that $G/H = \mathrm{Qd}(p)$.

Now let $S \in Syl_p(H)$. We have $G = HN_G(S)$ by the Frattini argument. It follows that $\mathrm{Qd}(p) \cong G/H \cong N_G(S)/N_H(S)$, and so $N_G(S) = G$ by the argument in the first paragraph. In particular, S is normal in H, and hence H has a Hall p'-subgroup K by the Schur-Zassenhaus theorem. Moreover, any two Hall p'-subgroup of H are conjugate in H by the conjugacy part of Schur-Zassenhaus theorem. Therefore, we obtain $G = HN_G(K)$ by [11, Thm 5.51]. This yields that $N_G(K)/N_H(K) \cong \mathrm{Qd}(p)$ in a similar way, and so $K \lhd G$.

We claim that $K = 1$. Assume the contrary. Write $\overline{G} = G/K$. Note that $\overline{G}/\overline{H} \cong G/H = \mathrm{Qd}(p)$ and $\overline{P} \cong P$. It follows that $\overline{G}$ satisfies the hypotheses, and so $\overline{G}$ p'-involves $\mathrm{Qd}(p)$ due to the fact that $|\overline{G}| < |G|$. Then we have $\overline{X}/\overline{Y} \cong \mathrm{Qd}(p)$ where $\overline{Y}$ is a p'-subgroup. Hence, we observe that Y is a p'-group as K is a p'-group. However, this is not possible as $X/Y \cong \mathrm{Qd}(p)$. This contradiction shows that $K = 1$, that is, H is a p-subgroup of G. It follows that $G \cong \mathrm{Qd}(p)$ by Lemma 2.1. This contradiction completes the proof. □

Theorem 1.1 is not true when $p = 3$ as the following example illustrates.

Example 2.2. First note that $SL(2,3) \cong Q_8 \rtimes C_3$ where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and C_3 permutes the elements i, j, k cyclically. Let $A = \langle a\rangle$ be a cyclic group of order 9 and Q be the quaternion group of order 8. Constitute T as the semidirect product $Q \rtimes A$ such that $C_A(Q) = \langle a^3\rangle$ and $A/\langle a^3\rangle \cong C_3$ acts on Q as above. Then we have $T/\langle a^3\rangle \cong SL(2,3)$. Note that T is a non-split extension of $SL(2,3)$ by C_3, and there is no such extension for $p > 3$.

Let E be a nonabelian group of order 27 with exponent 3. We can take a presentation for E as follows:

$$E = \langle x, y, z \mid x^3 = y^3 = z^3 = 1, [x, y] = z, [x, z] = [y, z] = 1\rangle.$$

We have an embedding of $T/\langle a^3\rangle \cong SL(2,3)$ into $Aut(E)$ which takes an element

$$\begin{bmatrix} u & v \\ s & t \end{bmatrix} \in SL(2,3)$$

to the automorphism of E defined by $x \to x^u y^s$ and $y \to x^v y^t$. Let φ denote the composition $T \to T/\langle a^3\rangle \to Aut(E)$. Define G to be the semidirect product of E by T using the homomorphism φ. Set $\overline{G} = G/\langle z^{-1}a^3\rangle$. We shall show that $\overline{G}$ is the desired counterexample.

We first study G. Let $P = EA$. Then P is generated by a, x, y. We can assume a maps to the element

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL(2,3),$$

otherwise we can replace a with a conjugate of a in $SL(2,3)$. Hence we have the relations ${}^a x = x$ and ${}^a y = xy$ together with the relations coming from E. Using these relations it is easy to see that the Frattini subgroup $\Phi(P)$ of P is the subgroup generated by the elements z, x, a^3. The quotient group $P/\Phi(P)$ is isomorphic to $C_3 \times C_3$ and it is generated by $\overline{a}$ and $\overline{y}$. If M is a maximal subgroup of P, then there are 4 possibilities, namely M is generated by $\Phi(P)$ and a single element in P which can be taken as one of the elements a, ay, a^2y, or y. Since $[a, x] = 1$, and $[y, x] \neq 1$, in the last 3 cases the maximal subgroup M is not abelian. Note that in these cases $\overline{M} = M/\langle z^{-1}a^3 \rangle$ is also nonabelian. The only case where M is an abelian group is when $M = \langle z, x, a \rangle$. In this case M is isomorphic to $C_3 \times C_3 \times C_9$, and the quotient group $\overline{M} = M/\langle z^{-1}a^3 \rangle$ is isomorphic to $C_3 \times C_9$.

Now let X be an elementary abelian 3-subgroup of $\overline{P} = P/\langle z^{-1}a^3 \rangle$ with $\mathrm{rk}(X) = 3$. Then X is a maximal subgroup of $\overline{P}$, hence it is of the form $\overline{M}$ for some maximal subgroup M of P. As we have shown above there are only 4 possibilities for $\overline{M}$, and it is either nonabelian or it is isomorphic to $C_3 \times C_9$. This contradicts the fact that $X \cong C_3 \times C_3 \times C_3$. Hence $\overline{P}$ has rank equal to 2, therefore $rk_3(\overline{G}) = 2$. It is clear that $\overline{G}/\langle \overline{z} \rangle \cong \mathrm{Qd}(3)$, hence $\overline{G}$ involves Qd(3).

Suppose that $\overline{G}$ $3'$-involves Qd(3). Then as $|\overline{G}| = 3|\mathrm{Qd}(3)|$, it follows that $\overline{G}$ has a subgroup H isomorphic to Qd(3). Since Qd(3) does not have a normal subgroup isomorphic to C_3, we have $H \cap \langle \overline{z} \rangle = 1$, which contradicts the fact that $\overline{G}$ has 3-rank two. Hence $\overline{G}$ is a rank 2 group which involves Qd(3) but does not $3'$-involve Qd(3).

For $p = 2$ there is a similar example.

Example 2.3. The group Qd(2) is isomorphic to the Symmetric group S_4, and its Sylow 2-subgroup is isomorphic to the Dihedral group of order 8. The mod 2 cohomology of S_4 is given by

$$H^*(S_4; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, y_2, c_3]/(x_1 c_3),$$

where the restriction of the 2-dimensional class y_2 to elementary abelian subgroups V_1 amd V_2 are both nonzero (see [1, Ex 4.4]). This shows that if we take the central extension

$$1 \to \mathbb{Z}/2 \to G \to S_4 \to 1$$

with extension class $y_2 \in H^2(S_4; \mathbb{F}_2)$, then G cannot include an elementary abelian subgroup isomorphic to $(\mathbb{Z}/2)^3$, hence $\mathrm{rk}(G) = 2$. It is also easy to see that G does not include $\mathrm{Qd}(2) \cong S_4$ as a subgroup.

3. Spherical fibrations and Glauberman's ZJ-theorem

Let S be a p-group, and $\mathcal{A}$ denote the set of all abelian subgroups in S. Let $d = \max\{|A| \mid A \in \mathcal{A}\}$. The *Thompson subgroup* $J(S)$ of S is defined as the subgroup generated by all subgroups $A \in \mathcal{A}$ with $|A| = d$. We denote the center of $J(S)$ by

$ZJ(S)$. Glauberman's ZJ-theorem [5, Thm B] says that if p is an odd prime and G does not involve $\mathrm{Qd}(p)$, then $N_G(ZJ(S))$ controls G-fusion in S.

There are three different definitions of Thompson's subgroup. Similar to $J(S)$, one can define $J_e(S)$ to be the subgroup of S generated by all elementary abelian p-subgroups of S with maximal order. Although it is commonly believed that Glauberman's ZJ-theorem [5, Thm B] is valid also for $J_e(S)$, we were not able to find a published reference for this. Because of this, we will be using an analog of Glauberman's theorem due to Stellmacher.

Theorem 3.1 (Stellmacher's ZJ-theorem)**.** *Let p be a prime (possibly $p = 2$), and G be a finite group with Sylow p-subgroup S. If G does not involve $Qd(p)$, then there exists a characteristic subgroup $W(S)$ of S such that*

$$\Omega(Z(S)) \leq W(S) \leq \Omega(Z(J_e(S)))$$

and $N_G(W(S))$ controls G-fusion in S.

Theorem 3.1 is a natural consequence of Stellmacher's normal ZJ-theorem (see [10, Thm 9.4.4] and [12]). However it is not easy to see how this implication works since the conditions and conclusions of these two theorems are stated differently. We explain below how Theorem 3.1 follows from [10, Thm 9.4.4] for the convenience of the reader. First we need a definition.

Definition 3.2. [6, pg 22] A group G is called *p-stable* if it satisfies the following condition: Whenever P is a p-subgroup of G, $g \in N_G(P)$ and $[P, g, g] = 1$ then the coset $gC_G(P)$ lies in $O_p(N_G(P)/C_G(P))$.

Proof of Theorem 3.1. By part (a) and (c) of [10, Theorem 9.4.4], we see that W is a section conjugacy functor (see the definition in [6, pg 15]). First assume that p is odd. Since G does not involve $\mathrm{Qd}(p)$, by [6, Proposition 14.7], every section of G is p-stable (according to Definition 3.2). It is easy to see that then every section of G is also p-stable according to the definition in [10, pg 255]. By part (b) of [10, Thm 9.4.4], this gives that if H is a section of G such that $C_H(O_p(H)) \leq O_p(H)$ and S is a Sylow p-subgroup of H, then $W(S)$ is a normal subgroup of H. Now Theorem 3.1 follows from [6, Thm 6.6].

For $p = 2$, Stellmacher's normal ZJ-theorem still holds under the condition that G does not involve $S_4 \cong \mathrm{Qd}(2)$ (see the main theorem and the remark after that in [12]). Hence the $p = 2$ case of Theorem 3.1 follows from [12] and [6, Thm 6.6]. □

A spherical fibration over the classifying space BG is a fibration $\xi : E \to BG$ whose fibre is homotopy equivalent to a sphere. A mod-p spherical fibration over BG is a fibration whose fibre is homotopy equivalent to a p-completed sphere $(S^n)^\wedge_p$ for some n. The Euler class of a mod-p spherical fibration is a cohomology class $e(\xi)$ in $H^{n+1}(BG; \mathbb{F}_p)$ defined using the Thom isomorphism (see [13, Sec 6.6] for details). A cohomology class $u \in H^i(BG; \mathbb{F}_p)$ is called *p-effective* if for every elementary abelian p-subgroup $E \leq G$ with $\mathrm{rk}(E) = \mathrm{rk}_p(G)$, the image of the restriction map $\mathrm{Res}^G_E(u)$ is non-nilpotent in the cohomology ring $H^*(E; \mathbb{F}_p)$.

One of the important steps in constructing free actions of a finite group G on a product of spheres is to construct a mod-p spherical fibration over BG with a

p-effective Euler class. To construct a mod-p spherical fibration with a p-effective Euler class, it is enough to construct a spherical (or mod-p spherical) fibration over the p-completed classifying space $BG^\wedge_p$. This is because given a fibration over $BG^\wedge_p$, we can first apply p-completion and then take the pull-back along the completion map $BG \to BG^\wedge_p$ (see [13, Sec 6.4]).

Theorem 3.3 (Jackson [9])**.** *Let p be an odd prime and G be a finite group with $rk_p(G) = 2$. If G does not p'-involve $Qd(p)$, then there is a mod-p spherical fibration over BG with a p-effective Euler class.*

Jackson proves Theorem 3.3 in two steps. Let S be a Sylow p-subgroup of G and χ a character of S. We say χ *respects fusion in G* if for every $x \in S$ and $g \in G$ satisfying $gxg^{-1} \in S$, the equality $\chi(x) = \chi(gxg^{-1})$ holds. A character of a group H is *p-effective* if for every elementary abelian p-subgroup $E \leq H$ with $\mathrm{rk}(E) = \mathrm{rk}_p(H)$, the restriction of χ to E does not include the trivial character as a summand, i.e., if $[\chi|_E, 1_E] = 0$. Jackson [9, Thm 45] proves that if G is a finite group satisfying the conditions of Theorem 3.3, then there is a p-effective character of S that respects fusion in G.

Jackson [9, Thm 16] also shows that if S has a p-effective character respecting fusion in G, then there is a complex vector bundle $E \to BG^\wedge_p$ with a p-effective Euler class. The sphere bundle of this vector bundle gives the desired spherical fibration over $BG^\wedge_p$. The construction of the vector bundle with a p-effective Euler class uses a mod-p homotopy decomposition of BG, and they are constructed by proving the vanishing of certain higher limits as obstruction classes.

We show below that there is a more direct construction of a spherical fibration with a p-effective Euler class using a characteristic subgroup of a Sylow p-subgroup in G as in Theorem 3.1.

Proposition 3.4. *Let p be a prime, and G a finite group with a Sylow p-subgroup S. Suppose that there is a nontrivial characteristic subgroup Z of S such that $Z \leq \Omega(ZJ_e(S))$ and that the normalizer $N_G(Z)$ controls G-fusion in S. Then there is a mod-p spherical fibration over BG with a p-effective Euler class.*

Proof. Let $N = N_G(Z)$. By the Cartan-Eilenberg theorem, the restriction map $H^*(G; \mathbb{F}_p) \to H^*(N, \mathbb{F}_p)$ is an isomorphism, hence the p-completion of the classifying space $BG^\wedge_p$ is homotopy equivalent to $BN^\wedge_p$. Therefore it is enough to construct the desired mod-p spherical fibration over $BN^\wedge_p$. We show below that there is a representation V of N that is p-effective. Then, the Borel construction

$$S(V) \to EN \times_N S(V) \to BN$$

for the unit sphere $S(V)$ gives a spherical fibration with a p-effective Euler class. We can also assume N acts trivially on $H_i(S(V); \mathbb{F}_p)$ by replacing V with $V \oplus V$ if necessary. Taking the p-completion of this fibration gives the desired mod-p spherical fibration. Note that to conclude that fibers are homotopy equivalent to $S(V)^\wedge_p$, we use the Mod-R fiber lemma by Bousfield and Kan [2, Thm 5.1, Chp VI].

Now we show how to construct V. If E is an elementary abelian subgroup of S of maximum rank, then $E \leq J_e(S)$, hence E and Z commute. Since the rank of E

is the maximum possible rank in G, and EZ is an elementary abelian p-group, we must have $Z \leq E$.

Let $\rho : Z \to \mathbb{C}^\times$ be a nontrivial one-dimensional representation of Z. Consider the induced representation $V = \mathrm{Ind}_Z^N \rho$. By Frobenius reciprocity, we have

$$[\mathrm{Res}_Z^N \mathrm{Ind}_Z^N \rho, 1] = [\rho, \mathrm{Res}_Z^N \mathrm{Ind}_Z^N 1].$$

As Z is normal in N,

$$\mathrm{Res}_Z^N \mathrm{Ind}_Z^N 1 = [N : Z]1.$$

So, $C_V(Z) = 0$, and therefore $C_V(E) = 0$ whenever $Z \leq E$. This shows that V is a p-effective character of N, hence completes the proof. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a finite group that does not involve $\mathrm{Qd}(p)$. Then, by Theorem 3.1, there exists a subgroup $W(S)$ of S satisfying the conditions of Proposition 3.4. Applying Proposition 3.4 we obtain the desired fibration. □

Remark 3.5. The argument above also shows that when G is a finite group which does not involve $\mathrm{Qd}(p)$, then there is a p-effective representation χ of a Sylow p-subgroup S which respects fusion in G. Note that since in this case $N = N_G(W(S))$ controls G-fusion in S, it is enough to construct this representation so that it respects fusion in N. To achieve this we can take $\chi = \mathrm{Res}_S^N V$ where V is the representation of N constructed above in the proof of Theorem 3.4.

In [13, Thm 1.2] it is shown that there is no mod-p spherical fibration over $B\mathrm{Qd}(p)$ with p-effective Euler class when p is an odd prime. This result can be extended to give a converse to Jackson's theorem.

Theorem 3.6 (Jackson [9], Okay-Yalçın [13])**.** *Let p be an odd prime and G be a finite group with $rk_p(G) = 2$. Then G does not p'-involve $Qd(p)$ if and only if there is a mod-p spherical fibration over BG with effective Euler class.*

Proof. One direction of this is Theorem 3.3 stated above. For the other direction, let $K \lhd H \leq G$ such that $H/K \cong \mathrm{Qd}(p)$ and K has order coprime to p. If there is a mod-p spherical fibration ξ_G over BG, we can pull it back to a mod-p spherical fibration ξ_H over BH via the map $BH \to BG$ induced by inclusion. Since $\mathrm{rk}_p(H) = 2$, if ξ_G has a p-effective Euler class, then ξ_H also has a p-effective Euler class.

By taking the p-completion of ξ_H, we get a mod-p spherical fibration over $BH_p^\wedge$ that has p-effective Euler class. Since K is a subgroup with coprime order to p, the quotient map $H \to H/K$ induces a homotopy equivalence $BH_p^\wedge \cong B(H/K)_p^\wedge$. Hence we obtain a mod-$p$ spherical fibration over $B\mathrm{Qd}(p)_p^\wedge$ with a p-effective Euler class. But there is no such fibration by [13, Thm 1.2]. □

For $p = 2$, the condition that G does not involve $\mathrm{Qd}(2) \cong S_4$ is not a necessary condition for the existence of a mod-2 spherical fibration over BG. The group $G = S_4$ acts on a 2-sphere $X = S^2$ with rank one isotropy, hence the Borel construction for this action $X \to EG \times_G X \to BG$ gives a spherical fibration with a p-effective Euler class. It is interesting to ask whether the condition "G does not involve

Qd(p)" is necessary for the existence of a mod-p spherical fibration over BG with a p-effective Euler class when G is group of arbitrary rank and p is an odd prime.

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