DIMENSION FUNCTIONS FOR SPHERICAL FIBRATIONS

CİHAN OKAY AND ERGÜN YALÇIN

Abstract. Given a spherical fibration ξ over the classifying space BG of a finite group G we define a dimension function for the m-fold fiber join of ξ, where m is some large positive integer. We show that the dimension functions satisfy the Borel–Smith conditions when m is large enough. As an application we prove that there exists no spherical fibration over the classifying space of $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ with p-effective Euler class, generalizing a result of [23] about group actions on finite complexes homotopy equivalent to a sphere. We have been informed that this result will also appear in [1] as a corollary of a previously announced program on homotopy group actions due to Jesper Grodal.

1. Introduction

This paper is motivated by a conjecture about group actions on products of spheres due to Benson and Carlson [5]. The conjecture states that the maximal rank of an elementary abelian p-group contained in a finite group is at most k if and only if there exists a finite free G-CW-complex X homotopy equivalent to a product of spheres $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$. When $k = 1$ this conjecture is proved by Swan [21]. The next case $k = 2$ is proved by Adem and Smith [2] for finite groups that do not involve $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ for any odd prime p.

An important technique developed in [2] for constructing free actions starts with a spherical fibration over BG whose Euler class is p-effective and uses fiber joins to construct a free action on a finite complex homotopy equivalent to a product of two spheres. One source of such a spherical fibration is a finite G-CW-complex $X \simeq \mathbb{S}^n$ with rank one isotropy. Ünlü [23] proved that for $G = \mathrm{Qd}(p)$ there exists no such finite G-CW-complex. The main goal of this paper is to extend this result by showing that there exists no spherical fibration over BG with p-effective Euler class when G is $\mathrm{Qd}(p)$. We also show that $\mathrm{Qd}(p)$ cannot act freely on a finite complex homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$. However, the case of the Benson–Carlson conjecture where the dimensions of the spheres are different remains open.

Given a spherical fibration $\xi : E \to BG$ over BG with fibers $\mathbb{S}^n$, there is an infinite-dimensional free G-space X_ξ, defined as the pull-back of ξ along the universal fibration $EG \to BG$, such that the Borel construction $EG \times_G X_\xi \to BG$ is fiber homotopy equivalent to ξ. Two G-spaces X and Y are said to be hG-equivalent if there is a zig-zag sequence of G-maps between X and Y that are weak equivalences (non-equivariantly). The fibre homotopy classes of n-dimensional spherical fibrations over BG are in one-to-one correspondence with hG-equivalence classes of G-spaces that are homotopy equivalent to $\mathbb{S}^n$ (see §6.3 for details). We will use this correspondence throughout the paper without further explanations.

Let G be a p-group and X be a finite dimensional G-CW-complex. We write $H(-)$ for mod-p cohomology. Classical Smith theory says that if $H(X) \cong H(\mathbb{S}^n)$ for some n then the fixed point space X^G also has the mod-p cohomology of a sphere. A systematic way of

studying fixed point subspaces is to define dimension functions n_X by setting

$$H(X^K) \cong H(\mathbb{S}^{n_X(K)-1})$$

for a subgroup $K \leq G$. It is a fundamental fact that n_X satisfies certain properties called the Borel–Smith conditions. Smith theory fails for infinite-dimensional complexes. The problem is that, up to homotopy, every action can be made free by taking a product with the universal contractible free G-space EG. One way around this problem is to consider homotopy fixed points $X^{hK} = \mathrm{Map}(EK, X)^K$ instead of ordinary fixed points. An important algebraic tool for studying cohomology of homotopy fixed points is Lannes' T-functor and its variant the Fix functor. Here a technical point is that X needs to be replaced by its Bousfield–Kan p-completion $X_p^\wedge$, and the theory only works for elementary abelian p-groups. Then a theorem of Lannes' relates the mod-p cohomology of homotopy fixed points $(X_p^\wedge)^{hV}$ to the algebraically defined object $\mathrm{Fix}_V(H(X_{hV}))$, where V is an elementary abelian p-subgroup of G, and $X_{hV} = EV \times_V X$.

Lannes' theory can be applied under certain conditions. We show that these conditions can be satisfied by replacing a given G-space $X \simeq \mathbb{S}^n$ with the p-completion of its m-fold join

$$X[m] = (\underbrace{X * \cdots * X}_{m})_p^\wedge.$$

For large m we prove that classical Smith theory holds for infinite-dimensional complexes, where the role of fixed points is played by homotopy fixed points.

Theorem 1.1. [1] *Let P be a finite p-group and $X \simeq (\mathbb{S}^n)_p^\wedge$ be a P-space. Then there exists a positive integer m such that $(X[m])^{hP} \simeq (\mathbb{S}^r)_p^\wedge$ for some r.*

We are informed that this result is going to appear in [1] and it is part of a program on homotopy group actions due to Jesper Grodal which was announced previously. Using this result we can define dimension functions for mod-p spherical fibrations. A mod-p spherical fibration is a fibration whose fiber has the homotopy type of a p-completed sphere. Given a mod-p spherical fibration $\xi : E \to BG$ and a p-subgroup $Q \leq G$, we can restrict the fibration ξ to a fibration $\xi|_{BQ} : E_Q \to BQ$ by taking the pull-back along the inclusion map $BQ \to BG$. This corresponds to restricting the G-action on X_ξ to a Q-action via the inclusion map. We define the integer $n_{\xi[m]}(Q)$ via the weak equivalence

$$(X_\xi[m])^{hQ} \simeq (\mathbb{S}^{n_{\xi[m]}(Q)-1})_p^\wedge$$

which is a consequence of Theorem 1.1. It turns out that for m large enough, $n_{\xi[m]}$ satisfies the Borel–Smith conditions when regarded as an integer-valued function on the set of p-subgroups of G (see Theorem 4.6). The dimension function can be made independent of m by considering a rational-valued dimension function defined as follows:

$$\mathrm{Dim}_\xi(Q) = \frac{1}{m} n_{\xi[m]}(Q) \qquad (m \gg 0)$$

for every p-subgroup $Q \leq G$.

The Euler class of a fibration is said to be p-effective if its restriction to elementary abelian p-subgroups of maximal rank is non-nilpotent. This is a condition on the Euler class of a spherical fibration that is required to obtain a free action of a rank two group on a product

of two spheres using the Adem–Smith method. As an application of the dimension function that we defined, we obtain the following.

Theorem 1.2. [1] *Assume $p > 2$. There exists no mod-p spherical fibration $\xi : E \to BQd(p)$ with a p-effective Euler class.*

We are informed that this result is also going to appear in [1] and it was previously announced as a theorem by Jesper Grodal. As a consequence of Theorem 1.2, we obtain that the Adem–Smith method of constructing free actions on finite complexes homotopy equivalent to a product of spheres does not work for $\mathrm{Qd}(p)$.

Another method for constructing free actions on a product of two spheres $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ is given by Hambleton and Ünlü [13]. This method applies only to the equidimensional case ($n_1 = n_2$). The following theorem shows that this method cannot be used for $\mathrm{Qd}(p)$ either.

Theorem 1.3. *Let $G = Qd(p)$. Then for any $n \geq 0$, there is no finite free G-CW-complex X homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$.*

Therefore if Benson–Carlson conjecture holds then in the construction of a complex $X \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ with free $\mathrm{Qd}(p)$-action the possibilities are narrowed down to distinct dimensional spheres with a more exotic action.

The general theory of homotopy group actions has been considered by Adem and Grodal [1]. They have informed us that Theorems 1.1 and 1.2 will also appear in their paper under preparation. The idea of using dimension functions for studying mod-p spherical fibrations goes back to Grodal and Smith's unpublished earlier work, although an outline of their ideas can be found in the extended abstract [12]. Theorems 1.1 and 1.2 can also be thought of as corollaries of a program on homotopy group actions due to Grodal. We are grateful to Adem and Grodal for sharing their ideas with us on the subject, and we are looking forward to reading their complete account on the subject. Here we offer our proofs for Theorems 1.1 and 1.2 for completeness and to cover a gap in the existing literature. We should also mention that a result stated by Assadi [4, Corollary 4] also implies Theorems 1.1 and 1.2. Unfortunately, no proofs were provided for this result either.

The organization of the paper is as follows. In Section 2 we compute $\mathrm{Fix}(HE)$ for a fibration $\xi : E \to B\mathbb{Z}/p$ whose fiber has the cohomology of a sphere. Our main result Theorem 1.1 (Theorem 3.11) is proved in Section 3, where we study the space of sections of a mod-p spherical fibration over the classifying space of a p-group. The dimension function for an m-fold join of a mod-p spherical fibration is defined in Section 4. We prove the non-existence result Theorem 1.2 (Theorem 4.9) in this section. In Section 5 we prove Theorem 1.3 (Theorem 5.1). We collect some results about mapping spaces, homotopy fixed points, and fiber joins in an appendix in Section 6.

Acknowledgement: We thank Alejandro Adem, Matthew Gelvin, and Jesper Grodal for their comments on the first version of this paper. The second author is supported by Tübitak 1001 project (grant no: 116F194).

2. Spherical fibrations and Lannes' T-functor

In this section we compute $\mathrm{Fix}(HE)$ for a mod-p spherical fibration $E \to B\mathbb{Z}/p$. More generally, we work with fibrations where the mod-p cohomology of the fiber is isomorphic to the cohomology of a sphere. We modify the argument of [22, Chapter 3, §4] for the classical

case, which works for group actions on finite dimensional complexes, and use the connection between Lannes' T-functor and localization, established in [9].

2.1. **Lannes' T-functor:** Let $\mathbf{U}$ (resp. $\mathbf{K}$) denote the category of unstable modules (resp. unstable algebras) over the mod-p Steenrod algebra $\mathcal{A}_p$. Let V denote an elementary abelian p-group and HV the mod-p cohomology ring of V. The tensor product functor $HV \otimes - : \mathbf{U} \to \mathbf{U}$ has a left adjoint $T^V : \mathbf{U} \to \mathbf{U}$ which is called the Lannes T-functor. Let $\mathbf{U}(HV)$ denote the category of unstable modules M with an HV-module structure such that the multiplication map $HV \otimes M \to M$ satisfies the Cartan formula. Let $f : HW \to HV$ denote the map induced by a subgroup inclusion $V \subset W$. Its adjoint $\hat{f} : T^V HW \to \mathbb{F}_p$ is determined by a ring homomorphism $\hat{f}_0 : (T^V HW)^0 \to \mathbb{F}_p$ in degree zero. We define

$$T_f^V(M) = \mathbb{F}_p \otimes_{(T^V HW)^0} T^V M,$$

where the $(T^V HW)^0$-module structure on $\mathbb{F}_p$ is the one determined by $\hat{f}_0$. Let S_f denote the multiplicatively closed subset of HV generated by the images of the Bocksteins of one-dimensional classes in HW that map non-trivially under f. The following is the main theorem of [9].

Theorem 2.1. *(Dwyer–Wilkerson [9]) Let W be an elementary abelian p-group, V a subgroup of W, and $f : HW \to HV$ the map induced by subgroup inclusion. Suppose that M is an object of $\mathbf{U}(HV)$ that is finitely-generated as a module over HV. Then there is a natural isomorphism*

$$T_f^V(M) \cong Un\, S_f^{-1}(M).$$

For an object M in $\mathbf{U}(HV)$ the Fix functor is defined by

$$\mathrm{Fix}_V(M) = \mathbb{F}_p \otimes_{T^V HV} T^V M$$

where $\mathbb{F}_p$ is regarded as a $T^V HV$-module via the adjoint $\hat{\varphi} : T^V HV \to \mathbb{F}_p$ of the identity map $\varphi : HV \to HV$, see [15, §4.4.3] for details. We record the following properties.

Proposition 2.2. *Let M be an object in $\mathbf{U}(HV)$.*

(1) The natural map $T_\varphi^V M \to HV \otimes Fix_V M$ is an isomorphism in $\mathbf{U}(HV)$.
(2) If M is a finitely generated HV-module then the localization of the natural map $M \to T_\varphi^V M$ with respect to S_φ is an isomorphism.

Proof. The first result is proved in [15, Proposition 4.5]. For the second result, the natural map is obtained as follows: Let $M \to HV \otimes T^V M$ denote the adjoint of the identity map $T^V M \to T^V M$. Composing this map with the unique algebra map $HV \to \mathbb{F}_p$ gives a map $M \to T^V M$. The desired map is obtained by applying the natural projection $T^V M \to T_\varphi^V M$ to the second factor. The fact that the resulting map is an isomorphism can be found in [9, Lemma 4.3, §5]. □

2.2. **Spherical fibrations over $B\mathbb{Z}/p$.** We will study fibrations $\xi : E \to B\mathbb{Z}/p$ where the cohomology HF of the fiber F is isomorphic to $H(\mathbb{S}^n)$ for some $n \geq 0$, and show that $\mathrm{Fix}(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$. Note that mod-p spherical fibrations satisfy this condition.

We start with recalling the mod-p cohomology ring of $\mathbb{Z}/p$. If $p = 2$, the cohomology ring $H(\mathbb{Z}/2)$ is a polynomial algebra $\mathbb{F}_2[t]$, where t is of degree one. When $p > 2$ we have

$H(\mathbb{Z}/p) = \mathbb{F}_p[t] \otimes \Lambda[s]$, where s is of degree one and $t = \beta s$. Here β is the Bockstein map. The set S_φ corresponding to the identity map $\varphi : H\mathbb{Z}/p \to H\mathbb{Z}/p$ is generated by the Bockstein of the one dimensional class in each case. If $S = \{1, t, t^2, \cdots\}$, then localization with respect to S is the same as localization with respect to S_φ. For simplicity of notation, when $V = \mathbb{Z}/p$ we will write $T = T^V$, $T_\varphi = T^V_\varphi$, and $\mathrm{Fix} = \mathrm{Fix}_V$.

Lemma 2.3. *For an arbitrary fibration $\xi : E \to BV$ we have $(Fix_V HE)^0 = 0$ if and only if $\xi^* : HV \to HE$ does not split in $\mathbf{K}$.*

Proof. $(\mathrm{Fix}_V HE)^0$ is isomorphic to $(T^V_\varphi HE)^0$ which has an $\mathbb{F}_p$-basis Z_φ given by the set of $\mathbf{K}$-maps $\alpha : HE \to HV$ such that $\alpha\xi^*$ is the identity map on HV, see [19, Theorem 3.8.6]. □

Now we are ready to prove our main result in this section.

Theorem 2.4. *Let $\xi : E \to B\mathbb{Z}/p$ be a fibration such that $HF \cong H(\mathbb{S}^n)$. Then $Fix(HE) \cong H(\mathbb{S}^r)$ for some $-1 \le r \le n$.*

Proof. The Serre spectral sequence of the fibration ξ has E_2-page given by

$$H(\mathbb{Z}/p) \otimes HF \Rightarrow HE,$$

which is non-zero only in two rows since $HF \cong H(\mathbb{S}^n)$. The spectral sequence is determined by the differential $d_{n+1} : E_2^{0,n} \to E_2^{n+1,0}$ whose image lies in the polynomial part of $H\mathbb{Z}/p$ [3, page 137]. First we assume that d_{n+1} is non-zero, i.e. ξ^* does not split. In this case t is nilpotent in HE. Hence the localization vanishes: $S^{-1}HE = 0$. By Theorem 2.1 we have $T_\varphi HE = 0$ and the first part of Proposition 2.2 implies that $\mathrm{Fix}\, HE = 0$. Next assume that $d_{n+1} = 0$ so that ξ^* splits. Localizing the natural map $HE \to T_\varphi HE$ with respect to S gives a diagram

$$\begin{tikzcd} HE \arrow[r] \arrow[d] & T_\varphi HE \arrow[d, hook] \\ S^{-1}HE \arrow[r, "\cong"] & S^{-1}(T_\varphi HE) \end{tikzcd} \tag{2.2.1}$$

Here the fact that the bottom map is an isomorphism is a consequence of the second part of Proposition 2.2. The right vertical monomorphism maps onto the unstable part of $S^{-1}(T_\varphi HE)$ as a consequence of Theorem 2.1 and the commutativity of the diagram. Since $HE \cong H(\mathbb{Z}/p) \otimes HF$ is a free $H(\mathbb{Z}/p)$-module generated by an element of degree n, the localization map $HE \to S^{-1}HE$ is a monomorphism. Note that in the spectral sequence multiplication by t is an isomorphism. After localizing the spectral sequence the two rows extend to negative degrees. Therefore comparing the spectral sequences we see that the localization map is an isomorphism in degrees $i \ge n$. Hence from Diagram 2.2.1 it follows that the natural map $H^iE \to (T_\varphi HE)^i \cong (H(\mathbb{Z}/p) \otimes \mathrm{Fix}(HE))^i$ is an isomorphism for $i \ge n$. Therefore we have

$$\oplus_{k=0}^i \mathrm{Fix}(HE)^k \cong \oplus_{k=0}^i H^k(F) = \mathbb{F}_p \oplus \mathbb{F}_p \quad \text{for } i \ge n. \tag{2.2.2}$$

One of the factors corresponds to a generator of $\mathrm{Fix}(HE)^0$, which is non-zero by Lemma 2.3. The other one corresponds to a generator of $\mathrm{Fix}(HE)$ in degree $r \le n$. Therefore $\mathrm{Fix}(HE)$ is isomorphic to $H(\mathbb{S}^r)$ where $0 \le r \le n$. □

3. Cohomology of homotopy fixed points

In this section we study the homotopy fixed point space or equivalently the space of sections of a fibration by applying Lannes' results. For the relationship between the T-functor and mapping spaces, our main references are [15] and [19]. See also the appendix at Section §6 for preliminaries on homotopy fixed points and space of sections.

3.1. **Lannes' theorems.** Lannes' T-functor gives an approximation to the cohomology of the mapping space $\mathrm{Map}(BV, Y)$, where Y is an arbitrary space. The evaluation map $\mathrm{ev} : BV \times \mathrm{Map}(BV, Y) \to Y$ induces a map in cohomology $HY \to HV \otimes H(\mathrm{Map}(BV, Y))$ whose adjoint is $T^V HY \to H(\mathrm{Map}(BV, Y))$. The adjoint map factors through

$$\hat{\mathrm{ev}} : T^V HY \to H(\mathrm{Map}(BV, Y_p^\wedge)). \tag{3.1.1}$$

In degree zero it is induced by the isomorphism $[BV, Y_p^\wedge] \to \mathbf{K}(HY, HV)$ defined by applying the cohomology functor [19, pg. 187]. It is convenient to work on a connected component associated to the homotopy class of a map $\alpha : BV \to Y_p^\wedge$. Let $\alpha^* : HY \to HV$ denote the homomorphism induced in cohomology. The component of $T^V HY$ at α^* is defined by

$$T^V(HY, \alpha^*) = \mathbb{F}_p \otimes_{(T^V HY)^0} T^V HY$$

where the module structure on $\mathbb{F}_p$ is given by the adjoint $T^V HY \to \mathbb{F}_p$ of α^*. Then $\hat{\mathrm{ev}}$ in 3.1.1 is the product of the maps

$$\hat{\mathrm{ev}}_\alpha : T^V(HY, \alpha^*) \to H(\mathrm{Map}(BV, Y_p^\wedge)_\alpha)$$

where α runs over the homotopy classes of maps $BV \to Y_p^\wedge$.

We need the notion of freeness for the next theorem due to Lannes. Let G denote the left adjoint of the forgetful functor $\mathbf{K} \to \mathbf{E}$ where $\mathbf{E}$ denotes the category of graded vector spaces over $\mathbb{F}_p$. For an object $K \in \mathbf{K}$ let ΣK^1 denote the graded vector space isomorphic to K^1 in degree one and zero in other degrees. There is an inclusion of graded vector spaces $\Sigma K^1 \to K$. Applying G to this map and composing with the counit $GK \to K$ of the adjunction gives a canonical map

$$\chi : G(\Sigma K^1) \to K. \tag{3.1.2}$$

An unstable algebra K is said to be *free in degrees* ≤ 2 if χ is an isomorphism in degrees < 2 and a monomorphism in degree 2. For a more explicit definition, see [15, pg. 25].

Theorem 3.1. *(Lannes* [15, Theorem 3.2.4]*) Assume that HY and $T^V HY$ are of finite type. If $T^V HY$ is free in degrees ≤ 2 then*

$$\hat{ev}_\alpha : T^V(HY, \alpha^*) \to H(Map(BV, Y_p^\wedge)_\alpha)$$

is an isomorphism of unstable algebras.

In the next section we will apply this theorem to mod-p spherical fibrations.

3.2. **Mod-p spherical fibrations.** A fibration whose fiber is homotopy equivalent to a p-completed sphere is called a mod-p spherical fibration. A source for such fibrations is the fiberwise completion of spherical fibrations. Let $\xi : E \to BV$ be a mod-p spherical fibration with connected fiber. There is a map of fibrations

$$\begin{array}{ccc} E & \longrightarrow & E_p^\wedge \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_p^\wedge} \\ BV & \longrightarrow & BV_p^\wedge \end{array}$$

where the horizontal maps are weak equivalences. In particular E is p-complete. Moreover, the diagram is a homotopy pull-back diagram. This implies that there is a weak equivalence $\mathrm{Sec}(\xi) \to \mathrm{Sec}(\xi_p^\wedge)$ between the space of sections of these fibrations induced by the p-completion map. Therefore in applying Lannes' theory we can ignore the p-completions up to weak equivalence. We are interested in the cohomology of the space of sections $\mathrm{Sec}(\xi)$. As it is explained in Section 6.1 and 6.2, the space of sections $\mathrm{Sec}(\xi)$ is weakly equivalent to the homotopy fixed point space X_ξ^{hV} where X_ξ is the V-space defined as the pull-back of ξ along the universal bundle $EV \to BV$. The space of homotopy sections $\mathrm{hSec}(\xi)$ is isomorphic to $BV \times \mathrm{Sec}(\xi)$ as simplicial sets (see §6.1). We will use Lannes' theory to study the cohomology of space of homotopy sections. Consider the diagram

$$\begin{array}{ccc} [BV, E] & \xrightarrow{\cong} & \mathbf{K}(HE, HV) \\ \downarrow & & \downarrow \\ [BV, BV] & \xrightarrow{\cong} & \mathbf{K}(HV, HV) \end{array}$$

induced by ξ, where the horizontal maps are bijections. Let Z_φ denote the subset of maps in $\mathbf{K}(HE, HV)$ which splits ξ^* induced in cohomology. The subset of maps in $[BV, E]$ that split ξ up to homotopy is in one-to-one correspondence with Z_φ. Then we have

$$T_\varphi^V HE = \prod_{\alpha^* \in Z_\varphi} T^V(HE, \alpha^*).$$

and the product of the evaluation maps $\hat{\mathrm{ev}}_\alpha$ gives a map

$$T_\varphi^V HE \to H(\mathrm{hSec}(\xi)). \tag{3.2.1}$$

By Theorem 3.1 this map is an isomorphism of unstable algebras if $T_\varphi^V HE$ is free in degrees ≤ 2. Note that the conditions that HE and $T^V HE$ are of finite type are satisfied in this case because of the spectral sequence calculation and by Theorem 2.4.

Theorem 3.2. *Let $\xi : E \to BV$ be a mod-p spherical fibration and X_ξ denote the pull-back of ξ along the universal fibration $EV \to BV$. Assume that $Fix_V(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$. If $r \neq 1$ then*

$$H(X_\xi^{hV}) \cong H(\mathbb{S}^r).$$

Proof. By the results in Section 6.2, we have $X_\xi^{hV} \simeq \mathrm{Sec}(\xi)$. Since $\mathrm{hSec}(\xi) \simeq BV \times \mathrm{Sec}(\xi)$, it is enough to show that the map in 3.2.1 is an isomorphism. When $r = -1$, the result follows from Lemma 2.3. For the cases $r = 0$ and $r > 1$ we check the freeness condition. In 3.1.2 it turns out that the object $G(\Sigma K^1)$ is isomorphic to HW, where W is the $\mathbb{F}_p$-dual of K^1.

Therefore χ is an isomorphism in degrees ≤ 2 if and only if $H^2W \to K^2$ is a monomorphism. We claim that $T^V_\varphi HE$ is free in degrees ≤ 2 when $r = 0$ and $r > 1$. If $r = 0$ then the set Z_φ contains two maps α_0, α_1, and $T^V_\varphi HE = T^V(HE, \alpha_0) \oplus T^V(HE, \alpha_1)$, where each component is isomorphic to HV. Note that for $K = HV$ the map χ is an isomorphism. Hence the freeness condition is satisfied. For $r > 1$ we have $T^V_\varphi HE = T^V(HE, \alpha)$ for a unique homotopy class of a map α. The freeness property holds since $(T^V_\varphi HE)^1 = H^1V$ and $H^2V \subset (T^V_\varphi HE)^2$. □

Remark 3.3. Note that in general $T^V_\varphi HE$ is not free in degrees ≤ 2 when $r = 1$. Hence in this case we cannot apply Lannes' Theorem 3.1 to calculate the cohomology of the homotopy fixed point space. See also [15, Theorem 4.9.3].

We turn to another theorem of Lannes to study the homotopy type of the homotopy fixed point space.

Theorem 3.4. [15, Corollary 3.4.3] *Assume that HY, T^VHY, and $H(Map(BV, Y)_\alpha)$ are of finite type. Then $T^V(HY, \alpha^*) \to H(Map(BV, Y)_\alpha)$ is an isomorphism of unstable algebras if and only if $(Map(BV, Y)_\alpha)^\wedge_p \to Map(BV, Y^\wedge_p)_\alpha$ is a homotopy equivalence.*

Using this theorem, the Fix calculation of Theorem 2.4, and Theorem 3.2 we can determine the homotopy type of the space of sections of a mod-p spherical fibration over $B\mathbb{Z}/p$.

Theorem 3.5. *Let $\xi : E \to B\mathbb{Z}/p$ be a mod-p spherical fibration such that $Fix(HE) \cong H(\mathbb{S}^r)$ where $r \neq 1$. Let X_ξ denote the pull-back of ξ along $E\mathbb{Z}/p \to B\mathbb{Z}/p$. Then*

$$X_\xi^{h\mathbb{Z}/p} \simeq (\mathbb{S}^r)^\wedge_p$$

where $-1 \leq r \leq n$.

Proof. In Theorem 2.4 we showed that $\mathrm{Fix}(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$. If $r \neq 1$ then Theorem 3.2 implies that $H(\mathrm{Sec}(\xi)) \cong H(\mathbb{S}^r)$. The section space is the product of mapping spaces $\mathrm{Map}(BV, E)_\alpha$ where α is a representative of a homotopy class such that α^* lies in Z_φ. Applying Theorem 3.4 to each component we obtain a homotopy equivalence

$$\mathrm{Sec}(\xi)^\wedge_p \to \mathrm{Sec}(\xi)$$

after identifying $\mathrm{Sec}(\xi) \simeq \mathrm{Sec}(\xi^\wedge_p)$ up to homotopy. Therefore $\mathrm{Sec}(\xi) \simeq X_\xi^{h\mathbb{Z}/p}$ is a p-complete space that has the cohomology of a sphere. □

3.3. **Fiber joins and the Fix functor.** Next we look at the relationship between the Fix functor and fiber joins to be able to go around the problem in Theorem 3.5 when $r = 1$. Let $\xi_1 : E_1 \to B\mathbb{Z}/p$ and $\xi_2 : E_2 \to B\mathbb{Z}/p$ be two fibrations with fibers F_1 and F_2, respectively. We assume that $HF_i \cong H(\mathbb{S}^{n_i})$ and $\xi_1^* : H\mathbb{Z}/p \to HE_1$ splits.

Lemma 3.6. *The natural map*

$$Fix(HE_1) \otimes Fix(HE_2) \to Fix\, H(E_1 \times_{B\mathbb{Z}/p} E_2)$$

is an isomorphism.

Proof. Consider the pull-back diagram of fibrations

$$\begin{tikzcd}
 & F_1 \arrow[r, equal] \arrow[d] & F_1 \arrow[d] \\
F_2 \arrow[r] \arrow[d, equal] & E_1 \times_{B\mathbb{Z}/p} E_2 \arrow[r, "p_1"] \arrow[d, "p_2"] & E_1 \arrow[d, "\xi_1"] \\
F_2 \arrow[r] & E_2 \arrow[r, "\xi_2"] & B\mathbb{Z}/p
\end{tikzcd}$$

We have $HE_1 = H\mathbb{Z}/p \otimes HF_1$. The differential d_{n+1} in the spectral sequence of ξ_2 is either zero or t^α. By comparing the spectral sequences we see that $H(E_1 \times_{B\mathbb{Z}/p} E_2)$ is either isomorphic to $HE_1 \otimes HF_2$ or $HE_1/(t^\alpha)$. Consider the natural map

$$\theta : HE_1 \otimes_{H\mathbb{Z}/p} HE_2 \to H(E_1 \times_{B\mathbb{Z}/p} E_2)$$

of unstable modules induced by p_1 and p_2. If $d_{n+1} = 0$ then the tensor product is isomorphic to $HE_1 \otimes HF_2$. If the differential is given by t^α then it becomes $HE_1 \otimes_{H\mathbb{Z}/p} (H\mathbb{Z}/p)/(t^\alpha) \cong HE_1/(t^\alpha)$. Therefore in both cases θ is an isomorphism of unstable modules. In fact it is a morphism in $\mathbf{U}(H\mathbb{Z}/p)$. Then the result follows from the isomorphism $\mathrm{Fix}(M_1 \otimes_{H\mathbb{Z}/p} M_2) \cong \mathrm{Fix}(M_1) \otimes \mathrm{Fix}(M_2)$, which is valid for $\mathbf{U}(H\mathbb{Z}/p)$-modules [15, Theorem 4.6.2.1]. □

Proposition 3.7. *Assume that $Fix\, H(E_i) \cong H(\mathbb{S}^{r_i})$ for some r_i. Then there is an isomorphism*

$$Fix\,(H(E_1 *_{B\mathbb{Z}/p} E_2)) \cong H(\mathbb{S}^{r_1+r_2+1}).$$

Proof. Consider the homotopy push-out square

$$\begin{tikzcd}
E_1 \times_{B\mathbb{Z}/p} E_2 \arrow[r] \arrow[d] & E_1 \arrow[d] \\
E_2 \arrow[r] & E_1 *_{B\mathbb{Z}/p} E_2
\end{tikzcd} \tag{3.3.1}$$

We can assume that $E_i = (X_i)_{h\mathbb{Z}/p}$ for some $\mathbb{Z}/p$-space X_i. The assignment $X \mapsto \mathrm{Fix}\, H(X_{h\mathbb{Z}/p})$ defines an equivariant cohomology theory on the category of $\mathbb{Z}/p$-spaces [15, §4.7]. Then associated to the push out diagram there is a Mayer–Vietoris sequence, which breaks into short exact sequences

$$0 \to \mathrm{Fix}\, H(E_1)^q \oplus \mathrm{Fix}\, H(E_2)^q \to (\mathrm{Fix}\, H(E_1) \otimes \mathrm{Fix}\, H(E_2))^q \to \mathrm{Fix}\, H(E_1 *_{B\mathbb{Z}/p} E_2)^{q+1} \to 0,$$

where we used Lemma 3.6 for the middle term. In degree zero we need to consider the reduced cohomology groups. Compare this sequence to the Mayer–Vietoris sequence of the homotopy push-out:

$$\begin{tikzcd}
\mathbb{S}^{r_1} \times \mathbb{S}^{r_2} \arrow[r] \arrow[d] & \mathbb{S}^{r_1} \arrow[d] \\
\mathbb{S}^{r_2} \arrow[r] & \mathbb{S}^{r_1} * \mathbb{S}^{r_2}
\end{tikzcd}$$

The result follows from $\mathbb{S}^{r_1} * \mathbb{S}^{r_2} \cong \mathbb{S}^{r_1+r_2+1}$. □

Let $\xi : E \to B$ be a fibration with fiber F. The fiberwise p-completion (§6.4) of ξ is a fibration $\xi^\wedge_{p/B} : E^\wedge_{p/B} \to B$ whose fiber is $F^\wedge_p$. We use the following notation (§6.5)

$$X[m] = (\underbrace{X * \cdots * X}_{m})^\wedge_p$$

and for a fibration $\xi : E \to B$ we define

$$E_{/B}[m] = (\underbrace{E *_B \cdots *_B E}_{m})^\wedge_{p/B}$$

and denote the associated fibration by $\xi[m] : E_{/B}[m] \to B$.

Corollary 3.8. *Let $X = X_\xi$ and r be defined as in Theorem 3.5. Then for all $m > 2$ we have*

$$(X[m])^{h\mathbb{Z}/p} \simeq \mathbb{S}^r[m].$$

Proof. By Corollary 6.3 we have a fiber homotopy equivalence $(X[m])_{h\mathbb{Z}/p} \simeq E_{/B\mathbb{Z}/p}[m]$. Since $X[m]$ is homotopy equivalent to a p-completed sphere, using Proposition 3.7 we obtain

$$\operatorname{Fix} H(E_{/B\mathbb{Z}/p}[m]) \cong \operatorname{Fix} H(E *_{B\mathbb{Z}/p} \cdots *_{B\mathbb{Z}/p} E) \cong H(\mathbb{S}^r[m]\,).$$

Therefore we can apply Theorem 3.5 to $\xi[m]$. □

Remark 3.9. According to Theorem 3.5, as long as $r \neq 1$ the statement of Corollary 3.8 holds with $m = 1$. The problem we faced for $r = 1$ can be handled by taking joins. When $r = 1$ it suffices to take $m = 2$ to obtain

$$(X[2])^{h\mathbb{Z}/p} \simeq (\mathbb{S}^3)^\wedge_p.$$

3.4. **Finite p-groups.** Next we extend our results to p-groups. Let P be a finite p-group and $Z \cong \mathbb{Z}/p$ be a subgroup of P contained in the center. Consider a mod-p spherical fibration $\xi : E \to BP$, and let $X = X_\xi$. We are interested in computing the homotopy type of the homotopy fixed point space X^{hP}. By transitivity of homotopy fixed points (§6.2) we have

$$X^{hP} \simeq Y^{hP/Z}$$

where $Y = \operatorname{Map}(EP, X)^Z \simeq X^{hZ}$. By replacing X with $X[k]$ for some k and using Corollary 3.8, we can ensure that X^{hZ} is homotopy equivalent to a p-completed sphere. Now we can consider the P/Z-space Y. But to be able to determine the homotopy type of $Y^{hP/Z}$ we may need to replace Y with $Y[l]$. At this step we need the following lemma.

Lemma 3.10. *Let X_1 and X_2 be P-spaces such that for $i = 1, 2$, $X_i \simeq (\mathbb{S}^{n_i})^\wedge_p$, and $X_i^{hZ} \simeq (\mathbb{S}^{r_i})^\wedge_p$ for some $r_i > 0$. There is a weak equivalence*

$$\alpha : (X_1^{hZ} * X_2^{hZ})^\wedge_p \to ((X_1 * X_2)^\wedge_p)^{hZ},$$

which is induced by a map of P/Z-spaces when the homotopy fixed point spaces are interpreted as mapping spaces.

Proof. We describe the map α. There is a natural map of P/Z-spaces

$$\alpha' : \operatorname{Map}(EP, X_1)^Z * \operatorname{Map}(EP, X_2)^Z \to \operatorname{Map}(EP, X_1 * X_2)^Z$$

defined by $\alpha'[f,g,t](z) = [f(z), g(z), t]$. Note that $\mathrm{Map}(EP, X)^Z$ is weakly equivalent to X^{hZ} via the natural map $EZ \to EP$. Hence we obtain

$$X_1^{hZ} * X_2^{hZ} \to (X_1 * X_2)^{hZ}.$$

Composing this with the natural map $(X_1 * X_2)^{hZ} \to ((X_1 * X_2)_p^\wedge)^{hZ}$, we obtain a map

$$X_1^{hZ} * X_2^{hZ} \to ((X_1 * X_2)_p^\wedge)^{hZ}.$$

Completion of this map at p gives the map α. Note that since $r_1 + r_2 + 1 > 1$ we can apply Theorem 3.5 to conclude that $((X_1 * X_2)_p^\wedge)^{hZ}$ is p-complete. To see that α is a weak equivalence it suffices to show that the map induced in mod-p cohomology is an isomorphism. A Mayer–Vietoris type of argument shows that

$$H((X_1^{hZ} * X_2^{hZ})_p^\wedge) \cong H(\mathbb{S}^{r_1+r_2+1}).$$

On the other hand, Proposition 3.7 implies that $\mathrm{Fix}\, H(X_1 * X_2)_{hZ} \cong H(\mathbb{S}^{r_1+r_2+1})$ and by Theorem 3.5 $((X_1 * X_2)_p^\wedge)^{hZ}$ is weakly equivalent to the p-completion of $\mathbb{S}^{r_1+r_2+1}$. □

Now we are ready to prove the main theorem of this section.

Theorem 3.11. *Let P be a finite p-group and $\xi : E \to BP$ be a mod-p spherical fibration. Then there exists a positive integer m such that $X[m]^{hP} \simeq (\mathbb{S}^r)_p^\wedge$ where $X = X_\xi$.*

Proof. We will use the transitivity property (§6.2) of homotopy fixed points:

$$(\mathrm{Map}(EP, X)^Z)^{hP/Z} \simeq X^{hP}$$

where $\mathrm{Map}(EP, X)^Z \simeq X^{hZ}$, and we will do induction on the order of P. We can assume X^{hZ} is non-empty, otherwise the result holds trivially. Using Corollary 3.8 we can replace X by $X[k]$ to ensure that $Y = \mathrm{Map}(EP, X[k])^Z \simeq X[k]^{hZ}$ has the homotopy type of a p-completed simply connected sphere. We regard Y as a P/Z-space. Since the order of P/Z is less than the order of P there exists, by the induction hypothesis, some l such that the homotopy fixed points $(Y[l])^{hP/Z}$ is weakly equivalent to a p-completed sphere. We claim that the homotopy fixed points of $X[kl]$ under the action of P is a p-completed sphere. To see this let $A = X[k]$. There is a weak equivalence

$$(A^{hZ})[l] \to (A[l])^{hZ}$$

that is induced by a map of P/Z-spaces when regarded as a map between the associated mapping spaces. This can be shown by using Lemma 3.10 and doing induction on l. Now consider the natural P-map

$$X[kl] \to (X[k])[l],$$

which is also a weak equivalence. Using these two maps we obtain a zig-zag of weak equivalences

$$(A^{hZ})[l] \to (A[l])^{hZ} \leftarrow X[kl]^{hZ}$$

through P/Z-maps. Thus we have

$$X[kl]^{hP} \simeq (Y[l])^{hP/Z}$$

and the result follows by induction. □

4. Dimension functions

In this section we will define dimension functions for spherical fibrations and show that they satisfy the Borel–Smith conditions after taking fiber joins.

4.1. **Dimension functions.** Let $C(G)$ denote the ring of integer valued functions defined on the set of all subgroups of G that are constant on G-conjugacy classes. Let $H \leq G$ be a subgroup. Given a finite G-CW-complex Y with $H(Y) \cong H(\mathbb{S}^n)$, the dimension function n_Y of Y is defined by $H(Y^H) \cong H(\mathbb{S}^{n_Y(H)-1})$. Thus Y gives rise to an element n_Y of $C(G)$.

We extend this definition to our situation. Let $\xi : E \to BG$ be a mod-p spherical fibration. For a p-subgroup $Q \leq G$ let X_Q denote the pull-back $\xi|_{BQ}$ along the universal fibration $EQ \to BQ$. By Theorem 3.11 there exists an m such that $(X_Q[m])^{hQ}$ has the homotopy type of $(\mathbb{S}^{r_Q})^\wedge_p$ for some r_Q for all p-subgroups $Q \leq G$. Note that by standard properties of homotopy fixed points and Theorem 3.5, we have $r_{Q'} \leq r_Q$ if $Q \leq Q'$ and $r_Q = r_{Q'}$ if Q is conjugate to Q' in G. Let $\mathcal{S}_p(G)$ denote the set of all p-subgroups of G. We define a function

$$n_{\xi[m]} : \mathcal{S}_p(G) \to \mathbb{Z} \quad \text{by} \quad n_{\xi[m]}(Q) = r_Q + 1,$$

which is constant on G-conjugacy classes and call it the dimension function associated to the fibration $\xi[m]$. Given a spherical fibration we can consider the dimension function associated to its fiberwise p-completion.

Remark 4.1. To associate a dimension function to a mod-p spherical fibration independent of m we can define a rational valued dimension function

$$\mathrm{Dim}_\xi(Q) = \frac{1}{m} n_{\xi[m]}(Q)$$

for all p-subgroups $Q \leq G$, where m is a positive integer large enough so that Theorem 3.11 holds.

4.2. **Dimensions and subgroups.** We will prove an important relation satisfied by the dimension functions. Let V be an elementary abelian p-group of rank two. Let $\xi : E \to BV$ be a mod-p spherical fibration and $X = X_\xi$. Assume that n_ξ is defined. (This can be achieved by replacing ξ with $\xi[m]$.) This means that the homotopy fixed points of X under the action of a subgroup of V is a p-completed sphere.

By the Thom isomorphism theorem for $W \leq V$ the reduced cohomology ring of the Thom space $\mathrm{Th}(\xi_W)$ of the fibration $\xi_W : (X^{hW})_{hV} \to BV$ is a free HV-module on a single generator $t(\xi_W)$. There is a map $X^{hV} \to X^{hW}$ defined as the composition

$$\mathrm{Map}(EV, X)^V \to \mathrm{Map}(EV, X)^W \to \mathrm{Map}(EW, X)^W$$

of the natural inclusion of the fixed points, and the map induced by $EW \to EV$. This map induces a diagram

$$\begin{tikzcd}
(X^{hV})_{hV} \arrow[r] \arrow[d, "\xi_V"] & (X^{hW})_{hV} \arrow[d, "\xi_W"] & \\
BV \arrow[r, equal] \arrow[d] & BV \arrow[d] & \\
\mathrm{Th}(\xi_V) \arrow[r, "\alpha"] & \mathrm{Th}(\xi_W) \arrow[r, "\beta"] & \Sigma(X_{W,V})_{hV}
\end{tikzcd} \tag{4.2.1}$$

where $X_{W,V}$ is the cofiber of $X^{hV} \to X^{hW}$. The bottom row is a cofibration sequence. In the long exact sequence of cohomology groups

$$\cdots \to \tilde{H}^i(\mathrm{Th}(\xi_W)) \xrightarrow{\alpha^*} \tilde{H}^i(\mathrm{Th}(\xi_V)) \xrightarrow{\beta^*} H^i(\,(X_{W,V})_{hV}) \to \cdots \tag{4.2.2}$$

we have $\alpha^*(t(\xi_W)) = e_{W,V}t(\xi_V)$ for some element $e_{W,V}$ in HV. Let us set $S_W = S_f$ where $f : HV \to HW$ is the map induced by a subgroup inclusion $W \leq V$.

Lemma 4.2. *Let $W \subset V$ be a subspace of codimension one. Then there is an isomorphism $H(\,(X_{W,V})_{hV}) \cong HV/(e_{W,V})$.*

Proof. Let Y denote the space of homotopy fixed points $\mathrm{Map}(EV, X)^W \simeq X^{hW}$. Let $V = W \times L$ be a splitting. Then $Y^{hL} \simeq X^{hV}$, and $(X^{hV})_{hV} \to (X^{hW})_{hV}$ induces a map in cohomology $H((X^{hW})_{hV}) \to H((X^{hV})_{hV})$. Note that $(X^{hW})_{hV} \simeq BW \times Y_{hL}$ and $(X^{hV})_{hV} \simeq BW \times (Y^{hL})_{hL}$. Therefore we obtain

$$HW \otimes H(Y_{hL}) \to HW \otimes H((Y^{hL})_{hL}),$$

which becomes an isomorphism after localizing with respect to S_L. This is a consequence of the isomorphism $T^L_\varphi H(Y_{hL}) \cong H((Y^{hL})_{hL})$ implied by Theorem 3.2 and the second part of Proposition 2.2 applied to $M = H(Y_{hL})$. Therefore the localization of $H((X^{hW})_{hV}) \to H((X^{hV})_{hV})$ with respect to S_V is an isomorphism. From the map between the cofiber sequences in 4.2.1 we see that the map between the cohomology rings of Thom spaces becomes an isomorphism after localizing with respect to S_V. Thus there is a diagram

$$\begin{array}{ccc} \tilde{H}(\mathrm{Th}(\xi_W)) & \xrightarrow{\alpha^*} & \tilde{H}(\mathrm{Th}(\xi_V)) \\ \downarrow & & \downarrow \\ S_V^{-1}\tilde{H}(\mathrm{Th}(\xi_W)) & \xrightarrow{\cong} & S_V^{-1}\tilde{H}(\mathrm{Th}(\xi_V)), \end{array}$$

where the vertical arrows are injective since $\tilde{H}(\mathrm{Th}(\xi_W))$ and $\tilde{H}(\mathrm{Th}(\xi_V))$ are HV-free. Therefore in 4.2.2 we have that α^* is injective and β^* is surjective. Then $H(\,(X_{W,V})_{hV})$ is the quotient of the map

$$\tilde{H}(\mathrm{Th}(\xi_W)) \to \tilde{H}(\mathrm{Th}(\xi_V))$$

induced by $t(\xi_W) \mapsto e_{W,V}t(\xi_V)$. □

Let us simply denote $e_{W,V}$ by e_V when W is the trivial group. Let t_L denote the generator of the polynomial part of HL. We regard t_L as an element of HV via the isomorphism $HV \cong HL \otimes HW$.

Lemma 4.3. *Assume that e_V belongs to the polynomial part of HV. We have $n_X(W) > n_X(V)$ if and only if t_L divides e_V. Moreover*

$$e_V = u \prod_W e_{W,V}$$

where $u \in \mathbb{F}_p$ is a unit and W runs over subspaces of codimension one in V such that $n_X(W) > n_X(V)$.

Proof. Note that $n_X(W) > n_X(V)$ if and only if $e_{W,V} = at_L^\alpha$ for some $\alpha > 0$ and $a \in \mathbb{F}_p$ is non-zero. Let β be the maximal natural number such that t_L^β divides e_V. Consider the last

cofiber sequences in Diagram 4.2.1 for the pair of subgroup inclusions given by $W \subset V$ and $1 \subset V$. There is a map between the cofiber sequences

$$\begin{array}{ccccc} \mathrm{Th}(\xi_V) & \longrightarrow & \mathrm{Th}(\xi_W) & \longrightarrow & \Sigma\,(X_{W,V})_{hV} \\ \| & & \downarrow & & \downarrow \\ \mathrm{Th}(\xi_V) & \longrightarrow & \mathrm{Th}(\xi_1) & \longrightarrow & \Sigma\,(X_{1,V})_{hV}\ . \end{array}$$

We claim that the map $S_W^{-1}H(X_{hV}) \to S_W^{-1}H((X^{hW})_{hV})$ is an isomorphism. This follows from the transitivity of the Borel construction. The map $((X^{hW})_{hW})_{hL} \to (X_{hW})_{hL}$ between Borel constructions with respect to the action of L induces a map between the E_2-pages

$$H(L, S_W^{-1}H(X_{hW})) \to H(L, S_W^{-1}H((X^{hW})_{hW}))$$

of the associated spectral sequences. Then the claim follows from the isomorphism

$$S_W^{-1}H(X_{hW}) \cong S_W^{-1}H((\,X^{hW})_{hW}).$$

Now comparing the diagrams for $W \subset V$ and $1 \subset V$ we see that localization of $H((X_{1,V})_{hV}) \to H((X_{W,V})_{hV})$ with respect to S_W is an isomorphism. Thus by Lemma 4.2 there is an isomorphism $S_W^{-1}HV/(e_V) \cong S_W^{-1}HV/(e_{W,V})$. This forces $\alpha = \beta$.

□

Proposition 4.4. *Assume e_V belongs to the polynomial part of HV. If $W_1, \cdots, W_s$ denote the subspaces of codimension one in V then*

$$n_X(1) - n_X(V) = \sum_{i=1}^{s}(n_X(W_i) - n_X(V)).$$

Proof. By Lemma 4.3 a codimension one subspace W contributes to the sum on the right-hand side if and only if t_L divides e_V. Therefore the result follows from comparing the degrees. Note that $|e_V| = n_X(1) - n_X(V)$ and $|e_{W,V}| = n_X(W) - n_X(V)$. □

Remark 4.5. In view of Corollary 6.6, by choosing m large enough in $\xi[m]$ we can achieve the property that e_V belongs to the polynomial part of HV.

4.3. **Borel–Smith functions.** An element τ of $C(G)$ is called a Borel–Smith function ([22, Definition 5.1]) if it satisfies the following conditions

(i) if $H \lhd K \leq G$ such that $K/H \cong \mathbb{Z}/p \times \mathbb{Z}/p$ and H_i/H denotes the cyclic subgroups then

$$\tau(H) - \tau(K) = \sum_{i=0}^{p}(\tau(H_i) - \tau(K)),$$

(ii) if $H \lhd K \leq G$ such that $K/H \cong \mathbb{Z}/p$ where $p > 2$ then $\tau(H) - \tau(K)$ is even, and

(iii) if $H \lhd L \lhd K \leq G$ such that $L/H \cong \mathbb{Z}/2$ then $\tau(H) - \tau(L)$ is even if $K/H \cong \mathbb{Z}/4$; $\tau(H) - \tau(L)$ is divisible by 4 if K/H is a generalized quaternion group of order $\geq 2^3$.

Let $C_b(G)$ denote the additive subgroup of Borel–Smith functions in $C(G)$. We also say a function $\mathcal{S}_p(G) \to \mathbb{Z}$ constant on the G-conjugacy classes satisfies Borel–Smith conditions if it satisfies (i), (ii), and (iii) on p-subgroups.

Theorem 4.6. *There exists a positive integer m such that $n_{\xi[m]} : \mathcal{S}_p(G) \to \mathbb{Z}$ satisfies the Borel–Smith conditions.*

Proof. Monotonicity is a consequence of Theorem 3.5. Condition (i) is proved in Proposition 4.4, where the hypothesis that e_V belongs to the polynomial part of HV holds by choosing m large enough. This is a consequence of Corollary 6.6. Conditions (ii) and (iii) can be achieved by taking m large. □

When G is a finite nilpotent group, Borel–Smith functions can always be realized as dimension functions of virtual representations. Let $RO(G)$ denote the Grothendieck group of real representations. There is an additive morphism $\dim : RO(G) \to C(G)$ which sends a real representation ρ to the function which sends a subgroup H to the dimension of the fixed subspace ρ^H. Let $C_{\mathrm{rep}}(G)$ denote the image of this homomorphism. A key fact is that if G is a finite nilpotent group then $C_b(G) = C_{\mathrm{rep}}(G)$. In the case of p-groups we can use honest representations when the Borel–Smith function is also monotone.

Theorem 4.7. [22, Theorem 5.13] *If P is a p-group and τ is a monotone Borel–Smith function then there is a real representation ρ such that $\tau = \dim \rho$.*

Up to fiber joins the dimension function of a mod-p spherical fibration can be realized by the dimension function of a real representation.

Corollary 4.8. *Let P be a finite p-group. Given a mod-p spherical fibration $\xi : E \to BP$ there is a positive integer m and a real representation ρ such that $n_{\xi[m]} = \dim \rho$.*

4.4. **Proof of Theorem 1.2.** A cohomology class in $H(G)$ is called *p-effective* if its restriction to maximal elementary abelian p-subgroups is non-nilpotent. Let $\mathrm{Qd}(p)$ denote the semi-direct product $(\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ where the special linear group acts in the obvious way.

Corollary 4.9. *Assume $p > 2$. There exists no mod-p spherical fibration $\xi : E \to BQd(p)$ with a p-effective Euler class.*

Proof. The idea of the proof follows [23, Theorem 3.3] but here we use dimension functions for mod-p spherical fibrations. Assume that there is a fibration ξ with an effective Euler class. Consider the dimension function $n_{\xi[m]}$ for some large m. Its Euler class is still effective by Corollary 6.6. Let P be a Sylow p-subgroup of $G = \mathrm{Qd}(p)$. The center $Z(P)$ is a cyclic group of order p. By Theorem 4.6 we can choose m large enough so that the dimension function of the restricted bundle $\xi[m]|_{BP}$ belongs to $C_b(P)$. Then Theorem 4.7 implies that there is a real representation ρ which realizes this dimension function. Since the Euler class is p-effective, the dimensions of subspaces of ρ fixed under a cyclic subgroup $C \leq P_i$ have the property that $\dim \rho^C = 0$ if and only if $C = Z(P)$ (see [23, Lemma 3.4]). Therefore on cyclic p-subgroups of P the dimension function $n_{\xi[m]}$ is zero only at $Z(P)$ but in G the center $Z(P)$ is conjugate to a non-central cyclic p-subgroup C of P. Then $n_{\xi[m]}(Z(P)) = n_{\xi[m]}(C)$, but this gives a contradiction. □

5. $Qd(p)$-action on $\mathbb{S}^n \times \mathbb{S}^n$

In this section we prove the following theorem.

Theorem 5.1. *Let $G = Qd(p)$. Then for any $n \geq 0$, there is no finite free G-CW-complex X homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$.*

If $p = 2$, then $G = \mathrm{Qd}(2)$ is isomorphic to the symmetric group S_4 that includes A_4 as a subgroup. In this case the theorem follows from a result of Oliver [18] which says that the group A_4 does not act freely on a finite complex X homotopy equivalent to a product of two equal dimensional spheres. Also note that for $n = 0$, the statement holds for obvious reasons. Hence it is enough to prove the theorem when p is an odd prime and $n \geq 1$.

Lemma 5.2. *Let G be a finite group generated by elements of odd order. Let X be a finite free G-CW-complex homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$ for some $n \geq 1$. Then n is odd, and the induced G-action on $H^*(X;\mathbb{Z})$ is trivial.*

Proof. It is enough to prove this for the case $G = \mathbb{Z}/p^k$, where p is an odd prime. By induction we can assume that the action of the maximal subgroup $H \leq G$ on cohomology is trivial. Consider the $G/H \cong \mathbb{Z}/p$ action on $H^*(X;\mathbb{Z})$. The only indecomposable $\mathbb{Z}$-free $\mathbb{Z}[\mathbb{Z}/p]$-modules are either 1-dimensional, $(p-1)$-dimensional, or p-dimensional [14, Theorem 2.6]. This gives that for $p > 3$, the G action on $H^*(X;\mathbb{Z})$ is trivial. For $p = 3$, the only nontrivial module can occur in dimension n, and in this case G/H acts on $H^n(X;\mathbb{Z})$ with the action $x \to -y$ and $y \to x - y$, where x, y are generators of $H^n(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Note that the trace of this action is -1, so by the Lefschetz trace formula $L(f) = 2 - (-1) = 3$ when n is odd, and $L(f) = 0 + (-1) = -1$ when n is even. In either case $L(f) \neq 0$, hence G cannot admit a free action on X if the G/H action on homology is nontrivial. If the action is trivial, then again by Lefschetz trace formula, n must be odd. □

The group $SL_2(p)$ is generated by elements of order p. For example, we can take

$$A = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

as a set of generators. Since $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ is a semidirect product of $(\mathbb{Z}/p)^2$ with $SL_2(p)$, it is also generated by elements of order p. Hence we conclude the following.

Proposition 5.3. *Let $G = Qd(p)$ where p is an odd prime. Suppose that there exists a finite free G-CW-complex X homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$ for some $n \geq 1$. Then n is odd and G acts trivially on $H^*(X;\mathbb{Z})$.*

To complete the proof of Theorem 5.1, we use the Borel construction. Let $G = \mathrm{Qd}(p)$ with p odd, and let X be a finite free G-CW-complex homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$ for some integer $n \geq 1$. By Proposition 5.3, the induced action of G on X is trivial and $n = 2k - 1$ for some $k \geq 1$. Consider the Borel fibration $X_{hG} \to BG$ where $X_{hG} = EG \times_G X$. There is an associated spectral sequence with E_2-term

$$E_2^{i,j} = H^i(G; H^j(X))$$

that converges to $H^{i+j}(X_{hG})$.

Note that since G acts freely on X we have $X_{hG} \simeq X/G$. From this one obtains that the cohomology ring $H^*(X_{hG})$ is finite-dimensional in each degree and vanishes above some degree. The first nonzero differential in the above spectral sequence takes the generators of $H^{2k-1}(X) = \mathbb{F}_p \oplus \mathbb{F}_p$ to the cohomology classes μ_1, μ_2 in $H^{2k}(G)$. These classes are called the k-invariants of the G-space X.

For any subgroup $H \leq G$, we can restrict the above spectral sequence to the one for the action of H on X. This follows from the fact that the Borel construction is functorial. The

k-invariants of this restricted action will be $Res^G_H \mu_1$ and $Res^G_H \mu_2$, where

$$Res^G_H : H^*(G) \to H^*(H)$$

denotes the homomorphism induced by inclusion of H into G.

Let V denote the (unique) normal elementary abelian subgroup $\mathbb{Z}/p \times \mathbb{Z}/p$ in G. Let θ_1, θ_2 denote the k-invariants of the restricted V-action on X. Note that the classes θ_i are restrictions of cohomology classes μ_1, μ_2 in $H^{2k}(G)$. By the Cartan–Eilenberg stable element theorem, the classes θ_i lie in the invariant subring $H^*(V)^{SL_2(p)}$. This invariant ring is described in detail in [16, Proposition 1.4.1, Claim 1.4.2]. If we write

$$H^*(V) = \mathbb{F}_p[x, y] \otimes \wedge(u, v),$$

then $H^{ev}(V)^{SL_2(p)} = \mathbb{F}_p[x, y]^{SL_2(p)} \otimes \wedge(vu)$ where $\mathbb{F}_p[x, y]^{SL_2(p)} = \mathbb{F}_p[\xi, \zeta]$ is a polynomial subalgebra generated by

$$\xi = \sum_{i=0}^{p} (x^{p-i} y^i)^{p-1} \quad \text{and} \quad \zeta = xy^p - yx^p.$$

For $i = 1, 2$, let $\theta_i = f_i(\xi, \zeta) + uvg_i(\xi, \zeta)$ for some polynomials f_i, g_i. Since θ_1 and θ_2 are integral classes, i.e., they are in the image of the map $H^*(V, \mathbb{Z}) \to H^*(V, \mathbb{F}_p)$ induced by mod-p reduction, we have $g_i = 0$ for $i = 1, 2$. This can be seen easily by applying the Bockstein operator $\beta : H^*(V) \to H^{*+1}(V)$ to the classes θ_i. Since $\beta(u) = x$ and $\beta(v) = y$, we obtain

$$0 = \beta(\theta_i) = \beta(uv)g_i = (xv - uy)g_i.$$

This gives $g_i = 0$. Hence the k-invariants θ_1, θ_2 lie in the polynomial subalgebra $\mathbb{F}_p[\xi, \zeta]$.

Let $I = (\theta_1, \theta_2)$ be the ideal in $H^*(V)$ generated by θ_1 and θ_2. By a theorem of Carlsson [8, Corollary 7], the cohomology ring $H^*(X_{hV}) \cong H^*(X/V)$ is isomorphic to $H^*(V)/I$. In fact it is proved that θ_1, θ_2 is a homogenous system of parameters, hence it gives a regular sequence (in any order it is taken). This makes the spectral sequence collapse at the E_{n+2}-page, and gives that $H^*(X_{hV}) \cong H^*(V)/I$ which is finite dimensional as a vector space.

For every $j \geq 0$, let $P^j : H^r(V) \to H^{r+2(p-1)j}(V)$ denote the Steenrod operation. From the isomorphism above, we obtain that the ideal I is closed under Steenrod operations, meaning that for every $j \geq 0$, we have $P^j(u) \in I$ for every $u \in I$. A slightly stronger Steenrod closeness condition also holds:

Lemma 5.4. *Let M denote the $\mathbb{F}_p$-vector space generated by θ_1 and θ_2. For every $m \in M$, and for $j \geq 0$, there exists α_1, α_2 in the invariant subring $\mathbb{F}_p[\xi, \zeta]$ such that $P^j(m) = \sum_i \alpha_i \theta_i$.*

Proof. Note that it is enough to show that the elements α_i can be chosen from $H^*(V)^{SL_2(p)}$. Since both $P^j(m)$ and θ_i belongs to the polynomial part of the invariant subring, this will imply α_i also belongs to $\mathbb{F}_p[\xi, \zeta]$. For this we will show that α_i can be taken as $\alpha_i = \mathrm{res}^G_V \lambda_i$ for some $\lambda_i \in H^*(G)$. Hence it is enough to show that the ideal $J \subset H^*(G)$ generated by μ_1, μ_2 is closed under Steenrod operations, where μ_1, μ_2 are the k-invariants of the G-action on $\mathbb{S}^n \times \mathbb{S}^n$. For this, first note that μ_1, μ_2 is a homogeneous system of parameters because its restriction to elementary abelian subgroups is a homogeneous system of parameters. If Z is the center of a Sylow p-subgroup of G, then $\mathrm{res}^G_Z \mu_i \neq 0$ for some i. Assume that $\mathrm{res}^G_Z \mu_1 \neq 0$, then by [7, Theorem 12.3.3], μ_1 is a nonzero divisor. This gives that the spectral sequence $E_2^{i,j} = H^i(G; H^j(X))$ described above collapses at E_{n+2}-page, in particular, $d_{2n+1} = 0$.

Hence $H^*(G)/J$ is a module over Steenrod algebra, giving that J is closed under Steenrod operations. $\square$

If $M \subset H^{2k}(V)$ is a subspace which lies inside the subring $\mathbb{F}_p[\xi, \zeta]$ and has the Steenrod closeness property given in the above lemma, then we say the subspace M is closed under Steenrod operations in the invariant subring.

Proposition 5.5. *Let $M \subset H^{2k}(V)$ be a nonzero subspace that lies inside the invariant subring $\mathbb{F}_p[\xi, \zeta]$. If M is closed under the Steenrod operations in the invariant subring, then M lies in the principal ideal generated by ζ.*

Proof. We will use an argument similar to the argument given by Oliver [18] for A_4-actions on $\mathbb{S}^n \times \mathbb{S}^n$. A nonzero homogeneous polynomial $f \in \mathbb{F}_p[\xi, \zeta]$ can be written as a sum

$$f = (f_0(\xi) + f_1(\xi)\zeta + \cdots + f_t(\xi)\zeta^t)\zeta^s$$

for some homogeneous polynomials $f_i(\xi) = a_i\xi^{d_i}$, where f_0 and f_t are nonzero, and $s \geq 0$. Note that the degree of ξ is $2p(p-1)$, the degree of ζ is $2(p+1)$, hence for dimension reasons $f_i = 0$ when p does not divide i. So we can conclude that every homogenous element f in $\mathbb{F}_p[\xi, \zeta]$ is of the form

$$f = \Big(\sum_{i=0}^{t'} f_i(\xi)\zeta^{ip}\Big)\zeta^s$$

for some homogenous polynomials $f_i(\xi) = a_i\xi^{d_i}$ with $a_i \in \mathbb{F}_p$ and $f_0(\xi) \neq 0$. By direct calculation, it is easy to see that $P^1(\zeta) = 0$ and $P^1(\xi) = \zeta^{p-1}$. So if $P^1(f) = 0$, then we have

$$\Big(\sum_{i=0}^{t'} P^1(f_i)\zeta^{ip}\Big)\zeta^s = 0$$

which gives $0 = P^1(f_i) = a_i d_i \xi^{d_i - 1}\zeta^{p-1}$ for all i. Hence for every i, we have $d_i = pd_i'$ for some d_i'. From this we conclude that f is of the form $(f')^p\zeta^s$ for some f' when $P^1(f) = 0$. Furthermore, if the degree of f is divisible by p, then $s = ps'$ for some s', and hence in this case we have $f = (f'')^p$ for some homogeneous polynomial f'' in $\mathbb{F}_p[\xi, \zeta]$. We will use these observations in the calculations below.

Now let $M \subset H^{2k}(V)$ be a subspace as in the proposition. If p does not divide k, then each $m \in M$ can be written as $m = \zeta m'$ for some $m' \in \mathbb{F}_p[\xi, \zeta]$. In this case we have $M \subset (\zeta)$ as desired. So lets assume that p divides k. Then every $m \in M$ is of the form

$$m = \Big(\sum_{i=0}^{t'} f_i(\xi)\zeta^{ip}\Big)\zeta^{s'p}$$

for some homogeneous polynomials $f_i(\xi) = a_i\xi^{d_i}$ with $a_i \in \mathbb{F}_p$ and $f_0(\xi) \neq 0$. Since M is closed under Steenrod operations in the invariant subring, we have $P^1(m) = \sum_i \alpha_i m_i$ for some $m_i \in M$ and $\alpha_i \in \mathbb{F}_p[\xi, \zeta]$. However, there are no $2(p-1)$ dimensional classes in the invariant subring, hence we must have $P^1(m) = 0$ for every $m \in M$. By the above observation this implies that every $m \in M$ is equal to $(m')^p$ for some (unique) $m' \in H^{2k/p}(V)$. Let $M' \subset H^{2k/p}(V)$ be the subspace formed by elements m' such that $(m')^p \in M$. Note that M' also lies in the invariant subring $\mathbb{F}_p[\xi, \zeta]$.

Let $r \geq 1$ be the largest integer such that there exists a subspace $M_r \subset H^{2k/p^r}(V)$ such that M_r lies in the invariant subring $\mathbb{F}_p[\xi, \zeta]$ and that every $m \in M$ is of the form $m = u^{p^r}$ for some $u \in M_r$. For every $j \geq 0$, we have

$$P^j(f^{p^b}) = \begin{cases} (P^a(f))^{p^b} & \text{if } j = ap^b \\ 0 & \text{otherwise} \end{cases}$$

So we have $P^{p^r}(u^{p^r}) = (P^1(u))^{p^r}$. Since M is closed under Steenrod operations in the invariant subring, we also have $P^{p^r}(m) = \sum_i \alpha_i m_i$ for some α_i in $\mathbb{F}_p[\xi, \zeta]$. Hence we have

$$(P^1(u))^{p^r} = \sum_i \alpha_i u_i^{p^r}$$

for some α_i in the invariant subring. We claim that the coefficients α_i are of the form $\alpha_i = (\alpha_i')^{p^r}$ for some α_i'.

To see this first note that P^{p^r} increases the dimension by $2p^r(p-1)$, so the dimension of α_i is $2p^r(p-1)$. If we apply P^1 to the above equation we get $0 = \sum_i P^1(\alpha_i) u_i^{p^r}$. The elements $P^1(\alpha_i)$ has degree $2(p^r+1)(p-1)$ and the elements $u_i^{p^r}$ has at least degree $2(p+1)p^r$, so by dimension reasons we have $P^1(\alpha_i) = 0$ for all i. Hence we have $\alpha_i = (\alpha_i')^p$ for some α_i'. Plugging this into the above equation and taking p-th root we get

$$(P^1(u))^{p^{r-1}} = \sum_i \alpha_i' u_i^{p^{r-1}}.$$

Repeating the same argument r times, we get $\alpha_i = (\alpha_i')^{p^r}$ for some α_i' that lies in the invariant subring. This gives $P^1(u) = \sum_i \alpha_i' u_i$ for some α_i'. Since the degree of α_i' is $2(p-1)$ and there is no invariant element in that dimension, we get $\alpha_i' = 0$. This gives $P^1(u) = 0$. From this we conclude that for all u we have $u = (u')^p$ for some u'. This contradicts with the assumption that r is the largest integer such that every $m \in M$ is of the form $m = u^{p^r}$ for some $u \in \mathbb{F}_p[\xi, \zeta]$. □

We are now ready to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $I = (\theta_1, \theta_2)$ denote the ideal generated by the k-invariants of the V-action on $X \simeq \mathbb{S}^n \times \mathbb{S}^n$. By [8, Corollary 7], there is an isomorphism

$$H^*(X/V) \cong H^*(V)/I,$$

hence $H^*(V)/I$ is a finite-dimensional vector space. Moreover the ideal I is closed under the Steenrod operations and it is of the form $M \cdot H^*(V)$ where $M \subset \mathbb{F}_p[x, y]^{SL_2(p)}$ is the subspace generated by θ_1, θ_2. By Lemma 5.4, M is closed under Steenrod operations in the invariant subring. Hence by Proposition 5.5 the ideal I is included in the principal ideal (ζ). This gives a contradiction because the fact that $H^*(V)/I$ is finite-dimensional implies that I cannot be included in a principal ideal generated by $\zeta = xy(x+y)\cdots(x+(p-1)y)$ by standard results in commutative algebra (see [8, Proposition 3]). □

6. Appendix

By a space we mean either a topological space or a simplicial set. The relation between the two is given by the singular simplicial set functor and the geometric realization functor. The category of simplicial sets is a model category with Quillen model structure with the

usual weak equivalences and Kan fibrations. The geometric realization functor carries a Kan fibration to a Serre fibration.

6.1. **Mapping spaces.** Let X and Y be simplicial sets. The mapping space $\mathrm{Map}(X, Y)$ is the simplicial set whose set of n-simplices is given by $\mathbf{S}(\Delta[n] \times X, Y)$, and the simplicial structure is induced by the ordinal maps $\Delta[n] \to \Delta[m]$. $\mathrm{Map}_B(X, Y)_f$ will denote the connected component of a map $f : X \to Y$, in other words the space of maps which are homotopic to f. Let $\mathbf{S}_{/B}$ denote the over category whose objects are maps $X \to B$ and whose morphisms are commutative triangles over B. Let $\mathrm{Map}_B(X, Y)$ denote the mapping space for the over category which is defined to be the simplicial set with n-simplices given by the set $\mathbf{S}_{/B}(\Delta[n] \times X, Y)$ with the simplicial structure defined similarly.

Let $\xi : E \to B$ be a Kan fibration of simplicial sets. Then ξ induces a fibration of mapping spaces

$$\xi_* : \mathrm{Map}(B, E) \to \mathrm{Map}(B, B)$$

whose fiber over the identity map $\mathrm{id} : B \to B$ is the mapping space $\mathrm{Map}_B(B, E)$. We will also denote the fiber by $\mathrm{Sec}(\xi)$ and call it the space of sections of ξ. There is a pull-back diagram

$$\begin{tikzcd} \mathrm{hSec}(\xi) \arrow[r] \arrow[d] & \mathrm{Map}(B, E) \arrow[d, "\xi_*"] \\ \mathrm{Map}(B, B)_{\mathrm{id}} \arrow[r] & \mathrm{Map}(B, B) \end{tikzcd}$$

where $\mathrm{hSec}(\xi)$ is the space of maps $B \to E$ such that the triangle

$$\begin{tikzcd} B \arrow[r] \arrow[d, equal] & E \arrow[dl, "\xi"] \\ B & \end{tikzcd}$$

commutes up to homotopy.

In Lannes' theory we will consider fibrations over the classifying space of an elementary abelian p-group V. The classifying space BV is a simplicial group with product $BV \times BV \to BV$ induced by the product on V. The mapping space $\mathrm{Map}(BV, BV)$ is isomorphic to $\mathrm{Hom}(V, V) \times BV$ as a simplicial set [17, Proposition 25.2], and the adjoint $BV \to \mathrm{Map}(BV, BV)$ of the product map identifies BV with the identity component of the mapping space. For any X the mapping space $\mathrm{Map}(BV, X)$ has an induced action of BV. Also the simplicial monoid $\mathrm{Map}(BV, BV)$ acts by pre-composition on the mapping space. The actions are equivariant with respect to the isomorphism $BV \to \mathrm{Map}(BV, BV)_{\mathrm{id}}$ of simplicial abelian groups [17, Proposition 25.3]. Given a fibration $\xi : E \to BV$ in the diagram of fibrations

$$\begin{tikzcd} BV \times \mathrm{Sec}(\xi) \arrow[r] \arrow[d] & \mathrm{hSec}(\xi) \arrow[d] \\ BV \arrow[r] & \mathrm{Map}(BV, BV)_{\mathrm{id}} \end{tikzcd}$$

the horizontal arrows are isomorphisms.

6.2. **Homotopy fixed points.** Let G be a discrete group and X be a simplicial set with G-action, also called a G-space. The homotopy orbit space X_{hG} is the quotient $(EG \times X)/G$ under the diagonal action. The homotopy fixed point space X^{hG} is the simplicial subset $\mathrm{Map}(EG, X)^G$ of G-equivariant simplicial set maps in $\mathrm{Map}(EG, X)$. Let $f : X \to Y$ be a map of G-spaces which is also a weak equivalence. Then the induced maps $X_{hG} \to Y_{hG}$ and $X^{hG} \to Y^{hG}$ are weak equivalences.

Next we describe a transitivity property of homotopy fixed points [10, Lemma 10.5]. Let H be a normal subgroup of G. There is a natural action of G/H on the mapping space $\mathrm{Map}(EG, X)^H$, and there is a weak equivalence

$$X^{hG} \simeq (\mathrm{Map}(EG, X)^H)^{hG/H}$$

where $\mathrm{Map}(EG, X)^H \simeq X^{hH}$.

Let $\xi : E \to BG$ be a fibration with fiber F. Consider the pull-back diagram

$$\begin{tikzcd} X_\xi \arrow[r] \arrow[d] & E \arrow[d, "\xi"] \\ EG \arrow[r] & BG \end{tikzcd}$$

along the universal principal G-fibration. Since X_ξ is a free G-space, the natural map $h : (X_\xi)_{hG} \to (X_\xi)/G = E$ is a weak equivalence, in fact a trivial fibration. There is a map of fibrations

$$\begin{tikzcd} X_\xi \arrow[r] \arrow[d] & F \arrow[d] \\ (X_\xi)_{hG} \arrow[r, "h"] \arrow[d, "\tilde{\xi}"] & E \arrow[d, "\xi"] \\ BG \arrow[r, equal] & BG \end{tikzcd}$$

where $\tilde{\xi}$ is the composition $\xi \circ h$. We will usually switch from an arbitrary fibration to the natural projection $(X_\xi)_{hG} \to BG$. As a consequence we have the following identifications:

$$(X_\xi)^{hG} = \mathrm{Map}(EG, X_\xi)^G = \mathrm{Sec}(\tilde{\xi}) \simeq \mathrm{Sec}(\xi).$$

The equivalence $\mathrm{Sec}(\tilde{\xi}) \simeq \mathrm{Sec}(\xi)$ is a consequence of the fact that the map

$$\mathrm{Map}(BG, Y) \to \mathrm{Map}(BG, Y')$$

induced by a trivial fibration $Y \to Y'$ is also a trivial fibration [11]. Note that both spaces are fibrations over $\mathrm{Map}(BG, BG)$. Pulling back along the subspace $\mathrm{Map}(BG, BG)_{\mathrm{id}} \to \mathrm{Map}(BG, BG)$ induces the required weak equivalence between the spaces of sections.

6.3. hG**-equivalence.** Let $\xi_i : E_i \to BG$ for $i = 1, 2$ be two fibrations. A map $E_1 \to E_2$ of fibrations over BG is called a fiber homotopy equivalence if there is a homotopy inverse over BG. Let X_{ξ_i} denote the associated G-spaces. Then f induces a map $X_{\xi_1} \to X_{\xi_2}$ which is a G-homotopy equivalence. Conversely one can start with two G-spaces X_1 and X_2 and compare the fibrations associated to their Borel constructions. In this case a weaker notion of equivalence is enough. We say X_1 and X_2 are hG-equivalent if there is a zig-zag of G-maps that are also weak equivalences. Given hG-equivalent G-spaces X_1 and X_2, the products

$EG \times X_1$ and $EG \times X_2$ are G-equivalent. Therefore $(X_1)_{hG}$ is fiber homotopy equivalent to $(X_2)_{hG}$. This implies that there is a one-to-one correspondence between fiber homotopy classes of fibrations over BG and hG-equivalence classes of G-spaces.

6.4. **Completion at a prime.** Let $X_p^\wedge$ denote the Bousfield–Kan completion of X at a prime p as defined in [6]. It comes with a natural map $X \to X_p^\wedge$. A space is called p-complete if this map is a weak equivalence. For example, the classifying space BP of a p-group is p-complete. A map $f : X \to Y$ induces an isomorphism $\tilde{H}_*(f, \mathbb{F}_p)$ if and only if its p-completion $f_p^\wedge : X_p^\wedge \to Y_p^\wedge$ is a weak equivalence. Note that any weak equivalence between p-completed spaces is a homotopy equivalence since the p-completion of a space is a fibrant simplicial set, i.e. a Kan complex. Let $\xi : E \to B$ be a fibration with fiber F. The p-completion $\xi_p^\wedge : E_p^\wedge \to B_p^\wedge$ is still a fibration. The fiber lemma [6, Chapter II 5.1] implies that if $\pi_1 B$ is a p-group and F is connected then the fiber of $\xi_p^\wedge$ is the p-completion of F. There is also a relative version of the completion construction which applies to a fibration $\xi : E \to B$, called the fiberwise completion at a prime p. We will denote the fiberwise p-completion by $\xi_{p/B}^\wedge : E_{p/B}^\wedge \to B$. This is a fibration whose fiber is given by $F_p^\wedge$. If ξ is a fibration over BP then up to homotopy $\xi_p^\wedge$ can be identified with $\xi_{p/B}^\wedge$. More explicitly, there is a map of fibrations

$$\begin{array}{ccc} E_{p/BP}^\wedge & \longrightarrow & E_p^\wedge \\ \downarrow & & \downarrow \\ BP & \longrightarrow & BP_p^\wedge \end{array}$$

where the horizontal maps are weak equivalences.

Proposition 6.1. *Let $\xi : E \to BP$ be a fibration with connected fiber and let $X = (X_\xi)_p^\wedge$ denote the p-completion of the P-space associated to ξ. Then there is a fiber homotopy equivalence*

$$\begin{array}{ccc} X_{hP} & \xrightarrow{\sim} & E_p^\wedge \\ \downarrow & & \downarrow \\ BP & = & BP \end{array}$$

Proof. This result is proved for an elementary abelian p-group in [15, Proposition 4.3.1]. For a general p-group we proceed as follows. Let X_ξ denote the pull-back of ξ along $EP \to BP$. It fits into a fibration sequence $F \to X_\xi \to EP$. After completion the sequence $F_p^\wedge \to (X_\xi)_p^\wedge \to EP_p^\wedge$ is a fiber sequence since $EP_p^\wedge$ is still contractible. Then the diagram

$$\begin{array}{ccc} (X_\xi)_p^\wedge & \longrightarrow & EP_p^\wedge \\ \downarrow & & \downarrow \\ E_p^\wedge & \longrightarrow & BP_p^\wedge \end{array}$$

is a homotopy pull-back diagram since $E_p^\wedge \to BP_p^\wedge$ is a fibration with fiber $F_p^\wedge$ by the fiber lemma. This implies that $X = (X_\xi)_p^\wedge$ is a P-covering of $E_p^\wedge$. Thus X is a free P-space and the desired map is given by the projection $X_{hP} \to X/P$. $\square$

6.5. **Fiber joins.** Next we will discuss the fiber join construction. First we look at the behavior with respect to the Borel construction and p-completion, and then we study the Euler class of fiber joins. The fiber join of two fibrations $\xi_1 : E_1 \to B$ and $\xi_2 : E_2 \to B$ is defined to be the homotopy push-out of

$$\begin{array}{ccc} E_1 \times_B E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & E_1 *_B E_2 \end{array} \tag{6.5.1}$$

where $E_1 \times_B E_2$ is the pull-back of the maps ξ_1 and ξ_2 over B. When B is a point this definition specializes to the join construction, and we simply write $E_1 * E_2$.

Proposition 6.2. *Let X and Y be G-spaces. Then there is a fiber homotopy equivalence*

$$\begin{array}{ccc} X_{hG} *_{BG} Y_{hG} & \xrightarrow{\sim} & (X * Y)_{hG} \\ \downarrow & & \downarrow \\ BG & = & BG\ . \end{array}$$

Proof. The map is induced by the Borel construction of the natural maps

$$X \to X * Y \leftarrow Y$$

and restricts to the identity map between the fibers. □

Given two spaces X, Y the join $X_p^\wedge * Y_p^\wedge$ is in general not p-complete [20, pg. 107]. For example, for spheres the p-completion of $(\mathbb{S}^n)_p^\wedge * (\mathbb{S}^m)_p^\wedge$ is given by $(\mathbb{S}^n * \mathbb{S}^m)_p^\wedge$. We use the following notation

$$X[m] = (\underbrace{X * \cdots * X}_{m})_p^\wedge.$$

A fiberwise version of this definition is as follows: Given a fibration $\xi : E \to B$ we define

$$E_{/B}[m] = (\underbrace{E *_B \cdots *_B E}_{m})_{p/B}^\wedge$$

and denote the associated fibration by $\xi[m] : E_{/B}[m] \to B$.

Corollary 6.3. *Let P be a p-group. If ξ is the fibration $E \to BP$ associated to the Borel construction of a P-space X, then $\xi[m]$ is fiber homotopy equivalent to $(X[m])_{hP}$.*

Proof. Using Proposition 6.2 and induction we see that there is a fiber homotopy equivalence

$$*_{BP}^n E \to (*^n X)_{hP}$$

over BP. Let Y denote the pull-pack of the fiber join $*_{BP}^n E$ along $EP \to BP$. Then Y is G-equivalent to $*^n X$. Applying Proposition 6.1 to $*^n X$ we obtain a fiber homotopy equivalence

$$((*^n X)_p^\wedge)_{hP} \to (*_{BP}^n E)_p^\wedge.$$

□

Remark 6.4. It is useful to observe that the natural map $X[mn] \to (X[m])[n]$ is a homotopy equivalence. Similarly $\xi[mn]$ is fiber homotopy equivalent to $(\xi[m])[n]$.

6.6. **Euler class.** We will study the Euler class of fiber joins of mod-p spherical fibrations. Note that the fiber join construction does not result in a mod-p spherical fibration until we fiberwise complete it at p. But the resulting fibration has a fiber whose mod-p cohomology is the mod-p cohomology of a sphere. Therefore for our purposes we consider a larger class of fibrations. Let $\xi : E \to B$ be a fibration whose fiber F satisfies $HF \cong H(\mathbb{S}^d)$ for some d. The Thom space $\mathrm{Th}(\xi)$ of ξ is defined to be the cofiber of the map ξ. Consider the diagram of cofibrations

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "\xi"] \\ * \arrow[r] \arrow[d] & B \arrow[d] \\ \Sigma F \arrow[r, "\mu"] & \mathrm{Th}(\xi) \ . \end{tikzcd}$$

The reduced cohomology ring $\tilde{H}(\mathrm{Th}(\xi))$ is a free HB-module generated on a generator $t(\xi)$ of degree $d+1$ called the Thom class of ξ. We can take $t(\xi)$ to be the dual of the image of a chosen generator under the map $H_{d+1}(\Sigma F) \to H_{d+1}(\mathrm{Th}(\xi))$ induced by μ. The image of the Thom class under the natural map $H^{d+1}(\mathrm{Th}(\xi)) \to H^{d+1}(B)$ is called the Euler class $e(\xi)$. Alternatively, $e(\xi)$ is the image of the transgression of the generator of $H^d(F)$ in the Serre spectral sequence of ξ.

We will need an alternative description of fiber join construction. Given fibrations $\xi_i : E_i \to B$ with fiber F_i where $i = 1, 2$, let us define a quotient space

$$E_1 \hat{*} E_2 = E_1 \times E_2 \times [0,1]/\sim$$

by the relations $(e_1, e_2, 0) \sim (e_1, e_2', 0)$ if $\xi_2(e_2) = \xi_2(e_2')$ and $(e_1, e_2, 1) \sim (e_1', e_2, 1)$ if $\xi_1(e_1) = \xi_1(e_1')$. Projecting onto each factor induces a fibration $\xi_1 \hat{*} \xi_2 : E_1 \hat{*} E_2 \to B \times B$ with fiber $F_1 * F_2$. The pull-back along the diagonal inclusion $\Delta : B \to B \times B$ is exactly the fiber join construction

$$\begin{tikzcd} F_1 * F_2 \arrow[r, equal] \arrow[d] & F_1 * F_2 \arrow[d] \\ E_1 *_B E_2 \arrow[r] \arrow[d] & E_1 \hat{*} E_2 \arrow[d, "\xi_1 \hat{*} \xi_2"] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

In more detail, the diagram 6.5.1 maps to the pull-back and induces a weak equivalence between the spaces given in the two definitions.

Proposition 6.5. *Let ξ_1 and ξ_2 be mod-p spherical fibrations over B. In the commutative diagram*

$$\begin{tikzcd} \Sigma(F_1 * F_2) \arrow[r, "\sim"] \arrow[d] & \Sigma F_1 \wedge \Sigma F_2 \arrow[d, "\Sigma\mu_1 \wedge \Sigma\mu_2"] \\ Th(\xi_1 \hat{*} \xi_2) \arrow[r, "\sim"] & Th(\xi_1) \wedge Th(\xi_2) \end{tikzcd}$$

the horizontal maps are weak equivalences.

Proof. Let $D(\xi)$ denote the fiber join of the identity map $B \to B$ (regarded as a fibration) with a mod-p spherical fibration $\xi : E \to B$. The Thom space $\mathrm{Th}(\xi)$ can be described as the quotient $D(\xi)/B$. We have the following identifications

$$\mathrm{Th}(\xi_1) \wedge \mathrm{Th}(\xi_2) = \frac{\mathrm{Th}(\xi_1) \times \mathrm{Th}(\xi_2)}{\mathrm{Th}(\xi_1) \vee \mathrm{Th}(\xi_2)} = \frac{D(\xi_1) \times D(\xi_2)}{(E \times D(\xi_2)) \cap (D(\xi_1) \times E)} \simeq \frac{D(\xi_1 \hat{*} \xi_2)}{\xi_1 \hat{*} \xi_2} = T(\xi \hat{*} \xi),$$

which are compatible with the equivalence $\Sigma F_1 \wedge \Sigma F_2 \simeq \Sigma(F_1 * F_2)$. □

An almost immediate consequence of Proposition 6.5 is that the Euler class of the fiber join of two mod-p spherical fibrations is the cup product of Euler classes of the individual fibrations.

Corollary 6.6. *If ξ_1 and ξ_2 are mod-p spherical fibrations, then*

$$e(\xi_1 *_B \xi_2) = e(\xi_1)e(\xi_2).$$

Proof. Proposition 6.5 implies that the Thom class of $\xi_1 \hat{*} \xi_2$ is the cross product of the Thom classes of ξ_1 and ξ_2. Looking at the corresponding diagram of cohomology groups associated to the diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad\Delta\quad} & B \times B \\ \downarrow & & \downarrow \\ \mathrm{Th}(\xi_1 *_B \xi_2) & \longrightarrow & \mathrm{Th}(\xi_1) \wedge \mathrm{Th}(\xi_2), \end{array}$$

we see that the Euler class of $\xi_1 *_B \xi_2$ is the cup product of the Euler classes of ξ_1 and ξ_2. □

References

[1] A. Adem and J. Grodal. Homotopical group actions and periodicity. *In preparation.*

[2] A. Adem and J. H. Smith. Periodic complexes and group actions. *Ann. of Math. (2)*, 154(2):407–435, 2001.

[3] C. Allday and V. Puppe. *Cohomological methods in transformation groups*, volume 32 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[4] A. H. Assadi. Some local-global results in finite transformation groups. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):455–458, 1988.

[5] D. J. Benson and J. F. Carlson. Complexity and multiple complexes. *Math. Z.*, 195(2):221–238, 1987.

[6] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[7] J. F. Carlson, L. Townsley, L. Valero-Elizondo, and M. Zhang, *Cohomology Rings of Finite Groups,* Kluwer Academic Publishers (2003).

[8] G. Carlsson. On the rank of abelian groups acting freely on $(S^n)^k$. *Invent. Math.*, 69(3):393–400, 1982.

[9] W. G. Dwyer and C. W. Wilkerson. Smith theory and the functor T. *Comment. Math. Helv.*, 66(1):1–17, 1991.

[10] W. G. Dwyer and C. W. Wilkerson. Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math. (2)*, 139(2):395–442, 1994.

[11] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[12] J. Grodal and J. H. Smith. Algebraic models for finite G-spaces. *Oberwolfach Report*, 45:2690–2692, 2007.

[13] I. Hambleton and Ö. Ünlü. Examples of free actions on products of spheres. *Q. J. Math.*, 60(4):461–474, 2009.

[14] A. Heller and I. Reiner. Representations of cyclic groups in rings of integers. I. *Ann. of Math. (2)*, 76:73–92, 1962.

[15] J. Lannes. Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire. *Inst. Hautes Études Sci. Publ. Math.*, (75):135–244, 1992. With an appendix by Michel Zisman.

[16] J. H. Long. The cohomology rings of the special affine group of $\mathbb{F}_p^2$ and of $\mathrm{PSL}(3,p)$. *Thesis*, 2008.

[17] J. P. May. *Simplicial objects in algebraic topology.* Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[18] R. Oliver. Free compact group actions on products of spheres. In *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, volume 763 of *Lecture Notes in Math.*, pages 539–548. Springer, Berlin, 1979.

[19] L. Schwartz. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994.

[20] D. P. Sullivan. *Geometric topology: localization, periodicity and Galois symmetry*, volume 8 of *K-Monographs in Mathematics.* Springer, Dordrecht, 2005. The 1970 MIT notes, Edited and with a preface by Andrew Ranicki.

[21] R. G. Swan. Periodic resolutions for finite groups. *Ann. of Math. (2)*, 72:267–291, 1960.

[22] T. tom Dieck. *Transformation groups*, volume 8 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1987.

[23] Ö. Ünlü. *Constructions of free group actions on products of spheres.* PhD thesis, University of Wisconsin–Madison, 2004.

Department of Mathematics, University of Western Ontario, London ON N6A 5B7
E-mail address: cokay@uwo.ca

Department of Mathematics, Bilkent University, Ankara, 06800, Turkey
E-mail address: yalcine@fen.bilkent.edu.tr