

FREE ACTIONS ON PRODUCTS OF REAL PROJECTIVE SPACES

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ABSTRACT. We prove that if $G = (\mathbb{Z}/2)^r$ acts freely and cellularly on a finite-dimensional CW-complex X homotopy equivalent to $\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_k}$ with trivial action on the mod-2 cohomology, then $r \leq \mu(n_1) + \cdots + \mu(n_k)$ where for each integer $n \geq 1$, $\mu(n) = 0$ if n is even, $\mu(n) = 1$ if $n \equiv 1 \pmod{4}$, and $\mu(n) = 2$ if $n \equiv 3 \pmod{4}$. This proves a homotopy-theoretic version of a conjecture of Cusick.

1. INTRODUCTION

Classifying all finite groups that can act freely on a product of spheres $X = S^{n_1} \times \cdots \times S^{n_k}$ is one of the most important open problems in transformation group theory. There is a conjecture that states that if $G = (\mathbb{Z}/p)^r$ acts freely on a product of k spheres, then $r \leq k$. Special cases of this conjecture were proved by Carlsson [5], [6], Adem and Browder [1], Yalçın [18], Hanke [10], Okutan and Yalçın [14], and others.

The techniques developed by Carlsson in [5] and [6] for studying free actions on products of spheres were used later to study free actions on other spaces such as products of real projective spaces [7], products of Milnor manifolds [16], and products of Dold spaces [9]. In the 1980s, a sequence of papers was published by Cusick on free actions on products of even-dimensional spheres and on products of real projective spaces (see [7], [8]). In one of these papers Cusick stated the following conjecture:

Conjecture 1.1 (Cusick [7]). *Suppose that X is a finite CW-complex with the mod-2 cohomology of $\prod_{i=1}^k \mathbb{R}P^{n_i}$. If X admits a free action of $G = (\mathbb{Z}/2)^r$ in such a way that the induced action on mod-2 cohomology is trivial, then*

$$r \leq \mu(n_1) + \cdots + \mu(n_k),$$

where for each integer $n \geq 1$,

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

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Cusick [7] proved Conjecture 1.1 in the case where $n_i \not\equiv 3 \pmod{4}$ for all $i = 1, \dots, k$. Later Yalçın [18] proved the conjecture for the case where X is homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ where all n_i are odd, under the additional assumption that the induced action on integral cohomology is trivial (see Remark 6.3). The equidimensional case without the assumption that the action on the mod-2 cohomology is trivial was settled by Adem and Yalçın [2]. In this paper we prove the following:

Theorem 1.2. *Suppose that $G = (\mathbb{Z}/2)^r$ and X is a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ such that the G -action on the mod-2 cohomology of X is trivial. Then*

$$r \leq \mu(n_1) + \dots + \mu(n_k),$$

where for each integer $n \geq 1$,

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Note that the assumption that X is homotopy equivalent to a product of real projective spaces is stronger than the assumption in Cusick's conjecture. Since we use integral cohomology and integral Bockstein operator, an assumption stronger than the mod-2 cohomology isomorphism is necessary for our proof. On the other hand, the assumption that X is finite-dimensional is weaker than that X is a finite CW-complex.

There are examples of free actions that show that the upper bound given in Conjecture 1.1 is sharp. To construct such actions, first observe that the quaternion group Q_8 of order 8 acts freely on S^3 via its identification as the group of unit quaternions. By taking joins of this action, we obtain a Q_8 -action on S^{4m+3} for all $m \geq 0$ which induces a free $\mathbb{Z}/2 \times \mathbb{Z}/2$ -action on $\mathbb{R}P^{4m+3}$ for all $m \geq 0$. There is also a free $\mathbb{Z}/2$ -action on $\mathbb{R}P^{2m+1}$ for any $m \geq 0$, induced by the free $\mathbb{Z}/4$ -action on S^{2m+1} defined by $(z_0, \dots, z_m) \rightarrow (iz_0, \dots, iz_m)$. Taking products of these actions we obtain a free $G = (\mathbb{Z}/2)^{k+2l}$ action on $X = \prod_{i=1}^k \mathbb{R}P^{2m_i+1} \times \prod_{i=1}^l \mathbb{R}P^{4m_i+3}$ with trivial action on the integral cohomology (see Example 5.5 for details).

There are also examples of free actions on products of real projective spaces that induce a nontrivial action on integral cohomology and trivial action on mod-2 cohomology. To construct such actions, consider the D_8 -action on S^{2m+1} obtained by taking joins of the D_8 -action on the circle S^1 defined by the symmetries of a square. This action induces a $\mathbb{Z}/2 \times \mathbb{Z}/2$ -action on $\mathbb{R}P^{2m+1}$ with nontrivial action on the integral cohomology when m is even. Combining this action with another action obtained via inflation from a free $\mathbb{Z}/2$ -action on $\mathbb{R}P^{2m+1}$, and taking products of these actions, we can obtain a free $(\mathbb{Z}/2)^{k+l}$ -action on $\prod_{i=1}^k \mathbb{R}P^{2m_i+1} \times \prod_{i=1}^l \mathbb{R}P^{4m_i+1}$ for any $k \geq l \geq 1$ with nontrivial action on integral cohomology whose mod-2 reduction is trivial (see Examples 6.1 and 6.2).

We prove Theorem 1.2 using the methods from [7] and [18]. The new input comes from a closer look at the interaction between the Serre spectral sequences with $\mathbb{F}_2$ - and $\mathbb{Z}$ -coefficients. Recall that if X is a G -CW-complex, there is an associated

Borel fibration $X \rightarrow X_G \rightarrow BG$ where $X_G = (EG \times X)/G$. For a commutative ring R , the Serre spectral sequence for the Borel fibration with coefficients in R is of the form

$$E_2^{p,q} = H^p(BG; H^q(X; R)) \Rightarrow H^{p+q}(X_G; R).$$

The Serre spectral sequence has a multiplicative structure and if $R = \mathbb{F}_2$ and the induced G -action on $H^*(X; \mathbb{F}_2)$ is trivial, then there is an isomorphism of bigraded $\mathbb{F}_2$ -algebras

$$E_2^{*,*} \cong H^*(BG; \mathbb{F}_2) \otimes H^*(X; \mathbb{F}_2).$$

If $G \cong (\mathbb{Z}/2)^r$, then we have

$$H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_r]$$

where $x_i = \pi_i^*(x)$, with x the generator of $H^1(\mathbb{Z}/2; \mathbb{F}_2)$. Similarly, for $X \simeq \prod_{i=1}^k \mathbb{R}P^{n_i}$, we have

$$H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \dots, t_k] / (t_1^{n_1+1}, \dots, t_k^{n_k+1})$$

where $t_i = \pi_i^*(t)$, with t the generator of $H^1(\mathbb{R}P^{n_i}; \mathbb{F}_2)$.

Let $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$ be the differential on the E_2 -page, and $\alpha_1, \dots, \alpha_k \in H^2(G; \mathbb{F}_2)$ be the 2-dimensional cohomology classes such that $d_2(1 \otimes t_i) = \alpha_i \otimes 1$ for each i . We call these cohomology classes the k -invariants of the G -action on X , a terminology that goes back to the obstruction theory.

It is proved by Cusick [7] that for every i where n_i is even, $\alpha_i = 0$, and for every i where $n_i \equiv 1 \pmod{4}$, $Sq^1(\alpha_i)$ lies in the ideal generated by $\alpha_1, \dots, \alpha_k$ in $H^*(G; \mathbb{F}_2)$. Using these Cusick proved the conjecture under the assumption that for all i , $n_i \not\equiv 3 \pmod{4}$ (see [7, Main Theorem]).

Using a result by Carlsson [5], it can be shown that if the G -action on X is free, then the k -invariants $\alpha_1, \dots, \alpha_k$ have no nontrivial common zeros when they are considered as quadratic polynomials in $\mathbb{F}_2$ -coefficients (see Proposition 2.4). This gives the inequality also for the case where $n_i \not\equiv 1 \pmod{4}$ for all i (see [7, Main Theorem]).

To prove the inequality in Theorem 1.2 under the assumption that X is homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ with trivial action on mod-2 cohomology, we first consider the case where the induced G -action on the integral cohomology of X is trivial. Under the assumption that the $\mathbb{F}_2$ -vector space generated by the k -invariants $\alpha_1, \dots, \alpha_k$ has an irreducible basis (see Definition 5.1), we prove that for all the indices i of the basis elements, $n_i \equiv 3 \pmod{4}$ (see Lemma 5.4). Together with the induction argument used in the proof of Theorem 1.2, this proves the desired inequality in the case where the integral cohomology action is trivial.

In Section 6, we consider the free actions with nontrivial induced G -action on the integral cohomology of X . We first give an example of such action in Example 6.2, and then analyze the integral Serre spectral sequence in this case. We prove that under the assumption that the $\mathbb{F}_2$ -vector space generated by k -invariants has an irreducible basis, all indices i of the basis elements satisfy $n_i \equiv 3 \pmod{4}$ (see Proposition 6.6). We show in Section 7 that this fact proves the inequality in Theorem 1.2 by induction.

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2. THE SERRE SPECTRAL SEQUENCE FOR THE BOREL CONSTRUCTION

Let G be a finite group and X a G -CW-complex. The Borel construction $X_G = EG \times_G X$ fits into the Borel fibration

$$X \rightarrow X_G \rightarrow BG$$

(see [4, Proposition II.2.4]). For a commutative ring R , the associated Serre spectral sequence is of the form

$$E_2^{p,q} = H^p(BG; \mathcal{H}^q(X; R)) \Rightarrow H^{p+q}(X_G; R),$$

where the local system is determined by the induced G -action on $H^q(X; R)$ (see [13, Theorem 5.2]).

When the G -action on the cohomology of X is trivial, the local coefficient system is trivial and $E_2^{p,q}$ is isomorphic to $H^p(BG; H^q(X; R))$. In this case we say the Borel fibration is simple. If R is a field and the Borel fibration is simple, then there is an isomorphism of R -algebras

$$E_2^{p,q} = H^p(BG; H^q(X; R)) \cong H^p(BG; R) \otimes_R H^q(X; R).$$

The Serre spectral sequence has a product structure defined by the composition

$$\begin{aligned} H^p(BG; \mathcal{H}^q(X; R)) \otimes H^s(BG; \mathcal{H}^t(X; R)) &\rightarrow H^{p+s}(BG; \mathcal{H}^q(X; R) \otimes \mathcal{H}^t(X; R)) \\ &\rightarrow H^{p+s}(BG; \mathcal{H}^{q+t}(X; R)), \end{aligned}$$

where both maps are defined by the cup product. In the case where the Borel fibration is simple and R is a field, the product structure on $E_2^{*,*} \cong H^p(BG; R) \otimes_R H^q(X; R)$ is defined by $(x \otimes y)(u \otimes v) = xu \otimes yv$, where xu and yv are the products defined by the cup products in $H^*(BG; R)$ and $H^*(X; R)$ (see [13, Theorem 5.6]).

The Serre spectral sequence of the Borel fibration is useful for finding obstructions to the existence of free actions on finite-dimensional spaces because of the following well-known observation:

Lemma 2.1. *Let G be a finite group and X a finite-dimensional free G -CW-complex. If $E_r^{*,*}$ denotes the r -th page of the Serre spectral sequence for the Borel fibration $X_G \rightarrow BG$ with $\mathbb{F}_2$ -coefficients, then $E_\infty^{p,q} = 0$ for all p, q with $p+q > \dim X$.*

Proof. If the G -action on X is free, then $X_G = EG \times_G X$ is homotopy equivalent to the orbit space X/G (see [5, Proposition 1]). Since X is a finite-dimensional G -CW-complex, X/G is also finite-dimensional with dimension equal to the dimension of X . This gives $H^n(X_G; \mathbb{F}_2) = 0$ for all $n > \dim X$, hence $E_\infty^{p,q} = 0$ for all p, q with $p+q > \dim X$. $\square$

Let $G = (\mathbb{Z}/2)^r$ and X be a finite-dimensional G -CW-complex whose mod-2 cohomology is isomorphic to the cohomology of $\prod_{i=1}^k \mathbb{R}P^{n_i}$. Suppose that G acts freely on X in such a way that the induced G -action on $H^*(X; \mathbb{F}_2)$ is trivial. Then there is an isomorphism of bigraded $\mathbb{F}_2$ -algebras

$$E_2^{*,*} \cong H^*(BG; \mathbb{F}_2) \otimes H^*(X; \mathbb{F}_2),$$

where

$$H^*(BG; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_r]$$

and

$$H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \dots, t_k] / (t_1^{n_1+1}, \dots, t_k^{n_k+1})$$

with $|x_i| = 1$ for all $i = 1, \dots, r$, and $|t_i| = 1$ for all $i = 1, \dots, k$.

Definition 2.2. Let $\alpha_1, \dots, \alpha_k \in H^2(G; \mathbb{F}_2)$ be the cohomology classes such that $d_2(1 \otimes t_i) = \alpha_i \otimes 1$ for $i = 1, \dots, k$. These cohomology classes are called the *k-invariants* of the G -action on X .

When it is convenient, we identify the vertical line $E_2^{0,*} \cong \mathbb{F}_2 \otimes H^*(X; \mathbb{F}_2)$ with $H^*(X; \mathbb{F}_2)$ and the horizontal line $E_2^{*,0} \cong H^*(BG; \mathbb{F}_2) \otimes \mathbb{F}_2$ with $H^*(BG; \mathbb{F}_2)$ and write the elements in $E_2^{p,q}$ as products uv where $u \in H^p(BG; \mathbb{F}_2)$ and $v \in H^q(X; \mathbb{F}_2)$. With these identifications, we sometimes write $d_2(t_i) = \alpha_i$.

In [7], Cusick proves the following:

Lemma 2.3 ([7, Proposition B]). (a) For every i where n_i is even, $\alpha_i = 0$.
(b) For every i where $n_i \equiv 1 \pmod{4}$, the Steenrod power $Sq^1(\alpha_i)$ lies in the ideal generated by $\alpha_1, \dots, \alpha_k$ in $H^*(G; \mathbb{F}_2)$.

Proof. Since the proofs are short, we repeat them here for the convenience of the reader.

(a) Let $n_i = 2m_i$ for some integer $m_i \geq 0$. Then $t_i^{2m_i+1} = 0$. This gives

$$0 = d_2(t_i^{2m_i+1}) = d_2(t_i)t_i^{2m_i} = \alpha_i t_i^{2m_i}.$$

On the E_2 -page, multiplication by $t_i^{2m_i}$ is injective. Hence $\alpha_i = 0$.

(b) Let $n_i = 4k_i + 1$ for some $k_i \geq 0$, and let $u_i := t_i^2$. Since $d_2(u_i) = d_2(t_i^2) = 0$, u_i survives to the E_3 -page. The differential $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$ is a transgression (see Definition 3.1), so it commutes with Steenrod operations (see [13, Theorem 6.8 and Corollary 6.9]). This gives

$$d_3(u_i) = d_3(Sq^1(t_i)) = Sq^1(d_2(t_i)) = Sq^1(\alpha_i).$$

Since $u_i^{2k_i+1} = 0$, we have

$$0 = d_3(u_i^{2k_i+1}) = d_3(u_i)u_i^{2k_i} = Sq^1(\alpha_i)u_i^{2k_i}$$

in $E_3^{3,4k_i}$. Hence there is an element $\theta \in E_2^{1,4k_i+1}$ such that $d_2(\theta) = Sq^1(\alpha_i)u_i^{2k_i}$. From the product structure of the E_2 -page, we see that $d_2(\theta)$ must be equal to $d_2(f)t_i^{4k_i}$ for some $f \in E_2^{1,1}$. Since multiplication with $u_i^{2k_i}$ is injective on the E_2 -page, we obtain $Sq^1(\alpha_i) = d_2(f)$. If $f = \sum_j l_j t_j$ where $l_j \in H^1(G; \mathbb{F}_2)$, then $Sq^1(\alpha_i) = d_2(f) = \sum_j l_j \alpha_j$. Hence $Sq^1(\alpha_i)$ lies in the ideal generated by $\alpha_1, \dots, \alpha_k$. $\square$

We also have the following observation for k-invariants.

Proposition 2.4. *Let $G = (\mathbb{Z}/2)^r$ and X be a finite-dimensional G -CW-complex whose mod-2 cohomology ring is isomorphic to $H^*(\prod_{i=1}^k \mathbb{R}P^{n_i}; \mathbb{F}_2)$. Suppose that G acts freely on X and the induced action on $H^*(X; \mathbb{F}_2)$ is trivial. Then the k-invariants $\alpha_1, \dots, \alpha_k \in H^2(G; \mathbb{F}_2)$ have no common zeros in $(\mathbb{F}_2)^r \setminus \{(0, \dots, 0)\}$ when they are considered as quadratic polynomials in the variables $x_1, \dots, x_r$.*

Proof. We repeat the argument given for the proof of the main theorem in [7]. A similar argument for free actions on products of spheres was given by Carlsson in [5].

Let $\alpha_1, \dots, \alpha_k \in H^2(G; \mathbb{F}_2)$ be the k-invariants of the action. Consider the k-invariants as quadratic polynomials in the variables $x_1, \dots, x_r$. Let $g_1, \dots, g_r$ be the basis for $G = (\mathbb{Z}/2)^r$ dual to the basis $x_1, \dots, x_r$. Let $C \leq G$ be a nontrivial cyclic subgroup of G generated by $g = g_1^{\lambda_1} \dots g_r^{\lambda_r}$, and $x \in H^1(C; \mathbb{F}_2)$ be the generator dual to g . Then $\text{res}_C^G \alpha_i$ is equal to the class $\alpha_i(\lambda_1, \dots, \lambda_r) \cdot x^2$. Hence $\alpha_1, \dots, \alpha_k$ have no common zeros in $(\mathbb{F}_2)^r \setminus \{(0, \dots, 0)\}$ as quadratic polynomials if and only if for every nontrivial cyclic subgroup $C \leq G$, there is an $i \in \{1, \dots, k\}$ such that $\text{res}_C^G \alpha_i \neq 0$.

Assume, to the contrary, that there exists a nontrivial cyclic subgroup $C \leq G$ such that $\text{res}_C^G \alpha_i = 0$ for all i . Consider the restriction of the G -action on X to the subgroup C . By the naturality of the Serre spectral sequence with respect to subgroup inclusions (see [5, page 1150]), in the Serre spectral sequence for $X_C \rightarrow BC$ we have $d_2(t_i) = \text{res}_C^G \alpha_i = 0$ for all i . By the product structure of the Serre spectral sequence, this implies that all the other differentials are also zero. Hence the Serre spectral sequence for the C -action on X collapses at the E_2 -page. This implies that $E_\infty^{p,0} = E_2^{p,0} \cong H^p(C; \mathbb{F}_2) \otimes H^0(X; \mathbb{F}_2)$ is nonzero for all $p \geq 0$, which contradicts the conclusion of Lemma 2.1. $\square$

3. COMPARISON WITH THE INTEGRAL SERRE SPECTRAL SEQUENCE

Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration where both B and F are path-connected. The Serre spectral sequence for the fibration $E \xrightarrow{\pi} B$ with coefficients in the abelian group A is of the form

$$E_2^{p,q}(A) = H^p(B; H^q(F; A)) \Rightarrow H^{p+q}(E; A).$$

We denote the r -th page of this spectral sequence by $E_r^{*,*}(A)$.

For every homomorphism $f : A \rightarrow A'$ between two abelian groups, there is a map of spectral sequences

$$f_* : E_r^{*,*}(A) \rightarrow E_r^{*,*}(A').$$

By this we mean that $f_* : E_r^{*,*}(A) \rightarrow E_r^{*,*}(A')$ is a homomorphism of bigraded abelian groups that commutes with the differentials $d_r^{*,*}(A)$ and $d_r^{*,*}(A')$ where $d_r^{p,q}(A) : E_r^{p,q}(A) \rightarrow E_r^{p+r, q-r+1}(A)$.

The vertical edge of the Serre spectral sequence is $E_2^{0,q} \cong H^0(B; H^q(F; A))$ which is isomorphic to the invariant subgroup $H^q(F; A)^G$ where $G = \pi_1(B, b_0)$. For each

$r \geq 2$, there is a injective map $E_r^{0,q} \hookrightarrow E_2^{0,q}$ and a projection $H^q(E; A) \twoheadrightarrow E_\infty^{0,q}$. The composition

$$H^q(E; A) \twoheadrightarrow E_\infty^{0,q} \hookrightarrow E_2^{0,q} \cong H^q(F; A)^G$$

is called the vertical edge homomorphism. Similarly, there is an isomorphism for the horizontal edge $E_2^{p,0} \cong H^p(B; H^0(F; A)) \cong H^p(B; A)$, and the horizontal edge homomorphism is defined as the composition

$$H^p(B; A) \cong E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(E; A).$$

The horizontal edge homomorphism coincides with the homomorphisms induced by $\pi : E \rightarrow B$.

For each $b \in B$, consider the diagram of maps

$$\begin{array}{ccc} H^{n+1}(B, b; A) & \xrightarrow{j^*} & H^{n+1}(B; A) \\ \pi^* \downarrow & \nearrow & \\ H^n(F; A) & \xrightarrow[\delta_A]{} & H^{n+1}(E, F; A) \end{array}$$

where δ_A is the connecting homomorphism for the sequence $F \rightarrow E \rightarrow (E, F)$ with coefficients in A , and j^* is the map induced by the inclusion $B \rightarrow (B, b)$.

Definition 3.1 ([13, page 186]). There is a map

$$\tau : \delta_A^{-1}(\text{Im } \pi^*) \rightarrow H^{n+1}(B; A)/j^*(\ker \pi^*)$$

defined by $\tau(x) = j^*(y)$, where $y \in H^{n+1}(B, b; A)$ such that $\pi^*(y) = \delta_A(x)$. This map is called a *transgression*. A cohomology class $x \in H^n(F; A)$ is *transgressive* if there is $y \in H^n(B, b; A)$ such that $\pi^*(y) = \delta_A(x)$.

The differential $d_{n+1} : E_{n+1}^{0,n} \rightarrow E_{n+1}^{n+1,0}$ between vertical and horizontal edges on the E_{n+1} -page coincides with the transgression τ (see [13, Theorem 6.8]). We prove the following proposition on the interaction between the transgression and the connecting homomorphism associated to a short exact sequence.

Proposition 3.2. *Let $0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$ be a short exact sequence of abelian groups, and let $\delta_F : H^n(F; A'') \rightarrow H^{n+1}(F; A)$ and $\delta_B : H^{n+1}(B; A'') \rightarrow H^{n+2}(B; A)$ be the connecting homomorphisms for the corresponding long exact cohomology sequences for the spaces F and B . If $x \in H^n(F; A'')$ is transgressive, then $\delta_F(x)$ is also transgressive and $\tau_A(\delta_F(x)) = -\delta_B(\tau_{A''}(x))$.*

Proof. Since the connecting homomorphisms are natural with respect to chain maps, from the definition of the transgression τ , we see that the equality in the proposition will follow once we show that

$$(1) \quad \delta_A \circ \delta_F = -\delta_{(E,F)} \circ \delta_{A''}.$$

Note that there is a commuting diagram of chain complexes

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^*(E, F; A) & \longrightarrow & C^*(E; A) & \longrightarrow & C^*(F; A) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^*(E, F; A') & \longrightarrow & C^*(E; A') & \longrightarrow & C^*(F; A') \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^*(E, F; A'') & \longrightarrow & C^*(E; A'') & \longrightarrow & C^*(F; A'') \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

and δ_A and $\delta_{A''}$ are the connecting homomorphisms for the top and bottom horizontal sequences, and $\delta_{(E,F)}$ and δ_F are the connecting homomorphisms for the left and right vertical sequences. The equality in (1) now follows from an easy diagram chase. This equality is proved as Proposition 9.20 in [15]. $\square$

Consider the exact sequence of abelian groups $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{m_2} \mathbb{F}_2 \rightarrow 0$, where m_2 is the mod-2 reduction map. The connecting homomorphism $H^n(X; \mathbb{F}_2) \rightarrow H^{n+1}(X; \mathbb{Z})$ induced by this short exact sequence, is called the *integral Bockstein operator*, denoted by β_0 . Note that the composition of β_0 with the induced map $m_2 : H^{n+1}(X; \mathbb{Z}) \rightarrow H^{n+1}(X; \mathbb{F}_2)$ gives the Bockstein operator $\beta : H^n(X; \mathbb{F}_2) \rightarrow H^{n+1}(X; \mathbb{F}_2)$. The following is a direct consequence of Proposition 3.2.

Corollary 3.3. *Let $F \rightarrow E \rightarrow B$ be a fibration with path-connected base and fiber spaces, and let $\tau_{\mathbb{Z}}$ and $\tau_{\mathbb{F}_2}$ denote the transgression maps in the Serre spectral sequences for the fibration with coefficients in $\mathbb{Z}$ and $\mathbb{F}_2$. Then if $x \in H^n(F; \mathbb{F}_2)$ is transgressive, $\beta_0(x) \in H^{n+1}(F; \mathbb{Z})$ is also transgressive and $\tau_{\mathbb{Z}}(\beta_0(x)) = \beta_0(\tau_{\mathbb{F}_2}(x))$.*

Remark 3.4. *Corollary 3.3 is well known to experts in the field. The proof given above provides a direct derivation using standard techniques from homological algebra.*

4. INTEGRAL COHOMOLOGY OF PRODUCTS OF REAL PROJECTIVE SPACES

Let $G = (\mathbb{Z}/2)^r$ and X be a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$. In this section we describe the integral cohomology rings of BG and X . To avoid the special cases, throughout the section we assume that $n_i \geq 2$ for all i . The general case will be handled in the proof of Theorem 1.2.

By Künneth's theorem, there is an isomorphism $H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_r]$ where $|x_i| = 1$ for all $i = 1, \dots, r$. For the integral cohomology ring of G , first observe that, by Künneth's theorem again, the integral cohomology groups of G have exponent 2 in positive dimensions. Using the long exact cohomology sequence

for the sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{q_2} \mathbb{Z}/2 \rightarrow 0$ of coefficients, we see that for each $n \geq 1$, there is a short exact sequence

$$0 \rightarrow H^n(G; \mathbb{Z}) \xrightarrow{m_2} H^n(G; \mathbb{F}_2) \xrightarrow{\beta_0} H^{n+1}(G; \mathbb{Z}) \rightarrow 0,$$

where $m_2 : H^*(G; \mathbb{Z}) \rightarrow H^*(G; \mathbb{F}_2)$ is the map induced by the mod-2 reduction map $q_2 : \mathbb{Z} \rightarrow \mathbb{Z}/2$. Note that since $m_2 : H^{n+1}(G; \mathbb{Z}) \rightarrow H^{n+1}(G; \mathbb{F}_2)$ is injective, the condition $\beta_0(x) = 0$ is equivalent to the condition that $\beta(x) = 0$. Hence, via the injective mod-2 reduction map, the positive-dimensional part of $H^*(BG; \mathbb{Z})$ identifies with the subring of $H^*(G; \mathbb{F}_2)$ consisting of elements $u \in H^*(G; \mathbb{F}_2)$ such that $\beta(u) = 0$.

To describe the integral cohomology of G as a ring, consider, for each nonempty subset $I \subseteq \{1, \dots, r\}$, the cohomology class $u_I \in H^*(G; \mathbb{Z})$ defined by

$$u_I = \beta_0 \left(\prod_{i \in I} x_i \right).$$

For $I, J \subseteq \{1, \dots, r\}$, let $I \Delta J$ denote the symmetric difference $(I \setminus J) \cup (J \setminus I)$.

Proposition 4.1 ([3, Proposition 6.1]). *Let $G \cong (\mathbb{Z}/2)^r$. Then $H^*(G; \mathbb{Z})$ is generated by the elements u_I , where $\emptyset \neq I \subseteq \{1, \dots, r\}$, subject to the relations $2u_I = 0$ and*

$$u_I u_J = \left(\prod_{j \in I \cap J} u_j \right) \left(\sum_{j \in I \Delta J} u_{(I \Delta J) \cup \{j\}} + \sum_{j \in J \setminus I} u_j u_{(I \Delta J) \setminus \{j\}} \right)$$

with no restriction on I and J except the assumption that I and J are non-empty.

The integral cohomology of $X \simeq \prod_{i=1}^k \mathbb{R}P^{n_i}$ can also be calculated using the long exact cohomology sequence for the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{m_2} \mathbb{Z}/2 \rightarrow 0$:

$$\begin{aligned} \dots \rightarrow H^{n-1}(X; \mathbb{F}_2) &\xrightarrow{\beta_0} H^n(X; \mathbb{Z}) \xrightarrow{\times 2} H^n(X; \mathbb{Z}) \xrightarrow{m_2} H^n(X; \mathbb{F}_2) \\ &\xrightarrow{\beta_0} H^{n+1}(X; \mathbb{Z}) \rightarrow \dots \end{aligned}$$

For the cohomology ring of a real projective space, we have

$$H^*(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[s, v]/(2s, s^{m+1}, sv, v^2) & \text{if } n = 2m + 1 \\ \mathbb{Z}[s]/(2s, s^{m+1}) & \text{if } n = 2m \end{cases}$$

where $|s| = 2$ and $|v| = 2m + 1$ (see [11, page 222]). A direct calculation shows that $m_2(s) = t^2$ and $m_2(v) = t^{2m+1}$, where $H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[t]/(t^{n+1})$. For each $n \geq 0$, we have $H^n(X; \mathbb{Z}) \cong F_n \oplus T_n$, where F_n is a free abelian group and T_n is the torsion subgroup. By Künneth's theorem, the torsion subgroup T_n has exponent 2. Using the long exact sequence above, we observe that there is a short exact sequence

$$(2) \quad 0 \rightarrow (F_n/2F_n) \oplus T_n \xrightarrow{m_2} H^n(X; \mathbb{F}_2) \xrightarrow{\beta_0} T_{n+1} \rightarrow 0$$

of $\mathbb{F}_2$ -vector spaces.

Consider the classifying map $q : X \rightarrow B\pi$, where $\pi = \pi_1(X, x_0)$. Since we assumed that $n_i \geq 2$ for all i , we have $\pi \cong (\mathbb{Z}/2)^k$, hence $H^*(B\pi; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \dots, t_k]$,

where $|t_i| = 1$. For each $\emptyset \neq I \subseteq \{1, \dots, k\}$, let

$$s_I = \beta_0 \left(\prod_{i \in I} t_i \right).$$

The integral cohomology $H^*(B\pi; \mathbb{Z})$ is generated by s_I , where $\emptyset \neq I \subseteq \{1, \dots, k\}$, subject to the relations $2s_I = 0$ and

$$(3) \quad s_I s_J = \left(\prod_{j \in I \cap J} s_j \right) \left(\sum_{j \in I \cap J} s_{(I \Delta J) \cup \{j\}} + \sum_{j \in J \setminus I} s_j s_{(I \Delta J) \setminus \{j\}} \right)$$

analogous to the relations given for u_I in Proposition 4.1. We have the following observation.

Lemma 4.2. *The induced map $q^* : H^*(B\pi; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$ is surjective on the torsion part of $H^*(X; \mathbb{Z})$.*

Proof. For $n \geq 1$, consider the commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (F_n/2F_n) \oplus T_n & \xrightarrow{m_2} & H^n(X; \mathbb{F}_2) & \xrightarrow{\beta_0} & T_{n+1} \longrightarrow 0 \\ & & \uparrow q^* & & \uparrow q^* & & \uparrow q^* \\ 0 & \longrightarrow & H^n(B\pi; \mathbb{Z}) & \xrightarrow{m_2} & H^n(B\pi; \mathbb{F}_2) & \xrightarrow{\beta_0} & H^{n+1}(B\pi; \mathbb{Z}) \longrightarrow 0. \end{array}$$

By Künneth's theorem $H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \dots, t_k]/(t_1^{n_1+1}, \dots, t_k^{n_k+1})$, hence the middle vertical map q^* is surjective. This implies that the map q^* on the right is also surjective. $\square$

For each $i \in \{1, \dots, k\}$, there is a map $\pi_i : X \rightarrow \mathbb{R}P^{n_i}$ obtained from the projection map to the i -th coordinate. For each i where n_i is odd, let $v_i = \pi_i^*(v)$ where v is the generator of $H^{n_i}(\mathbb{R}P^{n_i}; \mathbb{Z})$. For every i where n_i is even, we take v_i to be zero. By Künneth's theorem, the free part of $H^*(X; \mathbb{Z})$ is the exterior algebra generated by the classes $\{v_i\}_{i=1}^k$.

Proposition 4.3. *Let $X \simeq \prod_{i=1}^k \mathbb{R}P^{n_i}$, $n_i \geq 2$, where $n_i = 2m_i$ if n_i is even and $n_i = 2m_i + 1$ if n_i is odd. Then the integral cohomology of X is generated by the elements s_I , where $\emptyset \neq I \subseteq \{1, \dots, k\}$, and the elements v_i for all i , with degrees $|s_I| = |I| + 1$ and $|v_i| = n_i$ when n_i is odd, subject to the following relations:*

- (1) $2s_I = 0$ for all I ,
- (2) the relations for s_I given in Equation (3),
- (3) $v_i^2 = 0$ for all i , and $v_i = 0$ when n_i is even,
- (4) $v_i s_I = 0$ if $i \in I$, and $v_i s_I = s_i^{m_i} s_{I \cup \{i\}}$ if $i \notin I$, and
- (5) $(\prod_{i \in I} s_i^{m_i}) s_I = 0$ for all I .

Proof. The free part of $H^*(X; \mathbb{Z})$ is generated by the set of all v_i where n_i is odd, so the free part is generated by the given generating set. By Lemma 4.2, the induced map $q^* : H^*(B\pi; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$ is surjective on the torsion part and $H^*(B\pi; \mathbb{Z})$ is generated by the elements s_I where $\emptyset \neq I \subseteq \{1, \dots, k\}$. Hence $H^*(X; \mathbb{Z})$ is generated by the given set of generators.

The relations given in (1) and (2) come from the relations for the integral cohomology of $B\pi$. Since $v_i = \pi_i^*(v)$, the equality $v_i^2 = 0$ follows from $v^2 = 0$. The

products appearing in relations (4) and (5) are torsion classes. They hold because their mod-2 reductions hold in $H^*(X; \mathbb{F}_2)$.

To complete the proof, we need to show that there are no more relations. Note that the free part of $H^*(X; \mathbb{Z})$ is the exterior algebra generated by the classes v_i with n_i odd. Hence monomials in the v_i 's are linearly independent and satisfy no relations other than $v_i^2 = 0$ and graded commutativity. Since the torsion subgroup intersects the free part trivially, it remains only to analyze relations involving torsion classes.

Using the relations in (4), any monomial in s_I 's and v_i 's that has at least some nontrivial s_I term, reduces to an element in the image of $q^* : H^*(B\pi; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$. Modulo the relations in (2), every element in $H^*(B\pi; \mathbb{Z})$ can be expressed as a linear combination of elements of the form

$$(4) \quad (s_{j_1}^{a_1} \dots s_{j_p}^{a_p}) \cdot s_I,$$

where $I = \{i_1, \dots, i_q\}$ is a nonempty set such that $j_1 < \dots < j_p$, $i_1 < \dots < i_q$, and $j_p \leq i_q$. The elements of this form are linearly independent in $H^*(B\pi; \mathbb{Z})$ (see the proof of [3, Proposition 6.1]). Such an element maps to zero under q^* if and only if its mod-2 reduction is in the ideal $J = (t_1^{n_1+1}, \dots, t_k^{n_k+1})$. The mod-2 reduction of an element in the above form is

$$(t_{j_1}^{2a_1} \dots t_{j_p}^{2a_p}) \cdot \left(\prod_{i \in I} t_i \right) \cdot \left(\sum_{k \in I} t_k \right).$$

This polynomial lies in the ideal J if and only if each monomial in the sum has a factor $t_j^{b_j}$ with $b_j \geq n_j + 1$. Fix a $k \in I$ and consider the monomial

$$(t_{j_1}^{2a_1} \dots t_{j_p}^{2a_p}) \cdot \left(\prod_{i \in I} t_i \right) \cdot t_k.$$

Let $t_j^{b_j}$ be the factor in this monomial such that $b_j \geq n_j + 1$. Since $b_j \geq 3$, we must have $j = j_l$ for some l .

If $j_l \notin I$, then $2a_l \geq n_{j_l} + 1 \geq 2m_{j_l} + 1$. This implies $a_l \geq m_{j_l} + 1$. By relation (5) in Proposition 4.3 applied to $I = \{j_l\}$, we have $s_{j_l}^{m_{j_l}+1} = 0$. This gives $s_{j_l}^{a_l} = 0$, hence the monomial in Equation 4 is zero.

Assume that $j_l \in I$ and $j_l \neq k$. Then $2a_l + 1 \geq n_{j_l} + 1$. If $n_{j_l} = 2m_{j_l}$, this gives $a_l \geq m_{j_l}$. As a consequence of relation (4) together with $v_i = 0$ for even n_i , we have $s_i^{m_i} s_I = 0$ whenever $n_i = 2m_i$ and $i \in I$. So, $a_l \geq m_{j_l}$ gives that the monomial in Equation 4 is zero. If $n_{j_l} = 2m_{j_l} + 1$, then $2a_l + 1 \geq 2m_{j_l} + 2$ gives $a_l \geq m_{j_l} + 1$. This gives $s_{j_l}^{a_l} = 0$, so the monomial in Equation 4 is zero.

Finally, assume that $j_l \in I$, and $j_l = k$. Then $2a_l + 2 \geq n_{j_l} + 1 \geq 2m_{j_l} + 1$, which implies that $a_l \geq m_{j_l}$. We obtained that for each $k \in I$, either the monomial in Equation 4 is already zero modulo the stated relations, or there exists an index l with $j_l = k$ such that $a_l \geq m_k$. If the monomial is not already zero, then this gives that the monomial is divisible by

$$\left(\prod_{k \in I} s_k^{m_k} \right) s_I.$$

Since this factor is zero by the relation (5) in Proposition 4.3, the monomial in Equation 4 is zero. Hence an element of the form $q^*(u)$ is zero in $H^*(X; \mathbb{Z})$ if and only if u lies in the ideal generated by the relations stated in Proposition 4.3. This completes the proof. $\square$

5. FREE ACTIONS WITH TRIVIAL ACTION ON INTEGRAL COHOMOLOGY

Let $G = (\mathbb{Z}/2)^r$ and X be a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ with $n_i \geq 2$ for all i . Assume that the induced G -action on integral cohomology of X is trivial. Consider the Serre spectral sequence for the Borel fibration with mod-2 coefficients. Let $t_1, \dots, t_k$ be a basis for $H^1(X; \mathbb{F}_2)$ such that t_i corresponds the i -th term in the product, and let $\alpha_1, \dots, \alpha_k \in H^2(G; \mathbb{F}_2)$ be the corresponding k -invariants of the action. By Lemma 2.3, for every i where n_i is even, $\alpha_i = 0$.

Let V be the vector space generated by $\alpha_1, \dots, \alpha_k$, and $s = \dim V$. Since $r = 0$ case is trivial, we have $r \geq 1$. By Proposition 2.4, $\alpha_1, \dots, \alpha_k$ have no nontrivial common zeros in $(\mathbb{F}_2)^r$, so we must have $s \geq 1$. By changing the indexing, we can assume that $\alpha_1, \dots, \alpha_s$ is a basis for V .

Definition 5.1. We say that the basis $\{\alpha_1, \dots, \alpha_s\}$ for V is *irreducible* if G has no subgroup H of codimension $n \geq 1$ such that the $\mathbb{F}_2$ -vector space generated by $\{\text{res}_H^G \alpha_i\}_{i=1}^s$ has dimension $\leq s - n$.

We have the following lemma.

Lemma 5.2. *Let $\{\alpha_1, \dots, \alpha_s\}$ be an irreducible basis for V . Then, for every linear homogeneous polynomials $l_1, \dots, l_s$ such that $\sum_{i=1}^s l_i \alpha_i = 0$, we have $l_1 = \dots = l_s = 0$.*

Proof. Let $l'_1, \dots, l'_t$ be a basis for the vector space generated by $l_1, \dots, l_s$. For each $i = 1, \dots, s$, we have $l_i = \sum_{j=1}^t \gamma_{ij} l'_j$ for some $\gamma_{ij} \in \mathbb{F}_2$. Then the equation $\sum_{i=1}^s l_i \alpha_i = 0$ gives

$$\sum_{i=1}^s \left(\sum_{j=1}^t \gamma_{ij} l'_j \right) \alpha_i = \sum_{j=1}^t l'_j \left(\sum_{i=1}^s \gamma_{ij} \alpha_i \right) = 0.$$

Let $q_j := \sum_{i=1}^s \gamma_{ij} \alpha_i$ for $j = 1, \dots, t$. Since $l'_1, \dots, l'_t$ are linearly independent, they form a regular sequence in $\mathbb{F}_2[x_1, \dots, x_r]$. This follows from the fact that when l'_j are linearly independent, after a change of coordinates they become $x_1, \dots, x_t$, hence form a regular sequence.

Since $l'_1, \dots, l'_t$ form a regular sequence in $\mathbb{F}_2[x_1, \dots, x_r]$, the equation $\sum_{j=1}^t l'_j q_j = 0$ implies that $q_j \in (l'_1, \dots, l'_t)$ for every j . Let H be the subspace of G defined as $\cap_{i=1}^t \ker(l'_i)$. Then for every j , we have

$$0 = \text{res}_H^G q_j = \sum_{i=1}^s \gamma_{ij} \text{res}_H^G \alpha_i.$$

Since the matrix (γ_{ij}) has rank t , these are t linearly independent relations among $\text{res}_H^G \alpha_i$. Therefore the vector space generated by $\text{res}_H^G \alpha_1, \dots, \text{res}_H^G \alpha_s$ has dimension $\leq s - t$. Since H has codimension t in G , this contradicts the assumption that the

basis $\{\alpha_1, \dots, \alpha_s\}$ is irreducible unless $t = 0$. So, we must have $t = 0$, hence $l_i = 0$ for all $i = 1, \dots, s$. $\square$

Since $t_1, \dots, t_k$ is the basis for $H^1(X; \mathbb{F}_2)$. For each $i = s+1, \dots, k$, there exist $\lambda_{ij} \in \mathbb{F}_2$ such that $\alpha_i = \sum_{j=1}^s \lambda_{ij} \alpha_j$. Define

$$t'_i = t_i + \sum_{j=1}^s \lambda_{ij} t_j.$$

Then $d_2(t'_i) = 0$ for every $i \geq s+1$.

Definition 5.3. Let $\{t_1, \dots, t_s, t'_{s+1}, \dots, t'_k\}$ be a basis of $H^1(X; \mathbb{F}_2)$ such that $\{\alpha_i = d_2(t_i)\}_{i=1}^s$ is a basis for V , and $d_2(t'_i) = 0$ for $i = s+1, \dots, k$. Then $\{t_1, \dots, t_s, t'_{s+1}, \dots, t'_k\}$ is called a *normalized basis*. If $\{\alpha_1, \dots, \alpha_s\}$ is an irreducible basis, then we say that the normalized basis $\{t_1, \dots, t_s, t'_{s+1}, \dots, t'_k\}$ is irreducible.

We are ready to prove our main result in this section.

Lemma 5.4. Let $G = (\mathbb{Z}/2)^r$ and let X be a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$, $n_i \geq 2$, such that the induced G -action on the integral cohomology is trivial. Assume that there is a normalized basis

$$\{t_1, \dots, t_s, t'_{s+1}, \dots, t'_k\}$$

for $H^1(X; \mathbb{F}_2)$ which is irreducible. Then, for every $i \in \{1, \dots, s\}$, $n_i \equiv 3 \pmod{4}$.

Proof. By Lemma 2.3, we already know that n_i is odd for every $i \leq s$. So it remains to rule out the case $n_i \equiv 1 \pmod{4}$. Let $i \in \{1, \dots, s\}$ be a fixed index such that $n_i = 2m_i + 1$ where $m_i = 2k_i \geq 2$. We claim that in this case $\beta(\alpha_i) = 0$.

Consider the Serre spectral sequence $E_r^{p,q}(\mathbb{Z})$ for the Borel fibration with integer coefficients. Since $d_2(t_i) = \alpha_i$ is a transgression, by Corollary 3.3, $s_i := \beta_0(t_i)$ in $E_3^{0,2}(\mathbb{Z})$ is transgressive and

$$d_3(s_i) = \beta_0(d_2(t_i)) = \beta_0(\alpha_i).$$

Since $s_i^{m_i+1} = 0$, we have

$$0 = d_3(s_i^{m_i+1}) = (m_i + 1)d_3(s_i)s_i^{m_i} = \beta_0(\alpha_i)s_i^{m_i} = \beta(\alpha_i)s_i^{m_i}$$

in $E_3^{3,2m_i}(\mathbb{Z})$. Note that multiplication of an integral class u in $H^*(G; \mathbb{Z})$ with a torsion class s_I gives $m_2(u)s_I$. This is why $\beta_0(\alpha_i)$ becomes $\beta(\alpha_i)$.

Since $\beta(\alpha_i)s_i^{m_i} = 0$ in $E_3^{3,2m_i}(\mathbb{Z})$, there is an element $\theta_i \in E_2^{1,2m_i+1}(\mathbb{Z})$ such that $d_2(\theta_i) = \beta(\alpha_i)s_i^{m_i}$. We have

$$E_2^{1,2m_i+1}(\mathbb{Z}) \cong H^1(G; H^{2m_i+1}(X; \mathbb{Z})_f) \oplus H^1(G; H^{2m_i+1}(X; \mathbb{Z})_t).$$

Since $H^1(G; \mathbb{Z}) = 0$ and G acts trivially on the integral cohomology of X , we have $H^1(G; H^{2m_i+1}(X; \mathbb{Z})_f) = 0$. Thus

$$E_2^{1,2m_i+1}(\mathbb{Z}) \cong H^1(G; H^{2m_i+1}(X; \mathbb{Z})_t) \cong H^1(G; \mathbb{F}_2) \otimes H^{2m_i+1}(X; \mathbb{Z})_t.$$

By Proposition 4.3, every torsion class in $H^{2m_i+1}(X; \mathbb{Z})$ is a linear combination of monomials of the form

$$s_{j_1}^{b_1} \cdots s_{j_p}^{b_p} s_I$$

such that $2b_1 + \dots + 2b_p + |I| + 1 = 2m_i + 1$. Since the total degree $2m_i + 1$ is odd, $|I|$ must be even. Note that $d_2(s_i) = 0$ for all i , and $d_2(s_I) = \sum_{i \in I} \alpha_i s_{I \setminus \{i\}}$.

The only monomials of the above form whose image under d_2 involves $s_i^{m_i}$ is $s_i^{m_i-1} s_{ij}$ for some $j \neq i$. We can write θ_i as a sum $u + \theta'_i$ where

$$\theta'_i = \sum_{p,q \neq i} h_{pq} s_p s_i^{m_i-2} s_{iq} + \sum_{p < q, i \notin \{p,q\}} l_{pq} s_i^{m_i-1} s_{pq} + \sum_{j \neq i} l_j s_i^{m_i-1} s_{ij}$$

and u involves only the monomials having s_i^a with $a \leq m_i - 3$, or monomials involving $s_i^{m_i-2} s_{pq}$ with $i \notin \{p,q\}$, or monomials involving s_I with $|I| \geq 4$. Note that under these conditions $d_2(u)$ does not have any summand that involves $s_i^{m_i-1}$ or $s_i^{m_i}$.

Let $y_i = t_i$ for $i \leq s$ and $y_i = t'_i$ for $i \geq s+1$. For each i, j , define $w_{ij} := \beta_0(y_i y_j)$, and for each i , let $w_i := \beta_0(y_i)$. Note that for $i \leq s$, $w_i = s_i$ and for every $i, j \leq s$, $w_{ij} = s_{ij}$. For each i , let $\alpha'_i = d_2(y_i)$. Then for every i, j , we have $d_2(w_{ij}) = \alpha'_i w_j + \alpha'_j w_i$.

Since $\{y_1, \dots, y_k\}$ is a basis for $H^1(X; \mathbb{F}_2)$, $\{w_1, \dots, w_k\}$ is a basis for $H^2(X; \mathbb{Z})$. The change of basis from $\{t_1, \dots, t_k\}$ to $\{y_1, \dots, y_k\}$ induces an invertible change of basis between the degree-two products $\{t_i t_j\}_{i < j}$ and $\{y_i y_j\}_{i < j}$. Applying the integral Bockstein β_0 , which sends $t_i t_j$ to s_{ij} and $y_i y_j$ to w_{ij} , gives an invertible change of basis between $\{s_{ij}\}_{i < j}$ and $\{w_{ij}\}_{i < j}$. Hence $\{w_{ij}\}_{i < j}$ is an $\mathbb{F}_2$ -basis for $H^3(X; \mathbb{Z})_t$.

Using this basis, the equation for θ'_i can be written as

$$\theta'_i = u' + \sum_{p,q \neq i} h'_{pq} w_p s_i^{m_i-2} w_{iq} + \sum_{p < q, i \notin \{p,q\}} l'_{pq} s_i^{m_i-1} w_{pq} + \sum_{j \neq i} l'_j s_i^{m_i-1} w_{ij}$$

for some linear polynomials h'_{pq} , l'_{pq} and l'_j , and u' is a summand where $d_2(u')$ has no summand involving either $s_i^{m_i}$ or $w_j s_i^{m_i-1}$ for any j . The coefficients in this new expression come from regrouping the w_{ij} -terms, so l'_j may involve coefficients l_{pq} from the original sum. Comparing the coefficients of $s_i^{m_i}$ and $s_i^{m_i-1}$ in $d_2(\theta'_i)$, we obtain $\beta(\alpha_i) = \sum_{j \neq i} l'_j \alpha'_j$, and

$$\sum_{p,q \neq i} h'_{p,q} \alpha'_q w_p s_i^{m_i-1} + \sum_{p < q, i \notin \{p,q\}} l'_{pq} (\alpha'_p w_q + \alpha'_q w_p) s_i^{m_i-1} + \sum_{j \neq i} l'_j \alpha'_i w_j s_i^{m_i-1} = 0.$$

Note that in this calculation we use the fact that $w_i = s_i$.

The equation above is of the form $\sum_{j \neq i} A_j w_j s_i^{m_i-1} = 0$ for some $A_j \in H^3(G; \mathbb{F}_2)$. Since $m_i = 2k_i \geq 2$, Proposition 4.3 implies that $s_i^{m_i-1} \neq 0$. Moreover, for each j , the class $s_j s_i^{m_i-1}$ is an admissible monomial, and these monomials are distinct for different values of j . Since admissible monomials are linearly independent by the proof of Proposition 4.3, it follows that the classes $\{s_j s_i^{m_i-1}\}_{j=1}^k$ are linearly independent. Since $\{w_j\}_{j=1}^k$ is obtained from $\{s_j\}_{j=1}^k$ by an invertible linear change of basis, the classes $\{w_j s_i^{m_i-1}\}_{j=1}^k$ are also linearly independent. Hence $A_j = 0$ for all $j \neq i$.

From this we obtain that for each $j \neq i$,

$$\left(\sum_{q \neq i} h'_{j,q} \alpha'_q \right) + \left(\sum_{p \neq i, p < j} l'_{pj} \alpha'_p \right) + \left(\sum_{q \neq i, j < q} l'_{jq} \alpha'_q \right) + l'_j \alpha'_i = 0.$$

Note that since $d_2(t'_q) = \alpha'_q = 0$ for all $q \geq s+1$, the equation above involves α'_q only for $q \leq s$, and for every $q \leq s$, $\alpha'_q = \alpha_q$. Hence, by Lemma 5.2, all coefficients of the nonzero classes $\alpha_1, \dots, \alpha_s$ in this equation must vanish. In particular, $l'_j = 0$. Since this holds for all $j \neq i$ with $j \leq s$, and since $\alpha'_j = 0$ for $j \geq s+1$, we conclude that $\beta(\alpha_i) = \sum_{j \neq i} l'_j \alpha'_j = 0$.

To complete the proof, observe that if

$$\alpha_i = \sum_j a_j x_j^2 + \sum_{j < k} a_{jk} x_j x_k,$$

then $\beta(\alpha_i) = \sum_{j < k} a_{jk} \bar{u}_{jk} = 0$ where $\bar{u}_{jk} := \beta(x_j x_k)$ for $j < k$. This gives $a_{jk} = 0$ for all $j < k$. Hence $\alpha_i = \ell^2$ where $\ell = \sum_j a_j x_j$. If $\ell = 0$, then $\alpha_i = 0$, contradicting irreducibility. Thus $\ell \neq 0$, and we may take $H = \ker(\ell)$. Then $\text{res}_H^G \alpha_i = 0$. This gives that the dimension of the vector space generated by $\text{res}_H^G \alpha_1, \dots, \text{res}_H^G \alpha_s$ has dimension $\leq s-1$. This contradicts the assumption that the basis $\{\alpha_1, \dots, \alpha_s\}$ is irreducible. Therefore no such i can exist. Hence $n_i \equiv 3 \pmod{4}$ for every $i \in \{1, \dots, s\}$. $\square$

The following example shows that the assumption $n_i \equiv 1 \pmod{4}$ is necessary for the conclusion $\beta(\alpha_i) = 0$ in the proof of Lemma 5.4.

Example 5.5. Let Q_8 denote the quaternion group of order 8. Consider the Q_8 -action on $V = \mathbb{R}^4$ given by the embedding of Q_8 in $\mathbb{H} \cong \mathbb{R}^4$ as the group of unit quaternions. For every $m \geq 0$, this gives a Q_8 -action on $Y = S(\oplus_{m+1} V) \cong S^{4m+3}$. Here, and later in the paper, for every $\mathbb{R}G$ -module W , $S(W)$ denotes the unit sphere in W with respect to a G -invariant norm. Note that the central element $z \in Q_8$ of order 2 acts on Y with the antipodal map. We have $X = Y/\langle z \rangle \simeq \mathbb{R}P^{4m+3}$, and $G = Q_8/\langle z \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ acts freely on $X \cong \mathbb{R}P^{4m+3}$.

Since we identified Q_8 with the set of unit quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ in $\mathbb{H}$, the Q_8 -action on $\mathbb{R}^4$ can be described by

$$i \rightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } j \rightarrow \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Since both of these matrices have determinant one, the Q_8 -action on $H^{4m+3}(Y; \mathbb{Z})$ is trivial, hence the induced G -action on $H^{4m+3}(X; \mathbb{Z})$ is trivial. Let $\alpha = d_2(t)$ be the k-invariant of this action. Since the restriction of the $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ -action of $X = \mathbb{R}P^{4m+3}$ to any cyclic subgroup of G gives a free action of the $\mathbb{Z}/2$ -action on X , we conclude that $\alpha = x_1^2 + x_1 x_2 + x_2^2$, where x_1, x_2 are the generators of $H^*(G; \mathbb{F}_2)$. Observe that α is not a square of a 1-dimensional class.

Free $\mathbb{Z}/2$ -actions on $\mathbb{R}P^{4m+1}$ with trivial induced actions on integral cohomology are easy to construct as we show in the following example.

Example 5.6. Consider the $\mathbb{Z}/4$ -action on $W \cong \mathbb{R}^2$ given by rotation by $\pi/2$. Then for every $m \geq 1$, $\mathbb{Z}/4$ acts freely on $Y = S(\oplus_{m+1} W) \cong S^{2m+1}$. This gives a $\mathbb{Z}/2$ -action on $X = Y/(\mathbb{Z}/2) \cong \mathbb{R}P^{2m+1}$. Since the $\mathbb{Z}/4$ -action on W is orientation

preserving, the $\mathbb{Z}/2$ -action on the integral cohomology $H^{2m+1}(X, \mathbb{Z})$ is trivial. The k -invariant of this action is $\alpha = x^2$, where $H^*(\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x]$.

By taking products of the group actions given above, we can construct a $G = (\mathbb{Z}/2)^{k+2l}$ action on $X \cong \prod_{i=1}^k \mathbb{R}P^{2m_i+1} \times \prod_{i=1}^l \mathbb{R}P^{4m_i+3}$ with trivial induced action on integral cohomology. So the upper bound given in Theorem 1.2 is realizable by some free action.

6. FREE ACTIONS WITH NONTRIVIAL ACTION ON INTEGRAL COHOMOLOGY

We start with an example of a free $(\mathbb{Z}/2)^r$ -action on a product of real projective spaces with nontrivial action on integral cohomology such that the induced action on the mod-2 cohomology is trivial.

Example 6.1. Consider the D_8 -action on S^1 induced by the symmetries of a square. We can explain this action as the D_8 -action on the unit sphere $S(W)$, where W is the 2-dimensional irreducible real representation of D_8 . If we write $D_8 = \langle a, b \mid a^2 = b^2 = (ab)^4 = 1 \rangle$, then

$$\rho_W(a) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho_W(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $m \geq 0$ and $V = \oplus_{m+1} W$. Consider the D_8 -action on $Y = S(V) \cong S^{2m+1}$. The central element $z = (ab)^2$ in D_8 acts freely on Y with the antipodal action.

Let $X = Y/\langle z \rangle \cong \mathbb{R}P^{2m+1}$. Consider the induced $G = D_8/\langle z \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ -action on X . The G -action has no global fixed point, but the elements $\bar{a}$ and $\bar{b}$ each have fixed points. Thus we can conclude that the k -invariant of this action is equal to $\alpha = x_1 x_2$, where $H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]$. Note that the determinants of $\rho_V(a)$ and $\rho_V(b)$ are equal to $(-1)^{m+1}$. Hence, when m is even, the induced $\bar{a}$ or $\bar{b}$ action on $H^{2m+1}(X; \mathbb{Z}) \cong \mathbb{Z}$ is nontrivial.

Let $H = \langle \bar{ab} \rangle$. There is an isomorphism $H^{2m+1}(X; \mathbb{Z}) \cong \tilde{\mathbb{Z}}_H$ where

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G/H] \rightarrow \tilde{\mathbb{Z}}_H \rightarrow 0$$

is exact. The long exact cohomology sequence associated to this sequence gives

$$0 \rightarrow H^1(G, \tilde{\mathbb{Z}}_H) \xrightarrow{\delta} H^2(G; \mathbb{Z}) \rightarrow H^2(G; \mathbb{Z}[G/H]) \rightarrow \cdots$$

where $H^2(G; \mathbb{Z}[G/H]) \cong H^2(H, \mathbb{Z})$ by Shapiro's lemma. This gives that

$$H^1(G; \tilde{\mathbb{Z}}_H) \cong \mathbb{F}_2,$$

generated by $u_H := \delta^{-1}\beta_0(x_1 + x_2)$.

We have $\beta(x_1 x_2) = x_1 x_2 (x_1 + x_2) = m_2(u_{12})$, where $u_{12} \in H^3(G; \mathbb{Z})$ is the generator introduced in Proposition 4.1. Hence $\beta_0(\alpha) = u_{12} \neq 0$. Assume that $m = 2m'$ for some $m' \geq 1$. Then $X \cong \mathbb{R}P^{4m'+1}$. By the argument given in the proof of Lemma 5.4, in the integral Serre spectral sequence for the Borel construction, there is $\theta \in E_2^{1, 4m'+1}$ such that $d_2(\theta) = \beta_0(\alpha)s^{2m'} = u_{12}s^{2m'}$, where $s = \beta_0(t)$. Since the product $u_{12}s^{2m'}$ is nonzero, the element θ is nonzero. Note that $E_2^{1, 4m'+1} \cong H^1(G; H^{4m'+1}(X; \mathbb{Z})) \cong H^1(G; \tilde{\mathbb{Z}}_H) \cong \mathbb{F}_2$, hence we can conclude that θ is equal to u_H under the isomorphism above.

This example shows that there is a $(\mathbb{Z}/2)^r$ -action on $\mathbb{R}P^{4m+1}$ with nontrivial action on the integral cohomology such that the k-invariant α is not a square and $\beta_0(\alpha) \neq 0$. This example also shows that the differential $d_2 : E_2^{1,4m+1} \rightarrow E_2^{3,4m}$ may be nonzero when the action on the integral cohomology is nontrivial.

Example 6.2. In the example above the $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ -action on $Y_1 = \mathbb{R}P^{4m+1}$ is not free. However by taking the product with another action of G on $Y_2 = \mathbb{R}P^{2l+1}$, we can obtain a free action of G on $X = Y_1 \times Y_2$. To construct the G -CW-complex Y_2 , consider the $\mathbb{Z}/4$ -action on S^{2l+1} defined by $(z_0, \dots, z_l) \rightarrow (iz_0, \dots, iz_l)$. This gives a free $\mathbb{Z}/2$ -action on $\mathbb{R}P^{2l+1}$.

Let $q : G \rightarrow \mathbb{Z}/2$ denote the quotient map such that $q(\bar{a}) = q(\bar{b}) = 1$. Via q , there is a induced G -action on $Y_2 \cong \mathbb{R}P^{2l+1}$ with kernel $\langle \bar{a}\bar{b} \rangle \cong \mathbb{Z}/2$. The G -action on $X = Y_1 \times Y_2$ is free since $\langle \bar{a}\bar{b} \rangle$ acts freely on Y_1 . This gives a free $G = (\mathbb{Z}/2)^2$ action on $X \cong \mathbb{R}P^{4m+1} \times \mathbb{R}P^{2l+1}$ where the k-invariants are $\alpha_1 = x_1x_2$ and $\alpha_2 = (x_1 + x_2)^2$. Note that as polynomials in x_1 and x_2 , the k-invariants α_1 and α_2 have no nontrivial common zeros.

Remark 6.3. Note that in the example above, $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ acts freely on $X \simeq \mathbb{R}P^{4m+1} \times \mathbb{R}P^{2l+1}$ with k-invariants $\alpha_1 = x_1x_2$ and $\alpha_2 = (x_1 + x_2)^2$, and we have $\beta(\alpha_1) = x_1x_2(x_1 + x_2) \neq 0$. This contradicts a conclusion we had in the proof of [18, Lemma 8.1]. This lemma is correct only under the assumption that the G -action on the integral cohomology of X is trivial. Because of this, the statement of [18, Theorem 8.3] holds only under the same assumption. In the statements of [18, Lemma 8.1] and [18, Theorem 8.3], the assumption that G acts trivially on mod-2 cohomology should be replaced by the assumption that G acts trivially on the integral cohomology.

Before we state the main result of this section, we need to prove some lemmas.

Lemma 6.4. Let $G = (\mathbb{Z}/2)^r$ and $\alpha \in H^2(G; \mathbb{F}_2)$ be a cohomology class. If $\beta(\alpha) = \gamma\alpha$ for some $\gamma \in H^1(G; \mathbb{F}_2)$, then $\alpha = \eta(\gamma + \gamma)$ for some $\eta \in H^1(G; \mathbb{F}_2)$.

Proof. If $\gamma = 0$, then $\beta(\alpha) = 0$. Writing α as a quadratic polynomial, this implies that the mixed terms vanish, hence $\alpha = \eta^2$ for some $\eta \in H^1(G; \mathbb{F}_2)$. Thus $\alpha = \eta(\eta + \gamma)$.

So we can assume $\gamma \neq 0$. Let $H = \ker \gamma$, then $\beta(\text{res}_H^G \alpha) = \text{res}_H^G(\beta(\alpha)) = 0$. This implies that $\text{res}_H^G \alpha = (u')^2$ for some $u' \in H^1(H; \mathbb{F}_2)$. Let $u \in H^1(G; \mathbb{F}_2)$ be a 1-dimensional class such that $\text{res}_H^G u = u'$. Then $\text{res}_H^G(\alpha + u^2) = 0$ which gives that $\alpha = u^2 + \gamma\eta$ for some $\eta \in H^1(G; \mathbb{F}_2)$. Applying the Bockstein operator to both sides of the equality, we obtain $\gamma\alpha = \gamma\eta(\gamma + \eta)$. Since $H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_r]$ is an integral domain and $\gamma \neq 0$, we conclude $\alpha = \eta(\gamma + \eta)$. $\square$

We also need the following:

Lemma 6.5. Let X be a finite-dimensional G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ such that the induced G -action on the mod-2 cohomology of X is trivial. Let $H^n(X; \mathbb{Z}) = M_f \oplus M_t$ be a decomposition as an abelian group where M_f is a free subgroup and M_t is the torsion subgroup. Then both M_f and M_t are

G -invariant, and the decomposition $H^n(X; \mathbb{Z}) = M_f \oplus M_t$ is a decomposition of $\mathbb{Z}G$ -modules.

Proof. Let $M = H^n(X; \mathbb{Z})$ as a $\mathbb{Z}G$ -module, and let $M = M_f \oplus M_t$ be a decomposition as abelian groups. Since every element $g \in G$ takes a torsion element to a torsion element, M_t is a $\mathbb{Z}G$ -submodule of M .

Now we show that M_f is G -invariant. Let $m \in M_f$ and $g \in G$. By (2), there is a short exact sequence

$$0 \rightarrow M/2M \rightarrow H^n(X; \mathbb{F}_2) \rightarrow T_{n+1} \rightarrow 0,$$

so $M/2M$ is a trivial $\mathbb{F}_2G$ -module. Since $M = M_f \oplus M_t$ and M_t has exponent 2, we have $2M = 2M_f$. Since $M/2M$ is a trivial $\mathbb{F}_2G$ -module, we have $gm - m \in 2M = 2M_f$. Since $m \in M_f$, it follows that $gm \in M_f$. Thus M_f is G -invariant.

Since both M_f and M_t are $\mathbb{Z}G$ -submodules of M , the decomposition $M = M_f \oplus M_t$ is a decomposition of $\mathbb{Z}G$ -modules. $\square$

The main result of this section is the following:

Proposition 6.6. *Let $G = (\mathbb{Z}/2)^r$ and let X be a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$, $n_i \geq 2$, with trivial action on mod-2 cohomology. Assume that there is a normalized basis*

$$\{t_1, \dots, t_s, t'_{s+1}, \dots, t'_k\}$$

for $H^1(X; \mathbb{F}_2)$ which is irreducible. Then, for every $i \in \{1, \dots, s\}$, $n_i \equiv 3 \pmod{4}$.

Proof. By Lemma 2.3, for all i such that n_i is even, $\alpha_i = 0$. So for all $i \leq s$, n_i must be odd. Hence it is enough to prove that there is no $i \leq s$ such that $n_i \equiv 1 \pmod{4}$. Let $i \leq s$ be a fixed index such that $n_i = 2m_i + 1$ for some even $m_i \geq 2$. We claim that in this case $\beta(\alpha_i) = \ell_i \alpha_i$ for some $\ell_i \in H^1(G; \mathbb{F}_2)$.

Consider the Serre spectral sequence

$$E_2^{p,q}(\mathbb{Z}) = H^p(BG; H^q(X; \mathbb{Z})) \Rightarrow H^{p+q}(X_G; \mathbb{Z})$$

in integral coefficients for the Borel construction $X \rightarrow X_G \rightarrow BG$. Note that $H^p(BG; H^q(X; \mathbb{Z}))$ is isomorphic to the cohomology of the group G with coefficients in the $\mathbb{Z}G$ -module $H^q(X; \mathbb{Z})$. By Lemma 6.5, for every $q \geq 0$, we have $H^q(X; \mathbb{Z}) \cong H^q(X; \mathbb{Z})_f \oplus H^q(X; \mathbb{Z})_t$ as $\mathbb{Z}G$ -modules and G acts trivially on the torsion part since it has exponent 2. Hence

$$E_2^{p,q}(\mathbb{Z}) \cong H^p(G; H^q(X; \mathbb{Z})_f) \oplus H^p(G; H^q(X; \mathbb{Z})_t)$$

where the second summand is isomorphic to $H^p(G; \mathbb{F}_2) \otimes H^q(X; \mathbb{Z})_t$.

The integral spectral sequence $E_2^{p,q}(\mathbb{Z})$ has a product structure given by the map

$$\begin{aligned} H^p(G; H^q(X; \mathbb{Z}) \otimes H^{p'}(G; H^{q'}(X; \mathbb{Z})) &\rightarrow H^{p+p'}(G; H^q(X; \mathbb{Z}) \otimes H^{q'}(X; \mathbb{Z})) \\ &\rightarrow H^{p+p'}(G; H^{q+q'}(X; \mathbb{Z})) \end{aligned}$$

where the first map is defined by the cup product in the cohomology of G with non-trivial coefficients, and the second map is induced by the cup product in $H^*(X; \mathbb{Z})$. Note that the products of the cohomology classes in the summand $H^*(G; H^*(X; \mathbb{Z})_t)$

stays in this summand, and the Leibniz formula for the differentials holds for such products.

Let s_I , $\emptyset \neq I \subseteq \{1, \dots, k\}$, and v_j , $j = 1, \dots, k$, be the generators of the integral cohomology ring $H^*(X; \mathbb{Z})$ subject to relations given in Proposition 4.3. Since $\beta_0(t_i) = s_i$, by Corollary 3.3, s_i is transgressive and $d_3(s_i) = \beta_0(\alpha_i)$. By the relations in Proposition 4.3, we have $s_i^{m_i+1} = 0$. Since $s_i \in H^2(X; \mathbb{Z})_t$, by the remark above, we have

$$0 = d_3(s_i^{m_i+1}) = (m_i + 1)d_3(s_i)s_i^{m_i} = \beta(\alpha_i)s_i^{m_i}$$

in $E_3^{3,2m_i}$. Hence there is an element $\theta_i \in E_2^{1,2m_i+1}(\mathbb{Z})$ such that $d_2(\theta_i) = \beta(\alpha_i)s_i^{m_i}$.

Let $m_2 : E_2^{*,*}(\mathbb{Z}) \rightarrow E_2^{*,*}(\mathbb{F}_2)$ be the morphism of spectral sequences induced by the mod-2 reduction map. The mod-2 reduction map $m_2 : E_2^{p,q}(\mathbb{Z}) \rightarrow E_2^{p,q}(\mathbb{F}_2)$ is the composition

$$\begin{aligned} E_2^{p,q}(\mathbb{Z}) &\cong H^p(G; H^q(X; \mathbb{Z})_f) \oplus H^p(G; H^q(X; \mathbb{Z})_t) \xrightarrow{(m_2, m_2)} \\ &H^p(G; H^q(X; \mathbb{Z})_f \otimes \mathbb{F}_2) \oplus H^p(G; H^q(X; \mathbb{Z})_t) \hookrightarrow H^p(G; H^q(X; \mathbb{F}_2)). \end{aligned}$$

The $\mathbb{F}_2$ -vector space $H^q(X, \mathbb{F}_2)$ is generated by monomials $t_1^{a_1} \cdots t_k^{a_k}$ with $\sum_{i=1}^k a_i = q$. Since $H^*(X, \mathbb{Z})_f$ is generated by $v_1, \dots, v_k$, the cohomology group $H^q(X; \mathbb{Z})_f$ is generated by $v_I := \prod_{j \in I} v_j$ where I is a set of indices j such that n_j is odd and $\sum_{j \in I} n_j = q$. Hence the image of the mod-2 reduction map is generated by monomials $t_I := \prod_{j \in I} t_j^{n_j}$ where I is a set of indices with odd n_j such that $\sum_{j \in I} n_j = q$.

The image of $H^q(X; \mathbb{Z})_t$ under the mod-2 reduction is generated by mod-2 reductions of monomials of the form $s_{j_1}^{b_1} \cdots s_{j_p}^{b_p} s_I$ where $I = \{i_1, \dots, i_q\}$ with $j_1 < \cdots < j_p$, $i_1 < \cdots < i_q$ and $j_p < i_q$. Note that none of these give a monomial of the form t_I after mod-2 reduction. So, the monomials involved in the mod-2 reductions of $H^q(X; \mathbb{Z})_f$ and $H^q(X; \mathbb{Z})_t$ are distinct.

Since m_2 commutes with the differentials, we have

$$d_2(m_2(\theta_i)) = m_2(d_2(\theta_i)) = m_2(\beta(\alpha_i)s_i^{m_i}) = \beta(\alpha_i)t_i^{2m_i}$$

in $E_2^{3,2m_i}(\mathbb{F}_2)$. Let $M := H^{2m_i+1}(X; \mathbb{Z})$ and $M = M_f \oplus M_t$ be a decomposition of M as $\mathbb{Z}G$ -modules. We can write $\theta_i = (\theta_i^f, \theta_i^t)$, where $\theta_i^f \in H^1(G; M_f)$ and $\theta_i^t \in H^1(G; M_t)$. We have

$$d_2(m_2(\theta_i)) = d_2(m_2(\theta_i^f)) + d_2(m_2(\theta_i^t)) = \beta(\alpha_i)t_i^{2m_i}.$$

The image of

$$m_2 : H^1(G; M_f) \rightarrow H^1(G; M_f/2M_f) \cong H^1(G; \mathbb{F}_2) \otimes M_f/2M_f$$

is generated by elements of the form $\ell \otimes y$ where $\ell \in H^1(G; \mathbb{F}_2)$ and $y \in M_f/2M_f$. Since $M_f/2M_f$ is generated by monomials $t_I = \prod_{j \in I} t_j^{n_j}$, we can conclude that

$$m_2(\theta_i^f) = \sum_I \ell_I \otimes t_I$$

for some $\ell_I \in H^1(G; \mathbb{F}_2)$. Note that the sum above is over all sets of indices I such that for all j , n_j is odd, and $\sum_{j \in I} n_j = 2m_i + 1$. Applying d_2 to this sum, we

obtain

$$d_2(m_2(\theta_i^f)) = \sum_I \sum_{j \in I} \ell_I \alpha_j t_j^{2m_j} \prod_{h \in I, h \neq j} t_h^{n_h}$$

where $n_j = 2m_j + 1$ for all j . Since $m_j \geq 1$ for all j , the only term that can contribute to the $t_i^{2m_i}$ -component has index $I = \{i\}$.

Note that the sum above for $d_2(m_2(\theta_i^f))$ is the mod-2 reduction of

$$z = \sum_I \sum_{j \in I} \ell_I \alpha_j s_j^{m_j} \left(\prod_{h \in I, h \neq j} v_h \right) = \sum_I \sum_{j \in I} \ell_I \alpha_j s_j^{m_j-1} s_I \left(\prod_{h \in I, h \neq j} s_h^{m_h} \right)$$

The last equality follows from the relations given in Proposition 4.3. We can write $z = z' + \ell_i \alpha_i s_i^{m_i}$ where $\ell_i = \ell_{\{i\}}$. The equation $d_2(m_2(\theta_i^f)) + d_2(m_2(\theta_i^t)) = \beta(\alpha_i) t_i^{2m_i}$ gives

$$m_2(z' + \ell_i \alpha_i s_i^{m_i}) + m_2(d_2(\theta_i^t)) = m_2(\beta(\alpha_i) s_i^{m_i})$$

Since the mod-2 reductions of torsion basis monomials are linearly independent and span a direct summand of $H^q(X; \mathbb{F}_2)$, the induced map on $H^p(G; -)$ is injective on the torsion summand. Hence we can conclude that

$$z' + d_2(\theta_i^t) = (\ell_i \alpha_i + \beta(\alpha_i)) s_i^{m_i}.$$

As we did in the proof of Lemma 5.4, we can write θ_i^t as a sum $u + \theta'_i$ where

$$\theta'_i = \sum_{p, q \neq i} h_{pq} s_p s_i^{m_i-2} s_{iq} + \sum_{p < q, i \notin \{p, q\}} l_{pq} s_i^{m_i-1} s_{pq} + \sum_{j \neq i} l_j s_i^{m_i-1} s_{ij}$$

where $d_2(u)$ does not have any summand that involves $s_i^{m_i-1}$ or $s_i^{m_i}$. We claim that z' has no summand involving either $s_i^{m_i}$ or $w_j s_i^{m_i-1}$. The summands of z are indexed by sets I such that $\sum_{j \in I} n_j = n_i$. If $i \in I$, then we must have $I = \{i\}$, since all $n_j \geq 2$. This is precisely the term $\ell_i \alpha_i s_i^{m_i}$, which has been separated off. Hence every summand of z' has $i \notin I$, and therefore z' contains no factor involving s_i . In particular, z' does not contribute to the coefficients of $s_i^{m_i}$ or $w_j s_i^{m_i-1}$.

Using the argument given in the proof of Lemma 5.4, the coefficient comparison in the $s_i^{m_i}$ -terms gives

$$\ell_i \alpha_i + \beta(\alpha_i) = \sum_{j \neq i} l'_j \alpha'_j.$$

Applying Lemma 5.2, and comparing the coefficient of the $s_i^{m_i-1}$ -terms, we obtain that $l'_j = 0$ for all $j \neq i$ such that $j \leq s$. Since $\alpha'_j = 0$ for all $j \geq s+1$, we obtain $\ell_i \alpha_i + \beta(\alpha_i) = 0$.

By Lemma 6.4, this implies that $\alpha_i = \ell'_i(\ell'_i + \ell_i)$ for some linear polynomial ℓ'_i . Since $\alpha_i \neq 0$, we have $\ell'_i \neq 0$. Taking $H = \ker(\ell'_i)$ we see that

$$\dim_{\mathbb{F}_2} \langle \text{res}_H^G \alpha_1, \dots, \text{res}_H^G \alpha_s \rangle \leq s-1.$$

This contradicts the assumption that the normalized basis is irreducible. Since n_i is already known to be odd for $i \leq s$, it follows that $n_i \equiv 3 \pmod{4}$ for every $i \leq s$. $\square$

7. PROOF OF THE MAIN THEOREM

We are ready to prove Theorem 1.2. We first observe that it is enough to prove the theorem for the case $n_i \geq 2$ for all i .

Proposition 7.1. *Suppose that Theorem 1.2 holds for every $H = (\mathbb{Z}/2)^s$ and every finite-dimensional free H -CW-complex $X \simeq \prod_{i=1}^k \mathbb{R}P^{n_i}$ with $n_i \geq 2$ for all i , and with trivial H -action on mod-2 cohomology. Then Theorem 1.2 holds for every $G = (\mathbb{Z}/2)^r$ and every finite-dimensional free G -CW-complex $X \simeq \prod_{i=1}^{k'} \mathbb{R}P^{n'_i}$ with trivial G -action on mod-2 cohomology.*

Proof. Let $G \cong (\mathbb{Z}/2)^r$ and X be a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^{k'} \mathbb{R}P^{n'_i}$, with trivial G -action on mod-2 cohomology. If for all i , $n'_i \geq 2$, then the conclusion follows from the hypothesis. Otherwise, after reordering the factors, we can write $X \simeq M \times Y$ where

$$M = (S^1)^l \quad \text{and} \quad Y = \prod_{i=1}^k \mathbb{R}P^{n_i},$$

such that $l \geq 1$ is the number of indices with $n'_i = 1$, and $n_i \geq 2$ for all i . If $k = 0$, we take Y to be a point.

Note that $M = (S^1)^l$ is a compact solvmanifold (see [12]) and Y is a finite-dimensional CW-complex with a finite fundamental group. So we can use the argument given by Jo and Lee in the proof of [12, Theorem 2.2].

We have $\pi_1(X, x_0) \cong \Gamma \times F$, where $\Gamma \cong \mathbb{Z}^l$ and $F \cong (\mathbb{Z}/2)^k$. Since G acts freely on X , there is a group extension

$$1 \rightarrow \pi_1(X, x_0) \rightarrow \pi_1(X/G, \bar{x}_0) \rightarrow G \rightarrow 1.$$

Let $\pi = \pi_1(X/G, \bar{x}_0)$. Since F is the torsion subgroup of $\pi_1(X, x_0)$, it is characteristic in $\pi_1(X, x_0)$, hence it is normal in π . This gives a commuting diagram of groups

$$\begin{array}{ccccccc} & & F & \xlongequal{\quad} & F & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(X, x_0) & \longrightarrow & \pi & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \pi/F & \longrightarrow & G \longrightarrow 1. \end{array}$$

The 2-rank of a group K , denoted by $\text{rk}_2(K)$, is defined to be the largest r such that $(\mathbb{Z}/2)^r \leq K$. By [12, Theorem 1.3], we have

$$(5) \quad \text{rk}_2 G \leq \text{rk}_2(\pi/F) + l.$$

The sequence $1 \rightarrow \Gamma \rightarrow \pi/F \rightarrow G \rightarrow 1$ is associated to a sequence of covering maps:

$$\begin{array}{c} \overline{X} \simeq \widetilde{M} \times Y \simeq Y \\ \downarrow / \Gamma \\ X \simeq M \times Y \\ \downarrow / G \\ X/G \end{array}$$

where $\overline{X}$ is the Γ -cover of X , and $\widetilde{M} \simeq \mathbb{R}^l$ is the universal cover of M . To see these homotopy equivalences, note that under the homotopy equivalence $X \simeq M \times Y$, the subgroup $\Gamma \leq \pi_1(X)$ corresponds to the subgroup $\pi_1(M) \times 1 \leq \pi_1(M \times Y)$. Hence the associated Γ -cover has homotopy type $\widetilde{M} \times Y \simeq Y$.

Since X is a finite-dimensional free G -CW-complex, $\overline{X}$ is a finite-dimensional free (π/F) -CW-complex (see [17, Statement (N)]). Let $H \leq \pi/F$ be a subgroup isomorphic to $(\mathbb{Z}/2)^s$, where $s = \text{rk}_2(\pi/F)$. The restriction of the (π/F) -action on $\overline{X}$ to H is also a free action.

Now we will show that the induced H -action on mod-2 cohomology of $\overline{X}$ is trivial. Since $H \cap \Gamma = 1$, the composite $H \leq \pi/F \rightarrow G$ is injective. Let $p : \overline{X} \rightarrow X$ be the covering map. Since p is H -equivariant (where H acts on X through the injective homomorphism $H \hookrightarrow G$), the induced map

$$p^* : H^*(X; \mathbb{F}_2) \longrightarrow H^*(\overline{X}; \mathbb{F}_2)$$

is H -equivariant. Under the homotopy equivalences

$$X \simeq M \times Y \quad \text{and} \quad \overline{X} \simeq \widetilde{M} \times Y \simeq Y,$$

the map p^* identifies with the map induced by the covering $\widetilde{M} \times Y \rightarrow M \times Y$. Since the projection

$$H^*(M; \mathbb{F}_2) \otimes H^*(Y; \mathbb{F}_2) \longrightarrow H^*(Y; \mathbb{F}_2)$$

is surjective, p^* is also surjective. The G -action on $H^*(X; \mathbb{F}_2)$ is trivial, hence the induced H -action on $H^*(\overline{X}; \mathbb{F}_2)$ is also trivial.

If $k = 0$, then $\overline{X}$ is finite-dimensional and contractible. A nontrivial finite 2-group cannot act freely on a finite-dimensional contractible CW-complex. Hence we must have $H = 1$, and therefore $s = 0$. Thus $s \leq \sum_{i=1}^k \mu(n_i)$ holds trivially.

If $k > 0$, then the hypothesis applied to the free H -CW-complex $\overline{X} \simeq \prod_{i=1}^k \mathbb{R}P^{n_i}$ gives

$$s \leq \sum_{i=1}^k \mu(n_i).$$

Combining this with (5), we obtain

$$r = \text{rk}_2(G) \leq s + l \leq \left(\sum_{i=1}^k \mu(n_i) \right) + l = \sum_{i=1}^{k'} \mu(n'_i).$$

This completes the proof of the proposition. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $G = (\mathbb{Z}/2)^r$ and X be a finite-dimensional free G -CW-complex homotopy equivalent to $\prod_{i=1}^k \mathbb{R}P^{n_i}$ such that the G -action on the mod-2 cohomology of X is trivial.

Let $\alpha_1, \dots, \alpha_k$ be the k -invariants in the Serre spectral sequence in $\mathbb{F}_2$ -coefficients. Consider the $\mathbb{F}_2$ -vector space V generated by the k -invariants, and assume that the dimension of V is s . After reindexing the generators t_i , we can assume that $\{t_1, \dots, t_s\}$ is such that $\alpha_1, \dots, \alpha_s$ is a basis for V . We claim that under this setting the inequality

$$r \leq \sum_{i=1}^s \mu(n_i)$$

holds. Note that this implies the inequality in the conclusion of the theorem. We prove the claim by induction on r . The case $r = 0$ is immediate.

By Lemma 2.3, for each i where n_i is even, we have $\alpha_i = 0$. So n_i is odd for all $i \leq s$. By Proposition 7.1, it is enough to prove the theorem under the assumption that $n_i \geq 2$ for all i . Hence, for $i \leq s$, we can assume that $n_i = 2m_i + 1$ with some integer $m_i \geq 1$.

If there is a subgroup $H \leq G$ with codimension $n \geq 1$ such that the vector space W generated by $\{\text{res}_H^G \alpha_i\}_{i=1}^s$ has dimension $\leq s - n$, then we can reduce the G -action on X to the subgroup H . Then after reindexing, there is an integer $t \leq s - n$ such that $\{\text{res}_H^G \alpha_i\}_{i=1}^t$ is a basis for W . Applying the induction hypothesis to the free H -action on X gives $r - n \leq \sum_{i=1}^t \mu(n_i)$. From this we obtain

$$r \leq n + \sum_{i=1}^t \mu(n_i) \leq \sum_{i=1}^s \mu(n_i)$$

since $\mu(n_i) \geq 1$ for all odd n_i . Thus, whenever such a subgroup H exists, the theorem follows from the induction hypothesis. Therefore we may assume that the basis $\{\alpha_1, \dots, \alpha_s\}$ is irreducible (see Definition 5.1).

If $s = 0$, then all k -invariants are zero. By Proposition 2.4, this is possible only when $r = 0$, and the desired inequality holds. Hence assume that $s \geq 1$. By Proposition 6.6, for every $i \leq s$, $n_i \equiv 3 \pmod{4}$.

By Proposition 2.4, $\alpha_1, \dots, \alpha_s$ have no nontrivial common zeros, hence by [7, Proposition D], we have $r \leq 2s$. Since $n_i \equiv 3 \pmod{4}$ for all $i \leq s$, this gives the desired inequality $r \leq \sum_{i=1}^s \mu(n_i)$. Hence the proof is complete. $\square$

It is an open question whether the conclusion of Theorem 1.2 holds under the weaker assumption that X has the mod-2 cohomology of $\prod_{i=1}^k \mathbb{R}P^{n_i}$, i.e. whether Cusick's conjecture holds in the way it is stated. It is also an interesting question whether the conclusion of Theorem 1.2 still holds without the assumption that the induced action on the mod-2 cohomology is trivial.

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