STEENROD CLOSED PARAMETER IDEALS IN THE MOD-2 COHOMOLOGY OF A_4 AND SO(3)

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Abstract. In this paper, we classify the parameter ideals in $H^*(BA_4; \mathbb{F}_2)$ and in the Dickson algebra $H^*(B\,\mathrm{SO}(3); \mathbb{F}_2)$ that are closed under Steenrod operations. Consequently, we obtain restrictions on the dimensions m, n for which A_4 can act freely on $S^m \times S^n$. We construct free A_4-actions on finite CW-complexes homotopy equivalent to a product of two spheres realizing some of the ideals from our classification.

1. Introduction

It is a classical problem to classify all finite groups which can act freely on a sphere S^n. This problem is still open for some dimensions n. If we allow the dimension of the sphere to vary, then the problem is solved with the works of Smith, Swan, Milnor, and Madsen-Thomas-Wall (see [Ham15] for a survey). It is proved that a finite group G acts freely on S^n for some $n \geq 1$ if and only if all the subgroups of G with order p^2 and $2p$ are cyclic for all primes p. The first condition that every subgroup of order p^2 is cyclic is equivalent to the condition that the cohomology of the group G in $\mathbb{F}_p$-coefficients is periodic (see [AM04, Definition 6.1] for a definition). The dimension of the sphere S^n depends on the periodicity of the group cohomology $H^*(BG; \mathbb{Z})$, however there are also other obstructions which can make the dimension bigger than the periodicity. It requires delicate computations involving number theory to find the exact dimensions of spheres on which a finite group with periodic cohomology can act freely (see [Ham15]).

For every prime p, the *p-rank* of a finite group G is defined to be the maximum integer s such that $(\mathbb{Z}/p)^s$ injects into G, and the *rank* of G is the maximum of its p-ranks over all primes p dividing the order of G. It is conjectured by Benson and Carlson [BC87] that any rank r group acts freely on a finite CW-complex homotopy equivalent to a product of r spheres $S^{n_1} \times \cdots \times S^{n_r}$ for some $n_1, \dots, n_r \geq 1$. Although this conjecture is still open in full generality, some results are known for the existence of such actions. In particular, any rank 2 finite group which does not involve $\mathrm{Qd}(p)$ for any odd prime p admits such actions (see [Jac07] and [AS01]).

If a finite group G is known to act freely on a product of two spheres $S^n \times S^m$, it is interesting to ask which dimensions n and m are possible for such actions. In particular, one can ask if every rank 2 group acts freely on $S^n \times S^n$ for some n. This question was answered negatively by Oliver in [Oli79] where he proved that the alternating group on 4 elements A_4 can not act freely on a finite CW-complex

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X with $H^*(X;\mathbb{Z}) \cong H^*(S^n \times S^n;\mathbb{Z})$. Oliver proved this result by considering ideals in $H^*(BA_4;\mathbb{F}_2)$ which are closed under Steenrod operations. The cohomology ring

$$H^*(BA_4;\mathbb{F}_2) \cong H^*(B(\mathbb{Z}/2\times\mathbb{Z}/2);\mathbb{F}_2)^{C_3}$$

is isomorphic to the algebra $\mathbb{F}_2[u,v,w]/\langle u^3+v^2+vw+w^2\rangle$, where u has degree 2 and v,w have degree 3 (see Theorem 2.1). Oliver proved the following:

Theorem 1.1 ([Oli79, Lemma 1])**.** *Let $I \subset H^*(BA_4;\mathbb{F}_2)$ be a nonzero ideal generated by homogeneous elements of the same degree i. If I is closed under the Steenrod operations, then $I = \langle v^k\rangle$ for $k = i/3$.*

If there were a free A_4-action on a finite CW-complex homotopy equivalent to a product of spheres $S^n \times S^n$, then the kernel of the map $H^*(BA_4;\mathbb{F}_2) \to H^*(X/A_4;\mathbb{F}_2)$ would be such an ideal I for $i = n+1$ and $H^*(BA_4;\mathbb{F}_2)/I$ would be finite over $\mathbb{F}_2$. But for $k > 0$, the ring $H^*(BA_4;\mathbb{F}_2)/\langle v^k\rangle$ is not finite. This argument holds also for finite CW-complexes with mod-2 cohomology of $S^n \times S^n$ by Theorem 7.1.

In this paper we look at the question of Blaszczyk from [Bla13, Section 4]: which dimensions n and m are possible if $G = A_4$ acts freely on a finite CW-complex $X \simeq S^n \times S^m$ for $n \neq m$? To avoid trivial situations, we assume $n, m \geq 1$. We show in Proposition 7.4 that if such an action exists then the kernel of the homomorphism $H^*(BA_4;\mathbb{F}_2) \to H^*(X/A_4;\mathbb{F}_2)$ is a Steenrod closed ideal I in $H^*(BA_4;\mathbb{F}_2)$ with finite quotient, generated by the two k-invariants of the action. These k-invariants have degrees $m+1$ and $n+1$.

Our main result classifies the Steenrod closed parameter ideals in $H^*(BA_4;\mathbb{F}_2)$, i.e., the Steenrod closed ideals I generated by two homogeneous elements such that $H^*(BA_4;\mathbb{F}_2)/I$ is finite. These can be grouped into three cases.

First, there is the fibered case, where I has a system of parameters, such that one of the parameters generates a Steenrod closed ideal. By Theorem 1.1, that parameter is v^k for some k. We prove in Theorem 5.8 that every fibered ideal is of the form $\langle v^k, u^l\rangle$, where k is not larger than the highest power of 2 dividing l.

Secondly, there is the twisted case, where $I = \langle X, Y\rangle$ with $Y = \mathrm{Sq}^1(X)$. We prove in Theorem 3.15 that all such X can be computed using the recursion $x_1 = u$, $x_{n+1} = ux_n^2 + \mathrm{Sq}^1(x_n)^2$ for $n \geq 1$.

Thirdly, there is the mixed case. These are all other ideals. They can be obtained by applying certain operations to the ideals in the twisted case.

The ideal $\langle u, v\rangle$ is both fibered and twisted. Apart from this every Steenrod closed parameter ideal can be written uniquely as follows:

Theorem 1.2 (Theorem 6.10, Proposition 6.11)**.** *The set of Steenrod closed parameter ideals in $H^*(BA_4;\mathbb{F}_2)$ is the disjoint union of*

1. *the* fibered ideals *$\langle v^k, u^l\rangle$ with $l \geq 1$ and $1 \leq k \leq 2^t$, where 2^t is the largest power of 2 dividing l;*
2. *the* twisted ideals *$\langle x_n, \mathrm{Sq}^1(x_n)\rangle$ for $n \geq 2$, where x_n is recursively defined as $x_1 = u$ and $x_{n+1} = ux_n^2 + \mathrm{Sq}^1(x_n)^2$;*
3. *and* the mixed ideals *$\langle v^i x_n^{2^m}, x_{n+1}^{2^{m-1}}\rangle$, where $m \geq 1$, and either $n = 1$ and $1 \leq i < 2^{m-1}$, or $n \geq 2$ and $0 \leq i < 2^{m-1}$.*

Moreover, for a given pair of natural numbers there is at most one Steenrod closed parameter ideal I with parameters of these degrees.

Similar to Oliver's result, we obtain the following obstruction for the existence of free actions on a product of spheres:

Theorem 1.3 (Theorem 7.5, Theorem 7.7)**.** *Let $R = \mathbb{Z}$ or $\mathbb{F}_2$. Suppose that $G = A_4$ acts freely on a finite CW-complex X such that $H^*(X; R) \cong H^*(S^n \times S^m; R)$ with $1 \le n < m$. Then the classifying map induces a surjection $H^*(BG; \mathbb{F}_2) \to H^*(X/G; \mathbb{F}_2)$ whose kernel J is a Steenrod closed parameter ideal generated by the k-invariants of the spheres. Moreover, the following holds:*

(1) If $R = \mathbb{F}_2$, then the ideal J must be in one of the three types listed in Theorem 1.2.

(2) If $R = \mathbb{Z}$, then the ideal J must be in one of the three types listed in Theorem 1.2 except the twisted ones listed in (2).

The classification of the Steenrod closed parameter ideals provides restrictions for the degrees $m+1, n+1$ of the k-invariants, and hence also for the existence of free actions on a product of two spheres for any group containing A_4, such as finite simple groups of rank 2.

Theorem 1.4 (Corollary 7.8)**.** *Let $G = A_4$ and X be a finite, free G-CW-complex such that $H^*(X; \mathbb{Z}) \cong H^*(S^n \times S^m; \mathbb{Z})$ for some $n, m \ge 1$.*

Then the unordered pair $(n+1, m+1)$ must be one of the following:

(1) $(3k, 2l)$ where k is not larger than the highest power of 2 dividing l, or

(2) $(3i + 2^{s+r+1} - 2^{s+1}, 2^{s+r+1} - 2^s)$ for $s \ge 1$, $r \ge 1$ and $0 \le i < 2^{s-1}$.

Theorem 1.3 for $R = \mathbb{F}_2$ holds more generally for finite CW-complexes whose total mod-2 cohomology is four-dimensional and the top class is the product of two lower-dimensional classes. Sphere bundles over spheres with a free A_4-action satisfy this assumption. It is an interesting question which of the ideals listed in Theorem 1.2 can be realized in this way and which ideals can be realized by a free A_4-action on a finite CW-complex homotopy equivalent to a product of spheres. We use two methods of construction. The first construction uses fixity methods from [ADU04] and [UY10]. We extend them to the octonionic case and realize the following ideals.

Theorem 1.5 (Theorem 8.8)**.** *Any fibered ideal $I = \langle v^k, u^l \rangle$ with $k \le 8$ can be realized by a free A_4-action on the total space of an S^{2l-1}-bundle over S^{3k-1}. If $I \ne \langle v^c, u^c \rangle$ for all $c = 1, 2, 4, 8$, then I can be realized by a trivial bundle, and thus by a free A_4-action on a product of spheres.*

We use [MS03, Theorem 1.2] of Meyer-Smith in Proposition 8.11 to show that fibered ideals $\langle u^{2^r}, v^{2^r} \rangle$ for $2^r \ge 16$ are not realizable by a free A_4-action on a finite CW-complex X with four-dimensional mod-2 cohomology such that the top class is a product of two lower-dimensional classes.

In Section 9, we use the methods due to Adem-Smith introduced in [AS01] and show that for each $k \ge 1$ there is an $l_0 \ge 1$, depending on k, such that for every $s \ge 1$ the ideal $\langle v^k, u^{l_0 s} \rangle$ is realized by a finite, free G-CW-complex homotopy equivalent to $S^{3k-1} \times S^{2l_0 s - 1}$ (see Theorem 9.1).

Our constructions only realize fibered ideals. So the following is an interesting open problem:

Question 1.6. Are the mixed ideals given in Theorem 1.2 realizable by a finite, free G-CW-complex X homotopy equivalent to $S^n \times S^m$?

We visualize the classification and realizability results in Appendix A.

Blaszczyk proved in [Bla13, Proposition 4.3] that A_4 cannot act freely on $S^1 \times S^n$ for $n \geq 1$. We slightly generalize this result in Proposition 7.9 by showing that A_4 cannot act freely on any finite CW-complex with the same mod-2 cohomology algebra as $S^1 \times S^n$.

The classification given in Theorem 1.2 has implications for the Dickson algebra $D(2) = H^*(B\,\mathrm{SO}(3); \mathbb{F}_2) \cong \mathbb{F}_2[a,b]^{\mathrm{GL}_2(2)} \cong \mathbb{F}_2[u,v]$ which is a subalgebra of $H^*(BA_4; \mathbb{F}_2)$. Meyer and Smith classified in [MS03, Theorem V.1.1] all Steenrod closed parameter ideals in the Dickson algebra $D(2)$ generated by powers of u and v. We show in Corollary 6.17 that restriction of ideals from $H^*(BA_4; \mathbb{F}_2)$ to $H^*(B\,\mathrm{SO}(3); \mathbb{F}_2)$ induces a bijection on Steenrod closed parameter ideals. Hence Theorem 1.2 provides a full classification of the Steenrod closed parameter ideals in the Dickson algebra $D(2)$.

Another motivation for considering free A_4-actions on products of spheres is its similarity to the group $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(p)$, with $p > 2$. It is not known whether or not the rank 2 group $Qd(p)$ acts freely on a finite CW-complex homotopy equivalent to a product of two spheres. One of the first steps for constructing such free actions would be deciding on the k-invariants of such an action. The cohomology ring of $\mathrm{Qd}(p)$ is calculated in [Lon08] and its structure is much more complicated than the cohomology of A_4, however there are some similarities between these two groups. If we only focus on the restriction of the $\mathrm{Qd}(p)$-action to its elementary abelian subgroup $P \cong (\mathbb{Z}/p)^2$, then the k-invariants of this restricted action should generate a Steenrod closed parameter ideal in the invariant ring $H^*((\mathbb{Z}/p)^2; \mathbb{F}_p)^{\mathrm{SL}_2(p)}$. Using the methods of this paper one may attempt to classify all Steenrod closed parameter ideals in this invariant ring, and find good candidates for the k-invariants of free $\mathrm{Qd}(p)$-actions on a product of two spheres. It is shown in [OY18, Theorem 1.2] that free $\mathrm{Qd}(p)$-actions on a product of two spheres can not be constructed using spherical fibrations as we did in Theorem 9.1 for A_4, so most likely candidates for such ideals will be analogues of mixed ideals. Finding free actions of A_4 on a product of two spheres realizing the mixed ideals listed in our classification would be interesting from this point of view.

Convention. By an action of a finite group G on a CW-complex X, we mean that the G-space X admits the structure of a G-CW-complex.

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2. Recollection of invariant theory

We are interested in the mod-2 cohomology rings of the groups $\mathrm{SO}(3)$, $A_4 = (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes C_3$, and $\mathbb{Z}/2 \times \mathbb{Z}/2$. We denote the mod-2 cohomology ring of the group G by $H^*(BG)$. The group cohomology of $\mathbb{Z}/2 \times \mathbb{Z}/2$ in $\mathbb{F}_2$-coefficients is the polynomial ring $\mathbb{F}_2[a,b]$ and the other two are the invariant rings

$$H^*(B\,\mathrm{SO}(3)) \cong \mathbb{F}_2[a,b]^{\mathrm{GL}_2(2)},$$
$$H^*(BA_4) \cong \mathbb{F}_2[a,b]^{C_3},$$

where a fixed generator of the cyclic group C_3 of order 3 acts on the vector space $\mathbb{F}_2 a \oplus \mathbb{F}_2 b$ of homogeneous polynomials of degree 1 by $a \mapsto b$, $b \mapsto a+b$. The mod-2 cohomology of the group A_4 can be described as an algebra as follows:

Theorem 2.1 ([AM04, Chapter III, Theorem 1.3])**.** *We have*

$$H^*(BA_4) \cong \mathbb{F}_2[u,v,w]/\langle u^3 + v^2 + vw + w^2\rangle$$

where $\deg(u) = 2$ *and* $\deg(v) = \deg(w) = 3$*. Under the isomorphisms*

$$H^*(BA_4) \cong H^*(B(\mathbb{Z}/2)^2)^{C_3} \cong \mathbb{F}_2[a,b]^{C_3}$$

the generators correspond to

$$\begin{aligned} u &= a^2 + ab + b^2 \\ v &= a^2b + ab^2 = ab(a+b) \\ w &= a^3 + a^2b + b^3. \end{aligned}$$

The invariant ring $\mathbb{F}_2[a,b]^{\mathrm{GL}_2(2)}$ is the *Dickson algebra* $D(2)$ and identifies with the polynomial subring $\mathbb{F}_2[u,v] \subset H^*(BA_4)$; see [AM04, § III.2].

From now on let A be a graded, Noetherian commutative algebra over a field k such that $A_0 = k$.

Definition 2.2. A *parameter ideal* in A is an ideal generated by a system of homogeneous parameters, i.e., by homogeneous elements $a_1, \ldots, a_d$ of degree > 0 such that d is the Krull dimension of A and $A/\langle a_1, \ldots, a_d\rangle$ is finite over k.

If a finite group G acts on A, then the inclusion of invariants $A^G \subset A$ is a finite extension; see [Smi95, Theorem 2.3.1]. Thus A^G and A have the same Krull dimension. Since $H^*(B(\mathbb{Z}/2)^2) \cong \mathbb{F}_2[a,b]$ is of Krull dimension 2, so are the invariant rings $H^*(B\,\mathrm{SO}(3))$ and $H^*(BA_4)$.

The algebra $H^*(BA_4)$ is an invariant ring for the group C_3 whose order is coprime to the characteristic of the field $\mathbb{F}_2$. If the characteristic of k does not divide the order of G, then an application of the Reynolds operator $\mathcal{R}\colon A \to A^G$ defined by

$$\mathcal{R}(a) = \frac{1}{|G|}\sum_{g\in G} ga$$

provides the following result.

Lemma 2.3 (see [DK15, Lemma 2.6.10])**.** *Suppose that* $\operatorname{char} k$ *does not divide* $|G|$*. For any ideal* $I \subset A^G$*, we have* $AI \cap A^G = I$*. In particular* $I \subset A^G$ *is a parameter ideal if and only if* $AI \subset A$ *is a parameter ideal.*

For the $\mathrm{GL}_2(2)$-action on $\mathbb{F}_2[a,b]$, the characteristic of $\mathbb{F}_2$ divides the group order. Nevertheless, we have a similar result.

Lemma 2.4. *For any ideal* $I \subset H^*(B\,\mathrm{SO}(3))$*, we have*

$$H^*(BA_4)I \cap H^*(B\,\mathrm{SO}(3)) = I.$$

Proof. The ideal I is contained in its extension $H^*(BA_4)I$ and in $H^*(B\,\mathrm{SO}(3))$. We show the converse, i.e., that $H^*(BA_4)I \cap H^*(B\,\mathrm{SO}(3)) \subset I$. An element of the extension $H^*(BA_4)I$ is a finite sum $X = \sum_i p_i f_i$ with $p_i \in H^*(BA_4)$ and $f_i \in I$. As a graded module over $H^*(B\,\mathrm{SO}(3)) \cong \mathbb{F}_2[u,v]$, we have a direct sum

$$H^*(BA_4) \cong \mathbb{F}_2[u,v] \oplus \mathbb{F}_2[u,v]w.$$

Thus each p_i can be written uniquely as $p_i = q_i + q_i'w$ with $q_i, q_i' \in \mathbb{F}_2[u,v]$ and X decomposes as

$$X = (\sum_i q_i f_i) + (\sum_i q_i' f_i)w.$$

Suppose that $X \in H^*(B\,\mathrm{SO}(3))$. Then $\sum_i q_i' f_i = 0$ and hence $X = \sum_i q_i f_i$ lies in I. □

Recall that A is Cohen-Macaulay if it has a parameter ideal generated by a regular sequence. We have the following:

Lemma 2.5. *The rings $H^*(BA_4)$ and $H^*(B\,\mathrm{SO}(3))$ are Cohen-Macaulay rings.*

Proof. The polynomial algebra $H^*(B(\mathbb{Z}/2)^2) = \mathbb{F}_2[a,b]$ has a system of parameters given by the regular sequence a, b and thus is Cohen-Macaulay. In a Cohen-Macaulay ring a sequence of $d = \dim A$ elements $a_1, \ldots, a_d$ is a system of homogeneous parameters if and only if it is a regular sequence; see e.g. [NS02, Theorem A.3.5]. By [NS02, Proposition 5.1.1], two homogeneous elements a_1, a_2 in an invariant ring of $\mathbb{F}_2[a,b]$ form a regular sequence in $\mathbb{F}_2[a,b]$ if and only if they form a regular sequence in the invariant ring. Since $\{u, v\}$ is a system of parameters in $\mathbb{F}_2[a,b]$, it is a regular sequence in $\mathbb{F}_2[a,b]$, hence it is a regular sequence in $H^*(BA_4)$ and $H^*(B\,\mathrm{SO}(3))$. It follows that these rings are Cohen-Macaulay. □

All three algebras $H^*(B(\mathbb{Z}/2)^2)$, $H^*(BA_4)$, and $H^*(B\,\mathrm{SO}(3))$ are unique factorization domains; see [Nak82, Theorem 2.11] for $H^*(BA_4)$. This gives the following:

Lemma 2.6. *Let I be an ideal in $H^*(BA_4)$, $H^*(B\,\mathrm{SO}(3))$, or $H^*(B(\mathbb{Z}/2)^2)$ generated by homogeneous elements X and Y of positive degrees. Then the following are equivalent:*

(1) I is a parameter ideal.
(2) X, Y form a regular sequence.
(3) X and Y are coprime.

Proof. Let A denote any of the three cohomology rings. Since A is Cohen-Macaulay and of Krull dimension 2, the first two statements are equivalent. Since A does not have zero-divisors, elements X, Y of positive degree form a regular sequence if and only if multiplication by Y is injective on $A/\langle X\rangle$. This is equivalent to X, Y being coprime if A is a unique factorization domain. □

Our goal is to classify all parameter ideals in $H^*(B\,\mathrm{SO}(3))$ and in $H^*(BA_4)$ that are closed under Steenrod operations.

Definition 2.7. An ideal I in the mod-2 cohomology of a space is *Steenrod closed* if $\mathrm{Sq}(I) \subset I$, where Sq denotes the total Steenrod square.

The total Steenrod square for $H^*(B(\mathbb{Z}/2)^2) = \mathbb{F}_2[a,b]$ is the ring homomorphism determined by

$$\mathrm{Sq}(a) = a^2 + a, \quad \mathrm{Sq}(b) = b^2 + b.$$

Hence the total Steenrod squares of the generators u, v, w for $H^*(BA_4)$ can be computed as follows:

$$\begin{aligned}\mathrm{Sq}(u) &= u + v + u^2,\\ \mathrm{Sq}(v) &= v + uv + v^2,\\ \mathrm{Sq}(w) &= w + u^2 + u(v+w) + w^2.\end{aligned}$$

If $f\colon R \to S$ is a ring homomorphism and $I \subset S$ is an ideal in S, then the ideal $f^{-1}(I)$ is called contraction of I. For an ideal $J \subset R$, the ideal generated by the image $f(J)$ is called an extension of J.

Lemma 2.8. *An ideal $I \subset H^*(BA_4)$ is Steenrod closed if and only if its extension in $\mathbb{F}_2[a,b]$ is Steenrod closed. An ideal $I \subset H^*(B\,\mathrm{SO}(3))$ is Steenrod closed if and only if its extension in $H^*(BA_4)$ is Steenrod closed.*

Proof. Consider the ring homomorphisms

$$f\colon H^*(B\,\mathrm{SO}(3)) \to H^*(BA_4) \quad \text{and} \quad g\colon H^*(BA_4) \to \mathbb{F}_2[a,b]$$

defined by inclusions. Since both f and g preserve Steenrod operations, Steenrod closed ideals in these rings are closed under extension and contraction; see [NS02, Lemma 9.2.2]. By Lemma 2.3, the contraction of the extension $\mathbb{F}_2[a,b]I$, with respect to g, of an ideal $I \subset H^*(BA_4)$ is I itself, and the analogous statement holds for contractions of extensions of ideals $I \subset H^*(B\,\mathrm{SO}(3))$ with respect to f by Lemma 2.4. □

We will use repeatedly and often implicitly the following basic properties:

Proposition 2.9 (see [WW18, Chapter 1])**.** *Let $R = \mathbb{F}_2[a,b]$. The Steenrod operations satisfy the following properties:*

(1) $\mathrm{Sq}\colon R \to R$ *commutes with the* $\mathrm{GL}_2(2)$*-action.*
(2) For $f \in R$ and $s, i \geq 0$,

$$\mathrm{Sq}^i(x^{2^s}) = \begin{cases} (\mathrm{Sq}^j(x))^{2^s} & \text{if } i = 2^s j, \\ 0, & \text{else.} \end{cases}$$

(3) The Cartan formula

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y)$$

holds. Especially, Sq^1 *is a derivation and* $\mathrm{Sq}^1(x^2) = 0$ *for any x.*
(4) The sequence

$$\langle a,b\rangle \xrightarrow{\mathrm{Sq}^1} \langle a,b\rangle \xrightarrow{\mathrm{Sq}^1} \langle a,b\rangle$$

of $\mathbb{F}_2$-vector spaces is exact.

Remark 2.10. The induced sequence

$$\langle a,b\rangle \cap R^G \xrightarrow{\mathrm{Sq}^1} \langle a,b\rangle \cap R^G \xrightarrow{\mathrm{Sq}^1} \langle a,b\rangle \cap R^G$$

is exact for $G = C_3$ since the fixed point functor is exact if the characteristic of the field is coprime to the order of the group. It is not exact for $G = \mathrm{GL}_2(2)$, since there is no element x with $\mathrm{Sq}^1(x) = u^2$.

We will use the following observations to detect invariant elements.

Lemma 2.11. *Suppose that a finite group acts on an integral domain A.*

(1) For any equality $x = \lambda y$ with $x, y \in A^G$, $y \neq 0$ and $\lambda \in A$, it follows that $\lambda \in A^G$.
(2) If the characteristic of A is 2, then square roots are unique. In particular, $x^2 \in A^G$ for $x \in A$ implies $x \in A^G$.

Proof. (1) For any $g \in G$, we have

$$\lambda y = x = gx = (g\lambda)(gy) = (g\lambda)y.$$

Since A has no zero divisors, it follows that $\lambda = g\lambda$.

(2) Let $x, y \in A$. If $0 = x^2 - y^2 = (x-y)^2$, then $x = y$. If $x^2 \in A^G$, then $x^2 = g(x^2) = (gx)^2$ for all $g \in G$. Since square roots in A are unique, it follows that $x = gx$ for all $g \in G$, i.e., $x \in A^G$. □

3. Classification of twisted ideals

In this section, we classify the Steenrod closed parameter ideals in $H^*(BA_4)$ that have a system of parameters of the form $\{X, \mathrm{Sq}^1(X)\}$. We start with a lemma on parameter ideals.

Lemma 3.1. *Let $I \subset H^*(BA_4)$ be a Steenrod closed parameter ideal. Choose a homogeneous system $\{X, Y\}$ of parameters for I, then $|X| \neq |Y|$ and the parameter of lower degree as well as the degrees of the parameters are independent of the choice.*

Proof. Any Steenrod closed ideal generated by homogeneous elements of the same degree is of the form $\langle v^i \rangle$ by Theorem 1.1 and thus cannot be a parameter ideal. Hence X and Y must have different degrees. If $|X| < |Y|$, then any homogeneous element of I has degree at least $|X|$ and the homogeneous elements of degree $|X|$ form a one dimensional vector space over $\mathbb{F}_2$. Hence X is independent of the choice of system of parameters. If $\{X, Y'\}$ is another system of parameters, then $Y' = \lambda Y + \mu X$ for homogeneous coefficients $\mu, \lambda \in H^*(BA_4)$. The coefficient λ is nonzero since I is a parameter ideal. Hence $|Y| \leq |Y'|$ and analogously $|Y'| \leq |Y|$. Thus the degree of Y is independent of the choice as well. □

Lemma 3.2. *If $\langle X, Y \rangle$ is a Steenrod closed ideal in $H^*(BA_4)$ where X and Y are homogeneous elements with $|X| < |Y|$, then either $\mathrm{Sq}^1(X) = 0$ or $Y = \mathrm{Sq}^1(X)$.*

Proof. We can write $\mathrm{Sq}^1(X) = \lambda X + \nu Y$ for some elements λ and ν in $H^*(BA_4)$. Since $H^1(BA_4) = 0$, we have $\lambda = 0$. The element ν is of degree 0, so it is either 0 or 1. □

Definition 3.3. We call a Steenrod closed parameter ideal $I \subset H^*(BA_4)$ *twisted* if it has a system of parameters of the form $\{X, \mathrm{Sq}^1(X)\}$. Otherwise it is called *nontwisted*.

Remark 3.4. If $I \subset H^*(BA_4)$ is twisted, then I has a unique homogeneous system of parameters. Indeed, the parameter of lowest degree X is unique by Lemma 3.1 and the only possibility for the second parameter is $\mathrm{Sq}^1(X)$ by Lemma 3.2.

For cohomology classes in $H^*(B(\mathbb{Z}/2)^2)$ or $H^*(BA_4)$, we have the following observations (see [Oli79, Proof of Lemma 1]).

Lemma 3.5. *Suppose that X is a homogeneous element of degree n in $\mathbb{F}_2[a, b]$ which satisfies $\mathrm{Sq}^1(X) = 0$.*

(1) If n is even, then $X = x^2$ for some homogeneous $x \in \mathbb{F}_2[a, b]$.
(2) If n is odd, then $X = vx^2$ for some homogeneous $x \in \mathbb{F}_2[a, b]$.

If $X \in H^*(BA_4)$, then x is an element of $H^*(BA_4)$ in either case.

Proof. If n is even, then we can write $X = x^2 + abt^2$ for some $x, t \in \mathbb{F}_2[a, b]$. This gives

$$0 = \mathrm{Sq}^1(x^2 + abt^2) = (a^2b + ab^2)t^2,$$

which implies that $t = 0$. So X is the square of x. If $X \in H^*(BA_4) = \mathbb{F}_2[a, b]^{C_3}$, then it follows that $x \in H^*(BA_4)$ by Lemma 2.11.

If n is odd, we can write $X = ay^2 + bz^2$. Computing $\mathrm{Sq}^1(X)$, we get

$$0 = \mathrm{Sq}^1(X) = a^2y^2 + b^2z^2 = (ay + bz)^2,$$

which gives $ay = bz$. Hence $y = bx$ and $z = ax$ for some $x \in \mathbb{F}_2[a, b]$. It follows that $X = ab^2x^2 + a^2bx^2 = vx^2$. By Lemma 2.11, if $X \in H^*(BA_4) = \mathbb{F}_2[a, b]^{C_3}$, then $x \in H^*(BA_4)$. □

Any element in $\mathbb{F}_2[a, b]$ can be uniquely written in the form $x^2 + ay^2 + bz^2 + abt^2$, and if that element is concentrated in even or in odd degrees, half of the summands vanish.

Definition 3.6. We write

$$\kappa = (\kappa_1, \kappa_a, \kappa_b, \kappa_{ab})\colon \mathbb{F}_2[a, b] \to \mathbb{F}_2[a, b]^4$$

for the $\mathbb{F}_2$-linear map that sends $f = x^2 + ay^2 + bz^2 + abt^2$ to (x, y, z, t).

Remark 3.7. For each $* \in \{1, a, b, ab\}$, the map κ_* satisfies $\kappa_*(xy^2) = \kappa_*(x)y$. From the definition of κ we immediately get

$$\kappa_1(x) = \kappa_{ab}(abx),\ \ \kappa_a(x) = \kappa_{ab}(bx),\ \text{ and }\ \kappa_b(x) = \kappa_{ab}(ax).$$

The restriction of κ_{ab} to homogeneous elements of even degrees is the down Kameko map from [MS05, Definition 1.6.2].

We will now classify all twisted ideals as defined in Definition 3.3. The classification result states that all parameters of twisted ideals are obtained from $\{1, 0\}$ by a recursion formula.

Definition 3.8. For $n \geq 1$, let (x_n, y_n) be the sequence recursively defined by $(x_1, y_1) = (u, v)$ and

$$x_n = ux_{n-1}^2 + y_{n-1}^2 \ \text{ and }\ y_n = vx_{n-1}^2$$

for $n \geq 2$.

Note that since $\mathrm{Sq}(u) = u + v + u^2$ and $\mathrm{Sq}(v) = v + uv + v^2$, the ideal $\langle u, v\rangle$ is a Steenrod closed parameter ideal and since $\mathrm{Sq}^1(u) = v$, it is a twisted ideal. For $n = 2$, we have $(x_2, y_2) = (u^3 + v^2, vu^2)$, and by direct calculation one can show that $\langle x_2, y_2\rangle$ is a twisted ideal. Our first observation is that all pairs (x_n, y_n) obtained this way are parameters of a twisted ideal for all $n \geq 1$.

Lemma 3.9. *Let (μ_n) be the sequence recursively defined as $\mu_0 = 1$ and*

$$\mu_n = (1 + u + v)\mu_{n-1}^2 + x_n^2$$

for $n \geq 1$. For the sequence of pairs (x_n, y_n) defined in Definition 3.8, we have

$$\mathrm{Sq}(x_n) = \mu_{n-1}(x_n + y_n) + x_n^2,$$
$$\mathrm{Sq}(y_n) = v\mu_{n-1}x_n + ((u + 1)\mu_{n-1} + x_n)y_n + y_n^2.$$

In particular for all $n \geq 1$, the pairs (x_n, y_n) generate a Steenrod closed parameter ideal and $\mathrm{Sq}^1(x_n) = y_n$, thus $\langle x_n, y_n\rangle$ is twisted.

Proof. From the recursion formula in Definition 3.8, it is clear that x_n and y_n are coprime for all $n \geq 1$. Since their degrees are positive, they generate parameter ideals. We have that $\mathrm{Sq}^1(x_1) = \mathrm{Sq}^1(u) = v = y_1$ and $\mathrm{Sq}^1(x_n) = \mathrm{Sq}^1(u)x_{n-1}^2 = vx_{n-1}^2 = y_n$ for $n \geq 2$.

We show that the formulas for $\mathrm{Sq}(x_n)$ and $\mathrm{Sq}(y_n)$ hold by induction on n. The base case $n = 1$ follows immediately by inserting $x_1 = u$, $y_1 = v$, and $\mu_0 = 1$ to the equations. Now assume the above equations hold for some pair (x_n, y_n) for $n \geq 1$. We will show that they also hold for the pair (x_{n+1}, y_{n+1}). To make the equations simpler, we write $(x, y) = (x_n, y_n)$, $(X, Y) = (x_{n+1}, y_{n+1})$, $\mu = \mu_{n-1}$, and $\eta = \mu_n$. So we have $X = ux^2 + y^2$, $Y = vx^2$, and $\eta = (1 + u + v)\mu^2 + x^2$. This gives

$$\begin{aligned}\mathrm{Sq}(X) =& \mathrm{Sq}(ux^2 + y^2) = (u + v + u^2)\,\mathrm{Sq}(x)^2 + \mathrm{Sq}(y)^2 \\ =&(u + v + u^2)(\mu(x + y) + x^2)^2 + (v\mu x)^2 + ((u + 1)\mu + x)^2 y^2 + y^4 \\ =&(u + v)(\mu^2 x^2 + \mu^2 y^2 + x^4) + u^2\mu^2 x^2 + u^2\mu^2 y^2 + u^2 x^4 + v^2\mu^2 x^2 \\ &+ (u^2 + 1)\mu^2 y^2 + x^2 y^2 + y^4 \\ =&(\mu^2 + u\mu^2 + v\mu^2 + x^2)(ux^2 + y^2 + vx^2) + (ux^2 + y^2)^2 \\ =&\eta(X + Y) + X^2.\end{aligned}$$

Similarly for $\mathrm{Sq}(Y)$ we have

$$\begin{aligned}\mathrm{Sq}(Y) =& \mathrm{Sq}(vx^2) = (v + uv + v^2)(\mu(x + y) + x^2)^2 \\ =&v\mu^2 x^2 + v\mu^2 y^2 + vx^4 + uv\mu^2 x^2 + uv\mu^2 y^2 + uvx^4 \\ &+ v^2\mu^2 x^2 + v^2\mu^2 y^2 + v^2 x^4 \\ =&v(\mu^2 + u\mu^2 + v\mu^2 + x^2)(ux^2 + y^2) + (1 + u^2 + uv + v)\mu^2 vx^2 \\ &+ (x^2 + y^2)vx^2 + v^2 x^4 \\ =&v\eta X + ((u + 1)\eta + X)Y + Y^2.\end{aligned}$$ □

We record the following calculation for the classification of Steenrod closed parameter ideals in Section 6.

Lemma 3.10. *In $\mathbb{F}_2[u, v]/\langle v\rangle = \mathbb{F}_2[u]$, for every $n \geq 1$, we have $x_n \equiv u^{2^n - 1}$, $y_n \equiv 0$, and for $n \geq 0$,*

$$\mu_n \equiv \sum_{i=0}^{2^{n+1}-2} u^i = (1 + u^{2^{n+1}-1})/(1 + u).$$

Proof. Since $y_1 = v$ and $y_n = vx_{n-1}^2$, it follows that $y_n \equiv 0$ modulo $\langle v\rangle$. The other formulas hold by a straightforward induction. □

In addition to the inductive definition of x_n, y_n, we can express them explicitly as elements in $\mathbb{F}_2[a, b]$.

Remark 3.11. For $n \geq 1$, let $m = 2^{n+1} - 2$. Then for all $n \geq 1$,

$$x_n = \sum_{i=0}^{m} a^i b^{m-i} = (a^{m+1} + b^{m+1})/(a + b),$$

$$y_n = ab \sum_{i=0}^{m-1} a^i b^{m-i} = ab(a^m + b^m)/(a + b).$$

Note that for $n = 1$, we have $x_1 = u = a^2 + ab + b^2 = (a^3 + b^3)/(a + b)$ and $y_1 = v = ab(a + b) = ab(a^2 + b^2)/(a + b)$. The general case can be proved easily by induction.

Now we prove that every twisted ideal $\langle X, \mathrm{Sq}^1(X)\rangle$ satisfies $X = x_n$ for some $n \geq 1$. We do this by considering two separate cases.

Lemma 3.12. *Let $X \in H^*(BA_4)$ be a homogeneous element of even degree such that $\langle X, \mathrm{Sq}^1(X)\rangle$ is Steenrod closed. Then $\mathrm{Sq}^1(X) = 0$ or $X = uy^2 + \mathrm{Sq}^1(y)^2$ for some $y \in H^*(BA_4)$ and $\langle y, \mathrm{Sq}^1(y)\rangle$ is also Steenrod closed.*

Proof. Recall that $H^*(BA_4) \cong \mathbb{F}_2[a,b]^{C_3}$, where a generator of C_3 acts on $\mathbb{F}_2[a,b]$ via the automorphism φ given by $\varphi(a) = b$, $\varphi(b) = a+b$. Write X in the form $X = x^2 + aby^2$. Then

$$x^2 + aby^2 = X = \varphi(X) = \varphi(x)^2 + b(a+b)\varphi(y)^2 = \varphi(x)^2 + b^2\varphi(y)^2 + ab\varphi(y)^2.$$

Since the way to express X in the form above is unique, we get $\varphi(y) = y$ and hence $y \in H^*(BA_4)$. Using the Cartan formula, we get

$$\mathrm{Sq}^2(X) = \mathrm{Sq}^1(x)^2 + a^2b^2y^2 + ab\,\mathrm{Sq}^1(y)^2.$$

Since $\mathrm{Sq}^2(X) \in \langle X, \mathrm{Sq}^1(X)\rangle$ by assumption, we can write $\mathrm{Sq}^2(X)$ as a linear combination of X and $\mathrm{Sq}^1(X)$ with coefficients in $H^*(BA_4)$. For degree reasons, the coefficient of $\mathrm{Sq}^1(X)$ has to be zero. We thus get

$$\mathrm{Sq}^2(X) \in \{0, uX\}.$$

If $\mathrm{Sq}^2(X) = 0$, then

$$0 = \kappa_1(\mathrm{Sq}^2(X)) = \mathrm{Sq}^1(x) + aby,$$
$$0 = \kappa_{ab}(\mathrm{Sq}^2(X)) = \mathrm{Sq}^1(y).$$

Applying Sq^1 on both sides of the first equation yields

$$0 = \mathrm{Sq}^1(\mathrm{Sq}^1(x) + aby) = \mathrm{Sq}^1(\mathrm{Sq}^1(x)) + \mathrm{Sq}^1(ab)y + ab\,\mathrm{Sq}^1(y) = 0 + vy + 0 = vy.$$

As $\mathrm{Sq}^1(X) = vy^2$, it follows that $\mathrm{Sq}^1(X) = 0$. This completes the first case.

Now assume that $\mathrm{Sq}^2(X) = uX$. Then we have

$$\begin{aligned}\mathrm{Sq}^1(x)^2 + a^2b^2y^2 + ab\,\mathrm{Sq}^1(y)^2 &= (a^2+ab+b^2)(x^2+aby^2)\\ &= (a^2+b^2)x^2 + a^2b^2y^2 + ab(x^2+a^2y^2+b^2y^2).\end{aligned}$$

Applying κ_1 and κ_{ab} yields

$$\mathrm{Sq}^1(x) + aby = (a+b)x + aby,$$
$$\mathrm{Sq}^1(y) = x + ay + by.$$

Solving the last equation for x and inserting it into the definition of X gives

$$X = x^2 + aby^2 = (a^2+b^2)y^2 + \mathrm{Sq}^1(y)^2 + aby^2 = uy^2 + \mathrm{Sq}^1(y)^2.$$

So X has indeed the required form.

It remains to show that the ideal $\langle y, \mathrm{Sq}^1(y)\rangle$ in $H^*(BA_4)$ is Steenrod closed. By Lemma 2.8, it suffices to show that the extension $\langle y, \mathrm{Sq}^1(y)\rangle \subset \mathbb{F}_2[a,b]$ is Steenrod closed. Since the ideal $\langle X, \mathrm{Sq}^1(X)\rangle$ is Steenrod closed, we can find coefficients λ, μ such that $\mathrm{Sq}(X) = \lambda X + \mu\,\mathrm{Sq}^1(X)$. This gives

$$\mathrm{Sq}(u)\,\mathrm{Sq}(y)^2 + \mathrm{Sq}(\mathrm{Sq}^1(y))^2 = \lambda uy^2 + \lambda\,\mathrm{Sq}^1(y)^2 + \mu vy^2. \tag{3.13}$$

Let $* \in \{1, a, b, ab\}$. Using Remark 3.7, the map κ_* applied to the right-hand side of (3.13) yields

$$\kappa_*(\lambda u)y + \kappa_*(\lambda)\,\mathrm{Sq}^1(y) + \kappa_*(\mu v)y,$$

which is an element of $\langle y, \mathrm{Sq}^1(y)\rangle$. Since

$$\mathrm{Sq}(u) = u + v + u^2 = a^2 + ab + b^2 + a^2 b + ab^2 + a^4 + (ab)^2 + b^4,$$

κ_{ab} applied to the left-hand side of (3.13) yields $\mathrm{Sq}(y)$. Hence $\mathrm{Sq}(y)$ is in the ideal $\langle y, \mathrm{Sq}^1(y)\rangle$. Applying κ_1 to the left-hand side yields

$$(a + b + a^2 + ab + b^2)\,\mathrm{Sq}(y) + \mathrm{Sq}(\mathrm{Sq}^1(y)).$$

It follows that $\mathrm{Sq}(\mathrm{Sq}^1(y))$ also lies in the ideal $\langle y, \mathrm{Sq}^1(y)\rangle$. □

If $\mathrm{Sq}^1(X) = 0$, then $I = \langle X, \mathrm{Sq}^1(X)\rangle = \langle X\rangle$ is not a parameter ideal.

Lemma 3.14. *Let $X \in H^*(BA_4)$ be a homogeneous element of odd degree such that $\langle X, \mathrm{Sq}^1(X)\rangle$ is Steenrod closed. Then both X and $\mathrm{Sq}^1(X)$ are divisible by v, hence in this case $\langle X, \mathrm{Sq}^1(X)\rangle$ is not a parameter ideal.*

Proof. If an element X is divisible by v, e.g. $X = vz$, we get that

$$\mathrm{Sq}^1(X) = \mathrm{Sq}^1(vz) = \mathrm{Sq}^1(v)z + v\,\mathrm{Sq}^1(z) = 0 + v\,\mathrm{Sq}^1(z)$$

is also divisible by v. So it is enough to show that X is divisible by v. The last part will follow from Lemma 2.6. Since 0 is divisible by v, we may assume $X \neq 0$.

Write $X = ax^2 + by^2$ and thus $\mathrm{Sq}^1(X) = a^2x^2 + b^2y^2$. Now use again that $\mathrm{Sq}^2(X) \in \{0, uX\}$ for degree reasons. The Cartan-formula shows

$$\mathrm{Sq}^2(X) = a\,\mathrm{Sq}^1(x)^2 + b\,\mathrm{Sq}^1(y)^2.$$

Let us first look at the easier case of $\mathrm{Sq}^2(X) = uX$. In this case we have

$$a\,\mathrm{Sq}^1(x)^2 + b\,\mathrm{Sq}^1(y)^2 = (a^2 + ab + b^2)(ax^2 + by^2).$$

Applying κ_a and κ_b yields

$$\begin{aligned}\mathrm{Sq}^1(x) &= ax + bx + by,\\ \mathrm{Sq}^1(y) &= ax + ay + by.\end{aligned}$$

Now we can use that $\mathrm{Sq}^1 \circ \mathrm{Sq}^1 = 0$:

$$\begin{aligned}0 &= \mathrm{Sq}^1(ax + bx + by)\\ &= a^2x + a\,\mathrm{Sq}^1(x) + b^2x + b\,\mathrm{Sq}^1(x) + b^2y + b\,\mathrm{Sq}^1(y)\\ &= a^2x + (a + b)(ax + bx + by) + b^2x + b^2y + b(ax + ay + by)\\ &= abx + b^2y.\end{aligned}$$

Thus we get that $ax = by$. Hence x is divisible by b and we can substitute $x = bz$ to obtain $y = az$. Inserting this in the definition of X yields

$$X = (ab^2 + a^2b)z^2 = vz^2.$$

Hence X is divisible by v. In fact in this case we have $\mathrm{Sq}^1(X) = 0$.

Let us now look at the case of $\mathrm{Sq}^2(X) = 0$. From the equation

$$\mathrm{Sq}^2(X) = a\,\mathrm{Sq}^1(x)^2 + b\,\mathrm{Sq}^1(y)^2$$

we get $\mathrm{Sq}^1(x) = 0$ and $\mathrm{Sq}^1(y) = 0$.

If $|x|$ is odd, this means by Lemma 3.5 that $x = vz^2$ and $y = vt^2$ for some $z, t \in \mathbb{F}_2[a, b]$. Thus both X and $\mathrm{Sq}^1(X)$ would be divisible by v in $\mathbb{F}_2[a, b]$ and hence also in $H^*(BA_4)$ by Lemma 2.11.

If $|x|$ is even, then consider the ideal J generated by $ax + by$ and $\mathrm{Sq}^1(ax + by) = a^2x + b^2y$. We want to show that J is again Steenrod closed. Since

$$a^{n+1}x + b^{n+1}y = (a + b)(a^n x + b^n y) + ab(a^{n-1}x + b^{n-1}y)$$

it follows inductively that J contains all elements of the form $a^n x + b^n y$ for $n \geq 1$ and thus also all elements of the form

$$a^n by + ab^n x = a^{n+1}x + b^{n+1}y + (a^n + b^n)(ax + by).$$

Since $\mathrm{Sq}(X) = \mathrm{Sq}(ax^2 + by^2)$ is in $\langle X, \mathrm{Sq}^1(X)\rangle = \langle ax^2 + by^2, a^2x^2 + b^2y^2\rangle$, there exist homogeneous elements λ, μ such that

$$(a + a^2)\,\mathrm{Sq}(x)^2 + (b + b^2)\,\mathrm{Sq}(y)^2 = \mathrm{Sq}(ax^2 + by^2) = \lambda(ax^2 + by^2) + \mu(a^2x^2 + b^2y^2).$$

Applying κ_a and κ_b to both sides of the equation yields

$$\begin{aligned}\mathrm{Sq}(x) &= \kappa_1(\lambda)x + \kappa_a(\lambda b)y + \kappa_a(\mu)(ax + by),\\ \mathrm{Sq}(y) &= \kappa_b(\lambda a)x + \kappa_1(\lambda)y + \kappa_b(\mu)(ax + by).\end{aligned}$$

Since $\kappa_a(\lambda b) = \kappa_{ab}(\lambda)b$ and $\kappa_b(\lambda a) = \kappa_{ab}(\lambda)a$, we get that for any $n \geq 1$:

$$\begin{aligned}a^n\,\mathrm{Sq}(x) + b^n\,\mathrm{Sq}(y) =& \kappa_1(\lambda)(a^n x + b^n y) + \kappa_{ab}(\lambda)(a^n by + ab^n x)\\ &+ (a^n\kappa_a(\mu) + b^n\kappa_b(\mu))(ax + by)\end{aligned}$$

is in J. Since $\mathrm{Sq}(ax + by) = (a + a^2)\,\mathrm{Sq}(x) + (b + b^2)\,\mathrm{Sq}(y)$ and $\mathrm{Sq}(a^2x + b^2y) = (a^2 + a^4)\,\mathrm{Sq}(x) + (b^2 + b^4)\,\mathrm{Sq}(y)$, we see that J is Steenrod closed.

Note that $|X| > 1$ since $H^1(BA_4) = 0$. Thus the degree of $ax+by$ is smaller than the degree of X. Since $ax + by$ is the square root of $\mathrm{Sq}^1(X)$ it is also in $H^*(BA_4)$ by Lemma 2.11. It follows by induction that $v = ab(a + b)$ divides both $ax + by$ and $a^2x + b^2y$.

We show that v divides $X = ax^2 + by^2$. Since $ax^2 + by^2 = (x + y)(ax + by) + xy(a + b)$ and $v|(ax + by)$, it suffices to show that $v|xy(a + b)$. Since

$$v|(a^2x + b^2y)(x + y) - (ax + by)^2 = xy(a^2 + b^2)$$

and $v = ab(a + b)$ it follows that $ab|xy$ and hence $v|xy(a + b)$. ☐

We summarize the results of this section in the following theorem.

Theorem 3.15. *Let X be a homogeneous element in the cohomology ring $H^*(BA_4)$. The ideal $\langle X, \mathrm{Sq}^1(X)\rangle$ is a Steenrod closed parameter ideal if and only if there exists an $n \geq 1$ such that $X = x_n$, where x_n is defined recursively via $x_1 = u$ and $x_{n+1} = ux_n^2 + \mathrm{Sq}^1(x_n)^2$ for $n \geq 1$.*

Proof. By Lemma 3.9, we know that the ideal $\langle x_n, \mathrm{Sq}^1(x_n)\rangle$ is a Steenrod closed parameter ideal for any $n \geq 1$. Conversely, let $X \in H^*(BA_4)$ be a homogeneous element such that $\langle X, \mathrm{Sq}^1(X)\rangle$ is a Steenrod closed parameter ideal. In particular, we have $\mathrm{Sq}^1(X) \neq 0$. We show by induction on the degree of X that it is of the given form. If $|X|$ is odd, then Lemma 3.14 tells us that v divides X, and hence X and $\mathrm{Sq}^1(X)$ cannot be coprime. If $|X|$ is even, then it follows from Lemma 3.12 that $X = uY^2 + \mathrm{Sq}^1(Y)^2$ for some Y such that $\langle Y, \mathrm{Sq}^1(Y)\rangle$ is also Steenrod closed. If Y and $\mathrm{Sq}^1(Y)$ had a common divisor, the same divisor would also divide X and $\mathrm{Sq}^1(X)$. So Y and $\mathrm{Sq}^1(Y)$ are coprime. If $|Y| = 0$, then $X = u = x_1$. If $|Y| > 0$, then $\langle Y, \mathrm{Sq}^1(Y)\rangle$ is a parameter ideal and hence by induction assumption we have $Y = x_n$ for some $n \geq 1$ and thus $X = x_{n+1}$. ☐

4. Observations on nontwisted ideals

In this section we are interested in nontwisted Steenrod closed parameter ideals of $H^*(BA_4)$. By Lemma 3.2, these are the Steenrod closed parameter ideals such that Sq^1 of the parameter of lower degree is zero. The results will be used in the following two sections to classify all Steenrod closed parameter ideals.

Lemma 4.1. *Assume that $\langle X, Y\rangle$ is a Steenrod closed parameter ideal in $H^*(BA_4)$ where X and Y are homogeneous elements with $|X| < |Y|$ and that $\mathrm{Sq}^1(X) = 0$. There exists a homogeneous element $Y' \in H^*(BA_4)$ with $\langle X, Y\rangle = \langle X, Y'\rangle$ and $\mathrm{Sq}^1(Y') = 0$.*

Proof. Since $H^1(BA_4) = 0$, we have $\mathrm{Sq}^1(Y) = \lambda X$ for some $\lambda \in H^{|Y|+1-|X|}(BA_4)$. Since $(\mathrm{Sq}^1)^2 = 0$ and $\mathrm{Sq}^1(X) = 0$, we obtain

$$0 = \mathrm{Sq}^1(\lambda X) = \mathrm{Sq}^1(\lambda)X + \lambda\,\mathrm{Sq}^1(X) = \mathrm{Sq}^1(\lambda)X.$$

Since $H^*(BA_4)$ does not have zero divisors and $X \neq 0$, we have $\mathrm{Sq}^1(\lambda) = 0$ and by exactness of Sq^1, there exists $\lambda' \in H^{|Y|-|X|}(BA_4)$ such that $\mathrm{Sq}^1(\lambda') = \lambda$.

Setting $Y' = Y + \lambda' X$ provides a homogeneous element such that $\langle X, Y\rangle = \langle X, Y'\rangle$ and $\mathrm{Sq}^1(Y + \lambda' X) = \lambda X + \lambda X = 0$. □

We will be interested in Steenrod closed parameter ideals generated by squares. Therefore we introduce the following notation.

Definition 4.2 ([DK15, §2.6])**.** For an ideal I of a commutative ring R of characteristic 2, we define $I^{[2]}$ to be the ideal generated by all x^2 with $x \in I$.

If $I = \langle x, y\rangle$, then $I^{[2]} = \langle x^2, y^2\rangle$. Note that $I^{[2]}$ differs from $I^2 = \langle x^2, xy, y^2\rangle$ in general.

Lemma 4.3. *The operation $I \mapsto I^{[2]}$ is injective on parameter ideals in $H^*(BA_4)$.*

Proof. Let $I = \langle x, y\rangle$ and $J = \langle x', y'\rangle$ be parameter ideals in $H^*(BA_4)$ with $I^{[2]} = J^{[2]}$. By symmetry, it is enough to show that $J \subset I$. If $|x| = |y|$, then $|x'| = |y'|$. Hence $(x')^2$ and $(y')^2$ are $\mathbb{F}_2$-linear combinations of x^2 and y^2. In particular, the coefficients are squares themselves, say $(x')^2 = \alpha^2 x^2 + \beta^2 y^2$. It follows that $x' = \alpha x + \beta y$. Similarly, $y' \in I$ and hence $J \subset I$.

If $|x| \neq |y|$, then after renaming the parameters, we may assume that $|x| < |y|$ and $|x'| < |y'|$. For degree reasons and since we work over $\mathbb{F}_2$, we have $(x')^2 = x^2$ and $(y')^2 = y^2 + \lambda x^2$ for some homogeneous element $\lambda \in H^*(BA_4)$. By Lemma 2.11, squares are unique in $H^*(BA_4)$, thus $x^2 = (x')^2$ implies $x' = x$.

The equation $(y')^2 = y^2 + \lambda x^2$ gives that x^2 divides $(y' + y)^2$. Since $H^*(BA_4)$ is a unique factorization domain, we obtain that x divides $y + y'$. Let $\mu \in H^*(BA_4)$ be such that $y + y' = \mu x$. Then $y' = y + \mu x$ and $x' = x$, hence $J \subset I$. □

Lemma 4.4. *Let I be an ideal in $H^*(BA_4)$. Then $I^{[2]}$ is Steenrod closed if and only if I is Steenrod closed. Moreover, $I^{[2]}$ is a parameter ideal if and only if I is a parameter ideal.*

Proof. If I is Steenrod closed, then $I^{[2]}$ is Steenrod closed since $I^{[2]}$ is the extension of the ideal I with respect to the Frobenius homomorphism on $H^*(BA_4)$ and the Frobenius homomorphism commutes with Sq.

Conversely, suppose that $I^{[2]}$ is Steenrod closed. We show that I is Steenrod closed. If $x \in I$ then $\mathrm{Sq}(x^2) \in I^{[2]}$ by assumption and we can find a linear combination

$$\mathrm{Sq}(x)^2 = \mathrm{Sq}(x^2) = \sum_i \lambda_i y_i^2$$

with $y_i \in I$. Applying κ_1 yields with the help of Remark 3.7

$$\mathrm{Sq}(x) = \sum_i \kappa_1(\lambda_i) y_i.$$

Since the coefficients $\kappa_1(\lambda_i) \in \mathbb{F}_2[a,b]$ need not be in $H^*(BA_4) = \mathbb{F}_2[a,b]^{C_3}$, this only shows that the extension of I to $\mathbb{F}_2[a,b]$ is Steenrod closed. Nevertheless, it follows that I is Steenrod closed by Lemma 2.8.

The second statement holds since two homogeneous elements $x, y \in H^*(BA_4)$ are coprime if and only if x^2 and y^2 are so. □

The following is immediate from the above results.

Proposition 4.5. *If I is a Steenrod closed parameter ideal in $H^*(BA_4)$ generated by parameters of even degrees, then $I = J^{[2]}$ for some Steenrod closed parameter ideal J.*

Proof. Let X and Y be parameters of I with $|X| < |Y|$. Since X and Y have even degrees, I cannot be twisted. By Lemma 3.2, we have for a nontwisted $I = \langle X, Y\rangle$ that $\mathrm{Sq}^1(X) = 0$ and by Lemma 4.1, we can pick the second parameter Y' such that $\mathrm{Sq}^1(Y') = 0$. By Lemma 3.5, there exist x, y such that $X = x^2$ and $Y' = y^2$. Hence if we take $J = \langle x, y\rangle$, then $I = J^{[2]}$. The lemma now follows form Lemma 4.4. □

Now we consider the case where one of the degrees $|X|$ and $|Y|$ is odd. For this case the following lemma is useful.

Lemma 4.6. *Let $x, y \in H^*(BA_4)$ be homogeneous elements such that $|x| \neq 0$ and $\langle vx^2, y^2\rangle$ is a Steenrod closed parameter ideal. Then $\langle x, y\rangle$ is a Steenrod closed parameter ideal.*

Proof. If $\langle vx^2, y^2\rangle$ is a parameter ideal, then so is $\langle x, y\rangle$ since the former is contained in the latter. It remains to check that $\langle x, y\rangle$ is closed under the Steenrod operations. By Lemma 2.8, it suffices to show that its extension J in $\mathbb{F}_2[a,b]$ is Steenrod closed. Let us first look at

$$\mathrm{Sq}(y)^2 = \mathrm{Sq}(y^2) = \lambda v x^2 + \mu y^2.$$

Applying κ_1 yields

$$\mathrm{Sq}(y) = \kappa_1(\lambda v)x + \kappa_1(\mu)y$$

and thus $\mathrm{Sq}(y)$ lies in J. The trickier part is to show that $\mathrm{Sq}(x)$ belongs to J. Since $\langle vx^2, y^2\rangle$ is a Steenrod closed parameter ideal, we have

$$v(1+u+v)\,\mathrm{Sq}(x)^2 = \mathrm{Sq}(vx^2) = \alpha v x^2 + \theta y^2$$

for some α and θ. Since y is coprime to vx^2, θ has to be divisible by v. We thus get with $\theta = v\theta'$:

$$(1+u+v)\,\mathrm{Sq}(x)^2 = \alpha x^2 + \theta' y^2. \tag{4.7}$$

Recall that

$$1+u+v = 1+a^2+ab+b^2+a^2b+ab^2 = (1+a^2+b^2)+ab^2+a^2b+ab$$

and thus $\kappa_{ab}(1+u+v)=1$. Applying κ_{ab} to (4.7) yields $\mathrm{Sq}(x)=\kappa_{ab}(\alpha)x+\kappa_{ab}(\theta')y$ and thus $\mathrm{Sq}(x)\in J$. □

Recall from Lemma 3.1 that the degrees of parameters generating a Steenrod closed parameter ideal are independent of the system of parameters of the ideal.

Proposition 4.8. *For a nontwisted Steenrod closed parameter ideal $\langle X,Y\rangle$ in $H^*(BA_4)$ the following hold:*

(1) Both $|X|$ and $|Y|$ cannot be odd.
(2) If one of $|X|$, $|Y|$ is odd and >3, then there exists a Steenrod closed parameter ideal $\langle x,y\rangle$ such that $\langle X,Y\rangle=\langle vx^2,y^2\rangle$.
(3) If one of $|X|$, $|Y|$ is 3, then $\langle X,Y\rangle=\langle v,y^2\rangle$ for some homogeneous element $y\in H^*(BA_4)$.

Proof. We may assume that $|X|<|Y|$. Since the ideal $\langle X,Y\rangle$ is nontwisted, by Lemma 3.2 we have $\mathrm{Sq}^1(X)=0$, and by Lemma 4.1 we can pick the second generator Y' such that $\mathrm{Sq}^1(Y')=0$. By Lemma 3.5, if both $|X|$ and $|Y|$ were odd, then X and Y' would be divisible by v and thus they cannot form a system of parameters. If only one of the degrees of X or Y is odd, then $\langle X,Y\rangle=\langle vx^2,y^2\rangle$ by Lemma 3.5. We conclude that (3) holds and (2) follows from Lemma 4.6. □

Usually it takes a lengthy computation to check if two elements generate a Steenrod closed parameter ideal. The next lemma provides a helpful criterion in a special case.

Lemma 4.9. *Suppose $\langle X,Y\rangle$ is a Steenrod closed ideal generated by homogeneous elements X,Y in $H^*(BA_4)$ and let $k\geq 1$. Then $\langle v^kX,Y\rangle$ is Steenrod closed if and only if there exist coefficients λ,μ with*

$$\mathrm{Sq}(Y)=\lambda X+\mu Y$$

such that v^k divides λ. Furthermore if X,Y are coprime, this is equivalent to asking that for any choice of coefficients λ,μ as above, we have $\lambda\in\langle v^k,Y\rangle$

Proof. Suppose that there are such coefficients and let $\lambda=v^k\lambda'$. We then have

$$\mathrm{Sq}(Y)=\lambda' v^kX+\mu Y\in\langle v^kX,Y\rangle,$$
$$\mathrm{Sq}(v^kX)=v^k(1+u+v)^k\,\mathrm{Sq}(X)\in v^k\langle X,Y\rangle\subset\langle v^kX,Y\rangle$$

since $\mathrm{Sq}(X)\in\langle X,Y\rangle$. Conversely, assume that $\langle v^kX,Y\rangle$ is Steenrod closed. We then have

$$\mathrm{Sq}(Y)=\lambda' v^kX+\mu Y$$

for some λ',μ. This gives exactly the coefficients as above.

For the second part note that if X,Y are coprime, then any other choice of coefficients λ',μ' is of the form

$$\lambda'=\lambda+pY\text{ and }\mu'=\mu+pX.$$

This implies the second part of the statement. □

As a consequence we conclude the following.

Corollary 4.10. *Let $\langle X,Y\rangle$ be Steenrod closed and $k\geq 1$. If $\langle v^kX,Y\rangle$ is Steenrod closed then $\langle v^iX,Y\rangle$ is Steenrod closed for every $i\leq k$*

5. Classification of fibered ideals

Definition 5.1. A Steenrod closed parameter ideal $I \subset H^*(BA_4)$ is called *fibered*, if it has a system of parameters $\{X, Y\}$ such that $\langle X\rangle$ is Steenrod closed.

Note that we do not assume here that X is the generator of smaller degree. By Theorem 1.1, if $I = \langle X\rangle$ is a proper ideal in $H^*(BA_4)$ closed under Steenrod operations then there is a $k \geq 1$ such that $X = v^k$. This shows that fibered ideals are always of the form $\langle v^k, Y\rangle$. In this section we show that all fibered ideals are of the form $\langle v^k, u^l\rangle$ where k and l satisfy the following condition: if $l = 2^t c$ with c odd, then $k \leq 2^t$. This condition comes from an observation due to Meyer and Smith.

Theorem 5.2 ([MS03, Theorem V.1.1])**.** *The ideal $\langle v^k, u^l\rangle \subset H^*(BA_4)$ is Steenrod closed if and only if $l = 2^t c$ with c odd and $k \leq 2^t$.*

In the proof of our results in this section we do not use this theorem. Moreover, this theorem follows form the main theorem of this section (Theorem 5.8).

Lemma 5.3. *Let $X, Y \in H^*(BA_4)$ be two coprime, homogeneous elements such that $|X| < |Y|$ and let $\lambda \in H^{|Y|-|X|}(BA_4)$ be any homogeneous element. Then*

$$\langle vX^2, (Y + \lambda X)^2\rangle = \begin{cases} \langle vX^2, Y^2\rangle & \lambda \in \langle v\rangle \\ \langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle & \lambda \notin \langle v\rangle \end{cases},$$
$$\langle v(Y + \lambda X)^2, X^2\rangle = \langle vY^2, X^2\rangle.$$

Proof. First note that adding multiples of v to λ does not change the ideal:

$$\langle vX^2, (Y + (\lambda + \lambda' v)X)^2\rangle = \langle vX^2, Y^2 + \lambda^2 X^2 + \lambda'^2 v^2 X^2\rangle = \langle vX^2, Y^2 + \lambda^2 X^2\rangle.$$

So the case $\lambda \in \langle v\rangle$ is obvious.

Now assume that $\lambda \notin \langle v\rangle$. Write λ in the form $\lambda = p + wp'$ where $p, p' \in \mathbb{F}_2[u, v] \subset H^*(BA_4)$. We may leave out all monomials divisible by v and thus assume that $p, p' \in \mathbb{F}_2[u]$. Hence p and p' have even degree and for degree reasons, only one of them can be nonzero. They cannot both be zero, since then λ would be divisible by v. If $p \neq 0$, we have $p = u^{(|Y|-|X|)/2}$ since λ is homogeneous. Thus

$$\langle vX^2, Y^2 + \lambda^2 X^2\rangle = \langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle.$$

If $p' \neq 0$, we have $p' = u^{(|Y|-|X|-3)/2}$ and thus

$$\begin{aligned}\langle vX^2, Y^2 + \lambda^2 X^2\rangle &= \langle vX^2, Y^2 + w^2 u^{|Y|-|X|-3}X^2\rangle \\ &= \langle vX^2, Y^2 + (u^3 + v^2 + vw)u^{|Y|-|X|-3}X^2\rangle \\ &= \langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle.\end{aligned}$$

This proves the first part of the lemma. For the second part observe that

$$\langle v(Y + \lambda X)^2, X^2\rangle = \langle vY^2 + v\lambda^2 X^2, X^2\rangle = \langle vY^2, X^2\rangle. \qquad \square$$

As a consequence we obtain the following.

Lemma 5.4. *Let $I \subset H^*(BA_4)$ be a Steenrod closed parameter ideal with parameters $\{X, Y\}$ such that $|X| < |Y|$. Then the set of all ideals of the form $\langle vX'^2, Y'^2\rangle$, where $\{X', Y'\}$ is any system of parameters for I, consists of at most three elements, namely the elements $\langle vX^2, Y^2\rangle$, $\langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle$, $\langle X^2, vY^2\rangle$.*

Now we prove a technical lemma.

Lemma 5.5. *Let $X, Y \in H^*(BA_4)$ be two homogeneous elements such that $\langle X, Y\rangle$ is a Steenrod closed parameter ideal and $s := |Y|-|X|$ is nonnegative. Let $\alpha, \beta, \gamma, \delta \in H^*(BA_4)$ be elements satisfying*

$$\mathrm{Sq}(X) = \alpha X + \beta Y,$$
$$\mathrm{Sq}(Y) = \gamma X + \delta Y.$$

Then $\langle vX^2, Y^2 + u^s X^2\rangle$ is Steenrod closed if and only if

$$\gamma^2 + (u+v+u^2)^s\alpha^2 + u^s\delta^2 + (u+v+u^2)^s u^s\beta^2 \in \langle v, Y^2 + u^s X^2\rangle.$$

Proof. We know from Lemma 4.4 that $\langle X^2, Y^2\rangle = \langle X^2, Y^2 + u^s X^2\rangle$ is a Steenrod closed parameter ideal. In fact, we have

$$\begin{aligned}\mathrm{Sq}(Y^2 + u^s X^2) =& \mathrm{Sq}(Y)^2 + (u+v+u^2)^s\,\mathrm{Sq}(X)^2\\ =&(\gamma^2X^2 + \delta^2Y^2) + (u+v+u^2)^s(\alpha^2X^2 + \beta^2Y^2)\\ =&(\gamma^2 + (u+v+u^2)^s\alpha^2)X^2 + (\delta^2 + (u+v+u^2)^s\beta^2)Y^2\\ =&(\gamma^2 + (u+v+u^2)^s\alpha^2 + u^s\delta^2 + (u+v+u^2)^s u^s\beta^2)X^2 +\\ &(\delta^2 + (u+v+u^2)^s\beta^2)(Y^2 + u^sX^2).\end{aligned}$$

By Lemma 4.9, the ideal $\langle vX^2, Y^2 + u^sX^2\rangle$ is also Steenrod closed if and only if

$$\gamma^2 + (u+v+u^2)^s\alpha^2 + u^s\delta^2 + (u+v+u^2)^s u^s\beta^2 \in \langle v, Y^2 + u^s X^2\rangle. \qquad \square$$

To apply Lemma 5.5, we need to know the coefficients $\alpha, \beta, \gamma, \delta$. In the following lemma, we compute these coefficients for the fibered case.

Lemma 5.6. *Let $t, c \geq 0$ with c odd and $0 \leq k \leq 2^t$. If $(X, Y) := (v^k, u^{2^t c})$, then*

$$\mathrm{Sq}(X) = \alpha X \text{ and } \mathrm{Sq}(Y) = \gamma X + \delta Y$$

with

$$\begin{aligned}\alpha &= (1+u+v)^k,\\ \gamma &= \left(\sum_{j=0}^{c-1}\binom{c}{j}(u^{2^t}+u^{2^{t+1}})^j v^{2^t(c-j)-k}\right),\\ \delta &= (u^{2^t}+1)^c.\end{aligned}$$

Furthermore $\langle v^k, u^{2^t c}\rangle$ is a Steenrod closed parameter ideal if $k > 0$.

Proof. We have

$$\begin{aligned}\mathrm{Sq}(X) &= (v+uv+v^2)^k = (1+u+v)^k X,\\ \mathrm{Sq}(Y) &= (u+v+u^2)^{2^t c} = (u^{2^t}+u^{2^{t+1}}+v^{2^t})^c\\ &= \sum_{j=0}^{c}\binom{c}{j}(u^{2^t}+u^{2^{t+1}})^j v^{2^t(c-j)}\\ &= \left(\sum_{j=0}^{c-1}\binom{c}{j}(u^{2^t}+u^{2^{t+1}})^j v^{2^t(c-j)-k}\right)X + (u^{2^t}+1)^c Y.\end{aligned}$$

It is obvious that $v^k, u^{2^t c} \in \mathbb{F}_2[u,v]$ are coprime elements of positive degree for $k > 0$. Hence the ideal $\langle v^k, u^{2^t c}\rangle$ is a Steenrod closed parameter ideal. □

In the following lemma we consider the ideals of the form $\langle vX^2, Y^2\rangle$, $\langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle$, or $\langle vY^2, X^2 + u^{|X|-|Y|}Y^2\rangle$ as in Lemma 5.4 when $\langle X, Y\rangle$ is a fibered ideal with $(X,Y) = (v^k, u^l)$. To classify the fibered ideals in Theorem 5.8, we will use parts (1) and (2) of Lemma 5.7. The third statement is used in the next section, in the proof of Theorem 6.10.

Lemma 5.7. *Let $(X,Y) \coloneqq (v^k, u^{2^t c})$, where $t, c \geq 0$ with c odd, and $1 \leq k \leq 2^t$. Then the following hold:*

(1) $\langle vX^2, Y^2\rangle = \langle v^{2k+1}, u^{2^{t+1}c}\rangle$ is a Steenrod closed parameter ideal if and only if $k < 2^t$.
(2) If $|X| < |Y|$, then $\langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle$ is not a Steenrod closed parameter ideal.
(3) If $|X| > |Y|$, then the ideal $\langle vY^2, X^2 + u^{|X|-|Y|}Y^2\rangle$ is a Steenrod closed parameter ideal if and only if $k = 2^t$ and $c = 1$.

Proof. We prove (1). By Lemma 5.6, if $k < 2^t$, then $\langle vX^2, Y^2\rangle = \langle v^{2k+1}, u^{2^{t+1}c}\rangle$ is a Steenrod closed parameter ideal. If $k = 2^t$ we want to use Lemma 4.9 to conclude that $\langle vX^2, Y^2\rangle$ is not Steenrod closed. Note that $\langle X^2, Y^2\rangle$ is a Steenrod closed parameter ideal by Lemma 4.4 and thus we have to show that $\gamma^2 \notin \langle v, u^{2^{t+1}c}\rangle$, where γ is as in Lemma 5.6.

Modulo the ideal $\langle v, u^{2^{t+1}c}\rangle$, we have

$$\gamma \equiv \binom{c}{c-1}(u^{2^t} + u^{2^{t+1}})^{c-1} \equiv u^{2^t(c-1)}(1 + u^{2t})^{c-1}.$$

Hence $\gamma^2 \equiv u^{2^{t+1}(c-1)}(1 + u^{4t})^{c-1} \bmod \langle v, u^{2^{t+1}} c\rangle$. The summand with the lowest exponent is $u^{2^{t+1}(c-1)}$ and this summand is nonzero modulo $\langle v, u^{2^{t+1}c}\rangle$. Hence γ^2 is not in $\langle v, u^{2^{t+1}c}\rangle$.

Now we want to prove (2). Consider the ideal $I = \langle vX^2, Y^2 + u^s X^2\rangle$ for $s = |Y| - |X| \geq 0$. Let α, γ, δ be as in Lemma 5.6. By Lemma 5.5, the ideal I is Steenrod closed if and only if

$$z \coloneqq \gamma^2 + (u + v + u^2)^s \alpha^2 + u^s \delta^2$$

is in $\langle v, Y^2 + u^s X^2\rangle$. Since $k \geq 1$, we have $\langle v, Y^2 + u^s X^2\rangle = \langle v, u^{2^{t+1}c}\rangle$. Modulo this ideal, we have

$$z \equiv (u^{2^{t+1}} + u^{2^{t+2}})^{c-1} v^{2^{t+1}-2k} + (u + u^2)^s (1 + u)^{2k} + u^s (u^{2^{t+1}} + 1)^c.$$

First, consider the case $k < 2^t$. Then $\gamma \equiv 0$ and we obtain

$$\begin{aligned} z \equiv (u + u^2)^s (1 + u)^{2k} + u^s (u^{2^t+1} + 1)^c &\equiv u^s((1 + u)^{2k+s} + (u^{2^{t+1}} + 1)^c) \\ &\equiv u^s((1 + u)^{2^{t+1}c-k} + (1 + u^{2^{t+1}})^c) \end{aligned}$$

since $s = |Y| - |X| = 2^{t+1}c - 3k$. Multiplying with the unit $(1+u)^k$ of $\mathbb{F}_2[u]/\langle u^{2^{t+1}c}\rangle$ and writing $k = 2^l l'$ with l' odd yields

$$u^s(1 + u^{2^{t+1}})^c(1 + (1 + u^{2^l})^{l'}).$$

The summand with the lowest exponent is $u^s u^{2^l} = u^{s+2^l}$ and this summand is nonzero since the exponent $s+2^l \leq s+k = 2^{t+1}c-2k$ is smaller than $2^{t+1}c$. Hence z is not in the ideal $\langle v, u^{2^{t+1}c}\rangle$.

In the case $k = 2^t$, we get $s = 2^{t+1}c - 3 \cdot 2^t = (2c-3)2^t$. By assumption $s > 0$ and hence $c \geq 3$. Modulo the ideal $\langle v, u^{2^{t+1}c}\rangle$, we have

$$\begin{aligned} z \equiv &(u^{2^{t+1}} + u^{2^{t+2}})^{c-1} + (u+u^2)^s(1+u)^{2k} + u^s(u^{2^{t+1}}+1)^c \\ \equiv &u^s\Big(u^{2^t}(1+u)^{2^{t+1}(c-1)} + (1+u)^{2^t(2c-1)} + (1+u)^{2^{t+1}c}\Big). \end{aligned}$$

Multiplying with $(1+u)^{2^{t+1}}$ yields

$$\begin{aligned} (1+u)^{2^{t+1}} z \equiv &u^s(1+u)^{2^{t+1}c}\big(u^{2^t} + (1+u)^{2^t} + (1+u)^{2^{t+1}}\big) \\ \equiv &u^s(1+u)^{2^{t+1}c}u^{2^{t+1}} \equiv u^{(2c-1)2^t}(1+u)^{2^{t+1}c}. \end{aligned}$$

The nonzero term of lowest degree in this polynomial $u^{(2c-1)2^t}$ does not lie in the ideal $\langle v, u^{2^{t+1}c}\rangle$. This gives that $(1+u)^{2^{t+1}} z \not\equiv 0$ modulo $\langle v, u^{2^{t+1}c}\rangle$. From this we can conclude that $z \notin \langle v, u^{2^{t+1}c}\rangle$, hence I is not Steenrod closed.

Finally we want to prove (3). Since $r := |X| - |Y| = 3k - 2^{t+1}c \geq 0$, we have $c = 1$. By Lemma 5.5, we have that $\langle vY^2, X^2 + u^{|X|-|Y|}Y^2\rangle$ is Steenrod closed, if and only if

$$z' := 0^2 + (u+v+u^2)^r\delta^2 + u^r\alpha^2 + (u+v+u^2)^r u^r\gamma^2 \in \langle v, X^2 + u^rY^2\rangle.$$

Modulo $\langle v, X^2 + u^rY^2\rangle = \langle v, u^{|X|}\rangle = \langle v, u^{3k}\rangle$, we have

$$\begin{aligned} z' \equiv &(u+u^2)^r(u^{2^t}+1)^2 + u^r(1+u)^{2k} + (u+u^2)^r u^r v^{2^{t+1}-2k} \\ = &u^r\Big((1+u)^{r+2^{t+1}} + (1+u)^{2k} + u^r(1+u)^r v^{2^{t+1}-2k}\Big). \end{aligned}$$

If $k = 2^t$, then we have $r = 2^t$ and thus

$$z' \equiv u^{2^t}(1+u)^{2^t}\Big((1+u)^{2^{t+1}} + (1+u)^{2^t} + u^{2^t}\Big) = (1+u)^{2^t}u^{3\cdot 2^t} \equiv 0.$$

If $k < 2^t$, then putting $r = 3k - 2^{t+1}$ gives

$$z' \equiv u^{3k-2^{t+1}}\Big((1+u)^{3k} + (1+u)^{2k}\Big) = u^{3k-2^{t+1}}(1+u)^{2k}((1+u)^k + 1).$$

Writing $k = 2^l l'$ with l' odd yields

$$z' \equiv u^{3k-2^{t+1}}(1+u^{2^{l+1}})^{l'}\Big((1+u^{2^l})^{l'} + 1\Big).$$

The summand with the lowest exponent is $u^{3k-2^{t+1}}u^{2^l}$ and this summand is nonzero since the exponent $3k - 2^{t+1} + 2^l$ is strictly less than $3k$. □

Theorem 5.8. *An ideal $I \subset H^*(BA_4)$ is fibered, if and only if $I = \langle v^k, u^l\rangle$ with $l = 2^tc$ for c odd and $1 \leq k \leq 2^t$. Furthermore, every fibered ideal can be written uniquely as above.*

Proof. These ideals are Steenrod closed parameter ideals by Lemma 5.6. Uniqueness follows from Lemma 3.1 for degree reasons.

We proceed by induction. Let $\langle v^k, Y\rangle$ be a fibered ideal. We assume that if $\langle v^{k'}, Y'\rangle$ is fibered with $3k' + |Y'| < 3k + |Y|$, then the ideal $\langle v^{k'}, Y'\rangle$ has a system of parameters of the form $\{v^{k'}, u^{l'}\}$ as above.

Next we want to show that we can assume that $\mathrm{Sq}^1(Y) = 0$. If $3k < |Y|$, this follows from Lemma 4.1 as $\mathrm{Sq}^1(v^k) = 0$. If $|Y| < 3k$ and $\mathrm{Sq}^1(Y) \neq 0$, then $v^k = \mathrm{Sq}^1(Y)$ by Lemma 3.2 and thus the ideal is twisted. But the only twisted ideal, where one of the generators is of the form v^i is $\langle u, v\rangle$. This is an ideal of the desired form. We thus can assume that we have $\mathrm{Sq}^1(Y) = 0$ in general.

If $|Y|$ is odd, then v divides Y by Lemma 3.5, but then v^k and Y are not coprime. So we can assume $|Y|$ is even. By Lemma 3.5, this means that $Y = y^2$ for some y in $H^*(BA_4)$.

If $k = 2k'$, then we get by Lemma 4.4 that $\langle v^{k'}, y\rangle$ is also a Steenrod closed parameter ideal and hence $\langle v^{k'}, y\rangle = \langle v^{k'}, u^{l'}\rangle$ by induction assumption. Thus

$$\langle v^k, Y\rangle = \langle v^{k'}, y\rangle^{[2]} = \langle v^{k'}, u^{l'}\rangle^{[2]} = \langle v^{2k'}, u^{2l'}\rangle.$$

It remains to look at the case $k = 2k' + 1$. If $k' = 0$, we have $v^k = v$. Let us write $Y = p + wp'$ with $p, p' \in \mathbb{F}_2[u, v] \subset H^*(BA_4)$. We may leave out all monomials divisible by v and thus we can assume that $p, p' \in \mathbb{F}_2[u]$. Since Y has even degree we get $p' = 0$. The polynomial p cannot be zero, since otherwise $\langle v, Y\rangle = \langle v\rangle$ and this is not a parameter ideal. Hence $\langle v, Y\rangle = \langle v, u^{|Y|/2}\rangle$.

If $k' > 0$, then by Lemma 4.6, we have that $\langle v^{k'}, y\rangle$ is also a Steenrod closed parameter ideal and hence by induction assumption $\langle v^{k'}, y\rangle = \langle v^{k'}, u^{l'}\rangle$ for some $k' \leq 2^{t'}$ where $2^{t'}$ is the largest power of 2 dividing l'. Since $y \in \langle v^{k'}, u^{l'}\rangle$, there exist homogeneous elements μ, λ such that $y = \mu u^{l'} + \lambda v^{k'}$. By Lemma 3.1, the degrees of parameters are unique, hence we must have $|y| = |u^{l'}|$ and thus $\mu \in \mathbb{F}_2$. Since parameters are coprime, it follows that $\mu = 1$ and $y = u^{l'} + \lambda v^{k'}$. If $\lambda = 0$, then $\langle v^k, Y\rangle = \langle v^{2k'+1}, y^2\rangle = \langle v^{2k'+1}, u^{2l'}\rangle$. If $\lambda \neq 0$, then by Lemma 5.3, we get that $\langle v^k, Y\rangle = \langle v^{2k'+1}, (u^{l'} + \lambda v^{k'})^2\rangle$ is one of the ideals $\langle v^k, u^{2l'}\rangle$ or $\langle v^k, u^{2l'} + u^{2l'-3k'}v^{2k'}\rangle$. Since the latter is not Steenrod closed by Lemma 5.7(2), we have $\langle v^k, Y\rangle = \langle v^{2k'+1}, u^{2l'}\rangle$. By assumption, this ideal is a Steenrod closed parameter ideal and hence by Lemma 5.7(1), we get $k' < 2^{t'}$. Thus $k = 2k' + 1 \leq 2^{t'+1}$ and $2^{t'+1}$ is the largest power of 2 dividing $l = 2l'$. □

6. Classification of Steenrod closed parameter ideals

In Sections 3 and 5 we classified the twisted and fibered Steenrod closed parameter ideals, respectively. In this section, we will classify all Steenrod closed parameter ideals.

In Section 3 we have shown that every twisted ideal I is equal to $\langle x_n, y_n\rangle$ for some $n \geq 1$, where the sequence of pairs (x_n, y_n) is defined recursively by $(x_1, y_1) = (u, v)$ and $x_n = ux_{n-1}^2 + y_{n-1}^2$, $y_n = vx_{n-1}^2$ for $n \geq 2$. Starting with such a pair (x, y), we obtain new pairs using some transformations. One of the transformations we use is

$$D : (X, Y) \mapsto (X^2, Y^2)$$

defined by squaring both coordinates. We proved in Lemma 4.4 that a pair (X, Y) generates a Steenrod closed parameter ideal if and only if $D(X, Y)$ does so. Another transformation is a generalization of the recursion we defined above. It only applies to the images of the recursive application of the transformation D to (x, y). For each $m \geq 1$, we define

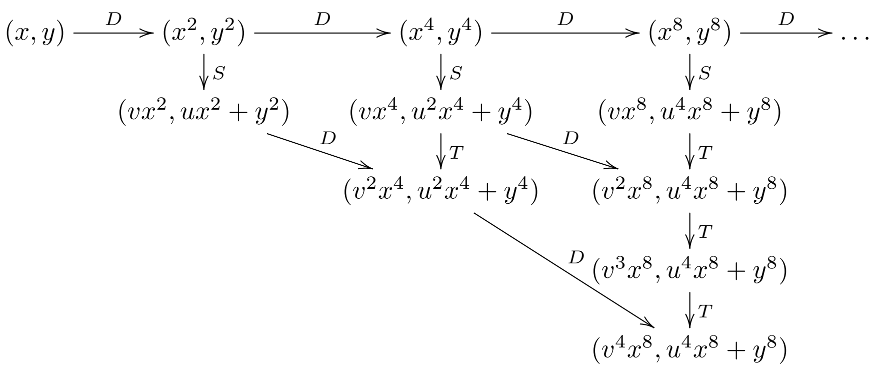
$$S\colon (x^{2^m}, y^{2^m}) \mapsto (vx^{2^m}, u^{2^{m-1}}x^{2^m} + y^{2^m}).$$

Note that $S(x^2, y^2) = (vx^2, ux^2 + y^2)$. Hence $S(x^2, y^2)$ generates the next twisted ideal.

Finally we define the transformation

$$T\colon (X, Y) \mapsto (vX, Y).$$

By Lemma 4.6, if (vX^2, Y^2) generates a Steenrod closed parameter ideal then so does (X, Y). For small values of m, we have the following table.



Note that all the pairs in the above diagram have a specific form. We give a specific name for all the pairs obtained this way.

Definition 6.1. A pair of homogeneous classes (X, Y) in $H^*(BA_4)$ is called a *mixed pair* if

$$(X, Y) \coloneqq (v^i x_n^{2^m}, u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m}),$$

for some $n \geq 1$, $m \geq 1$, and $0 \leq i \leq 2^{m-1}$, where x_n and μ_n are as in Lemma 3.9 satisfying that $\langle x_n, \mathrm{Sq}^1(x_n)\rangle$ is a Steenrod closed parameter ideal.

Remark 6.2. For $n \geq 1$ we have

$$u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m} = (ux_n^2 + y_n^2)^{2^{m-1}} = x_{n+1}^{2^{m-1}}.$$

So, a mixed pair can also be expressed as a pair $(v^i x_n^{2^m}, x_{n+1}^{2^{m-1}})$ as it is done in Theorem 1.2. Also note that when $i = 2^{m-1}$, the mixed pair becomes the $(y_{n+1}^{2^{m-1}}, x_{n+1}^{2^{m-1}})$ pair. In particular, $(vx_n^2, ux_n^2 + y_n^2) = (y_{n+1}, x_{n+1})$, so in this case the ideal generated by a mixed pair is a twisted ideal. We exclude the case $i = 2^{m-1}$ from the subsequent statements about mixed pairs to avoid overlaps.

Another overlap occurs when $n = 1$ and $i = 0$. In this case the ideal generated by a mixed pair is equal to $\langle v^{2^m}, u^{2^m}\rangle$ which is a fibered ideal. We exclude this case in our definition of mixed ideals.

Definition 6.3. For $m \geq 1$, the ideal generated by a mixed pair $(v^i x_n^{2^m}, x_{n+1}^{2^{m-1}})$, with either $n = 1$ and $1 \leq i < 2^{m-1}$, or $n \geq 2$ and $0 \leq i < 2^{m-1}$, is called a *mixed ideal*.

The main aim of this section is to prove that every mixed ideal is a Steenrod closed parameter ideal and all the Steenrod closed parameter ideals which are not fibered or twisted are mixed ideals. We start with proving one direction of our claim.

Lemma 6.4. *Let* $(X,Y) := (v^i x_n^{2^m}, u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m})$ *be a mixed pair with* $n \geq 1$*,* $m \geq 1$*, and* $0 \leq i < 2^{m-1}$*. Then we have*

$$\mathrm{Sq}(X) = \alpha X + \beta Y \text{ and } \mathrm{Sq}(Y) = \gamma X + \delta Y,$$

with

$$\begin{aligned}\alpha &= (1+u+v)^i(\mu_{n-1}^{2^m} + x_n^{2^m} + \mu_{n-1}^{2^m} u^{2^{m-1}}),\\ \beta &= (v+uv+v^2)^i \mu_{n-1}^{2^m},\\ \gamma &= \mu_n^{2^{m-1}} v^{2^{m-1}-i}, \text{ and } \delta = \mu_n^{2^{m-1}} + Y.\end{aligned}$$

Furthermore, X *and* Y *are coprime and hence* $\langle X, Y\rangle$ *is a Steenrod closed parameter ideal.*

Proof. By Lemma 3.9, we have

$$\begin{aligned}\mathrm{Sq}(x_n) &= (\mu_{n-1} + x_n)x_n + \mu_{n-1}y_n\\ \mathrm{Sq}(y_n) &= v\mu_{n-1}x_n + ((u+1)\mu_{n-1} + x_n + y_n)y_n.\end{aligned}$$

This gives

$$\begin{aligned}\mathrm{Sq}(X) &= \mathrm{Sq}(v)^i \,\mathrm{Sq}(x_n)^{2^m}\\ &= (v+uv+v^2)^i\big((\mu_{n-1}^{2^m} + x_n^{2^m})x_n^{2^m} + \mu_{n-1}^{2^m} y_n^{2^m}\big)\\ &= (v+uv+v^2)^i\Big((\mu_{n-1}^{2^m} + x_n^{2^m})x_n^{2^m} + \mu_{n-1}^{2^m}(u^{2^{m-1}} x_n^{2^m} + Y)\Big)\\ &= (1+u+v)^i(\mu_{n-1}^{2^m} + x_n^{2^m} + \mu_{n-1}^{2^m} u^{2^{m-1}})X + (v+uv+v^2)^i \mu_{n-1}^{2^m} Y.\end{aligned}$$

Hence the formula for $\mathrm{Sq}(X)$ given in the lemma holds. To verify the formula for $\mathrm{Sq}(Y)$, observe that

$$v^{2^{m-1}-i}X = v^{2^{m-1}} x_n^{2^m} = y_{n+1}^{2^{m-1}} \text{ and } Y = x_{n+1}^{2^{m-1}} \tag{6.5}$$

by Remark 6.2. So we have:

$$\begin{aligned}\mathrm{Sq}(Y) = \mathrm{Sq}(x_{n+1})^{2^{m-1}} &= (\mu_n^{2^{m-1}} + x_{n+1}^{2^{m-1}})x_{n+1}^{2^{m-1}} + \mu_n^{2^{m-1}} y_{n+1}^{2^{m-1}}\\ &= (\mu_n^{2^{m-1}} + Y)Y + \mu_n^{2^{m-1}} v^{2^{m-1}-i} X.\end{aligned}$$

Since $n \geq 1$, the generators X, Y have degrees bigger than 0. It remains to show that X and Y are coprime. Any common divisor would either be a divisor of x_n, in which case it would also divide $\mathrm{Sq}^1(x_n)$, which contradicts the assumption that x_n and $\mathrm{Sq}^1(x_n)$ are coprime; or v would be a common divisor. Since v divides $\mathrm{Sq}^1(x_n)$ by Lemma 3.10, it would also divide x_n which contradicts again that x_n and $\mathrm{Sq}^1(x_n)$ are coprime. □

Thus we have shown that any mixed ideal as defined in Definition 6.3 is a Steenrod closed parameter ideal. Using the formulas in Lemma 6.4, we immediately obtain the following:

Lemma 6.6. *Let* $(X,Y) := (v^i x_n^{2^m}, u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m})$ *be a mixed pair with* $n \geq 1$*,* $m \geq 1$*, and* $0 \leq i < 2^{m-1}$*. Modulo* v *we have* $X \equiv 0$ *for* $i > 0$ *and*

$X \equiv u^{2^{n+m}-2^m}$ *for* $i=0$*, and* $Y \equiv u^{2^{n+m}-2^{m-1}}$*. Moreover, the residue classes of the coefficients from Lemma 6.4 are given by*

$$\begin{aligned}\alpha \equiv& (1+u)^i (u^{2^{n+m}-2^{m-1}}+1)/(u^{2^{m-1}}+1),\\ \beta \equiv& \begin{cases}(u^{2^{n+m}-2^m}+1)/(u^{2^m}+1) & i=0\\ 0 & i>0\end{cases},\\ \gamma \equiv& 0, \ \textit{and} \ \ \delta \equiv (u^{2^{n+m}}+1)/(u^{2^{m-1}}+1).\end{aligned}$$

Proof. By Lemma 3.10, we have

$$x_n \equiv u^{2^n-1}, \ y_n \equiv 0, \ \text{and } \mu_n \equiv (u^{2^{n+1}-1}+1)/(u+1)$$

and thus $Y \equiv u^{2^{m-1}}(u^{2^n-1})^{2^m} \equiv u^{2^{n+m}-2^{m-1}}$ and $X \equiv 0$ for $i>0$ and $X \equiv u^{2^{n+m}-2^m}$ for $i=0$. Now the result follows from plugging these into the formulas in Lemma 6.4 and reducing everything mod v. □

We have the following lemma for mixed pairs.

Lemma 6.7. *Let* $(X,Y) := (v^i x_n^{2^m}, u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m})$ *be a mixed pair with* $n \geq 1$*,* $m \geq 1$*, and* $0 \leq i < 2^{m-1}$*. Then the following hold:*

1. *(1)* $\langle vX^2, Y^2\rangle$ *is a Steenrod closed parameter ideal.*
2. *(2) If* $|X| < |Y|$*, then* $\langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle$ *is not a Steenrod closed parameter ideal.*

Proof. Note that the ideal

$$\langle vX^2, Y^2\rangle = \langle v^{2i+1}x_n^{2^{m+1}}, u^{2^m}x_n^{2^{m+1}} + \mathrm{Sq}^1(x_n)^{2^{m+1}}\rangle$$

is also of the form above. Since $0 \leq i < 2^{m-1}$ and $m \geq 1$, we have $2i+1 < 2^m$. Thus by Lemma 6.4, we can conclude that $\langle vX^2, Y^2\rangle$ is a Steenrod closed parameter ideal.

Now consider $I = \langle vX^2, Y^2 + u^s X^2\rangle$ with $s = |Y|-|X|$. Write $\mathrm{Sq}(X) = \alpha X + \beta Y$, $\mathrm{Sq}(Y) = \gamma X + \delta Y$. By Lemma 5.5, the ideal I is Steenrod closed if and only if the coefficient

$$z := \gamma^2 + (u+v+u^2)^s \alpha^2 + u^s\delta^2 + (u+v+u^2)^s u^s \beta^2$$

lies in the ideal $\langle v, Y^2 + u^s X^2\rangle$. By Lemma 6.6, this ideal simplifies to

$$\langle v, Y^2 + u^s X^2\rangle = \begin{cases}\langle v\rangle & i=0\\ \langle v, u^{|Y|}\rangle & i>0.\end{cases}$$

It is helpful to record the degrees of X, Y using that $|x_n| = 2^{n+1}-2$.

$$|X| = 3i + 2^{m+n+1} - 2^{m+1}, \quad |Y| = 2^{n+m+1} - 2^m, \quad s = |Y| - |X| = 2^m - 3i.$$

First, we consider the case $i > 0$. We have mod $\langle v, u^{|Y|}\rangle$:

$$z \equiv (u+u^2)^s(1+u)^{2i}(u^{2^{n+m+1}-2^m}+1)/(u^{2^m}+1) + u^s(u^{2^{n+m+1}}+1)/(u^{2^m}+1).$$

It suffices show that the coefficient multiplied with the unit $1+u^{2^m}$ in $\mathbb{F}_2[u]/\langle u^{|Y|}\rangle$ is nonzero. We have

$$(1+u^{2^m})z \equiv u^s((1+u)^{2i+s}(u^{2^{n+m+1}-2^m}+1) + (u^{2^{n+m+1}}+1)) \equiv u^s((1+u)^{2^m-i}+1)$$

since $u^{2^{n+m+1}} \equiv u^{2^{n+m+1}-2^m} \equiv 0 \bmod u^{|Y|}$. Now write $i = l'2^l$ with l' odd and thus $l < m-1$. Then

$$(1+u)^{2^m-i} = (1+u^{2^l})^{2^{m-l}-l'} = 1 + u^{2^l} + \cdots$$

since $2^{m-l} - l'$ is odd and it suffices to show that u^{s+2^l} is nonzero in $\mathbb{F}_2[u]/\langle u^{|Y|}\rangle$. Note that $|X| - i = 2i + 2^{m+n+1} - 2^{m+1} > 0$. We thus have $s + 2^l \leq s + i < s + |X| = |Y|$ and hence the coefficient is indeed nonzero in $\mathbb{F}_2[u]/\langle u^{|Y|}\rangle$. Hence z does not lie in the ideal $\langle v, Y^2 + u^s X^2\rangle$

Secondly, consider the case $i = 0$. Then by Lemma 6.6 we have that $Y \equiv u^{2^{m+n}-2^{m-1}}$ and $X \equiv u^{2^{m+n}-2^m}$. Thus $Y^2 + u^s X^2 \equiv 0 \bmod v$ and hence the ideal $\langle vX^2, Y^2 + u^s X^2\rangle$ cannot be a parameter ideal, since both generators are divisible by v. □

With swapped roles for Y and X, we do not obtain any Steenrod closed parameter ideals.

Lemma 6.8. *Let $(X, Y) := (v^i x_n^{2^m}, u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m})$ be a mixed pair with $n \geq 1$, $m \geq 1$, and $0 \leq i < 2^{m-1}$. Then the following hold:*

(1) The ideal $\langle vY^2, X^2\rangle$ is not a Steenrod closed parameter ideal.
(2) If $|X| > |Y|$, then the ideal $\langle vY^2, X^2 + u^{|X|-|Y|}Y^2\rangle$ is not a Steenrod closed parameter ideal.

Proof. Consider $I = \langle vY^2, X^2\rangle$. If $i > 0$, then both generators are divisible by v and hence the ideal cannot be a parameter ideal. If $i = 0$, we have $\langle vY^2, X^2\rangle = \langle v\,\mathrm{Sq}^1(x_n)^{2^{m+1}}, x_n^{2^{m+1}}\rangle$ and this ideal is not Steenrod closed, because it does not contain $\mathrm{Sq}^{2^{m+1}}(X^2) = \mathrm{Sq}^1(x_n)^{2^{m+1}}$. Otherwise it would be a multiple of $x_n^{2^{m+1}}$ for degree reasons. This contradicts the assumption that $\langle x_n, \mathrm{Sq}^1(x_n)\rangle$ is a parameter ideal.

Now consider the case of $I = \langle vY^2, X^2 + u^s Y^2\rangle$ with $s = |X| - |Y|$. We are applying Lemma 5.5 to the pair Y, X. Thus X in Lemma 5.5 corresponds to Y here. Write $\mathrm{Sq}(Y) = \delta Y + \gamma X$, $\mathrm{Sq}(X) = \beta Y + \alpha X$ with the coefficients from Lemma 6.4. Then the ideal I is Steenrod closed if and only if

$$z := \beta^2 + (u+v+u^2)^s \delta^2 + u^s \alpha^2 + (u+v+u^2)^s u^s \gamma^2$$

lies in $\langle v, X^2 + u^s Y^2\rangle$. We will show that this is not the case. Since $s > 0$, we have $i > 0$ and it follows from Lemma 6.6 that

$$\langle v, X^2 + u^s Y^2\rangle = \langle v, u^{|X|}\rangle.$$

It is helpful to record the degrees of X, Y using that $|x_n| = 2^{n+1} - 2$:

$$|X| = 3i + 2^{m+n+1} - 2^{m+1}, \quad |Y| = 2^{n+m+1} - 2^m, \quad s = |X| - |Y| = 3i - 2^m.$$

We have mod $\langle v, u^{|X|}\rangle$:

$$z \equiv (u+u^2)^s (u^{2^{n+m+1}} + 1)/(u^{2^m} + 1) + u^s (1+u)^{2i} (u^{2^{n+m+1}-2^m} + 1)/(u^{2^m} + 1).$$

It suffices to show that this element is nonzero in $\mathbb{F}_2[u]/\langle u^{|X|}\rangle$. Multiplication with the unit $u^{2^m} + 1$ in $\mathbb{F}_2[u]/\langle u^{|X|}\rangle$ yields

$$\begin{aligned}(u^{2^m} + 1)z \equiv& (u+u^2)^s (u^{2^{n+m+1}} + 1) + u^s (1+u)^{2i} (u^{2^{n+m+1}-2^m} + 1) \\ \equiv& u^s ((1+u)^{2i} ((1+u)^s + u^{2^{n+m+1}-2^m} + 1))\end{aligned}$$

since $u^s u^{2^{n+m+1}} \equiv 0 \bmod u^{|X|}$. Write $s = l'2^l$ with l' odd. Then $(1+u)^s = (1+u^{2^l})^{l'}$ and from $2i = 2(s + 2^m)/3 = 2^{l+1}(l' + 2^{m-l})/3$ it follows that $(1+u)^{2i} = (1 + u^{2^{l+1}})^{(l'+2^{m-l})/3}$. We conclude that the smallest power of u that appears in the sum is $u^s \cdot u^{2^l}$. This summand is nonzero in $\mathbb{F}_2[u]/\langle u^{|X|}\rangle$ since $2^l \leq s = 3i - 2^m < 3 \cdot 2^{m-1} - 2^m \leq 2^{m+n+1} - 2^m = |Y|$ and hence $|u^s \cdot u^{2^l}| < |Y| + s = |X|$. □

We need the following lemma in the proof of our main theorem.

Lemma 6.9. *Let $\langle x, \mathrm{Sq}^1(x)\rangle$ be a twisted Steenrod closed parameter ideal. Then none of the ideals $\langle vx^2, \mathrm{Sq}^1(x)^2\rangle$ and $\langle x^2, v\,\mathrm{Sq}^1(x)^2\rangle$ is a Steenrod closed parameter ideal.*

Proof. The ideal $\langle x^2, v\,\mathrm{Sq}^1(x)^2\rangle$ is not Steenrod closed: if $\mathrm{Sq}^2(x^2) = \mathrm{Sq}^1(x)^2$ was in $\langle x^2, v\,\mathrm{Sq}^1(x)^2\rangle$, then it would be a multiple of x^2 for degree reasons. We would thus have $\mathrm{Sq}^1(x)^2 = \lambda x^2$. This contradicts the assumption that x and $\mathrm{Sq}^1(x)$ are coprime. The ideal $\langle vx^2, \mathrm{Sq}(x)^2\rangle$ is not a parameter ideal, because vx^2 and $\mathrm{Sq}^1(x)^2$ are not coprime as v divides $\mathrm{Sq}^1(x)$ by Lemma 3.10 and Theorem 3.15. □

The main result of this section is the following classification theorem.

Theorem 6.10. *The set of Steenrod closed parameter ideals in $H^*(BA_4; \mathbb{F}_2)$ consists of*

(1) the fibered ideals *$\langle v^k, u^l\rangle$ with $l \geq 1$ and $1 \leq k \leq 2^t$, where 2^t is the largest power of 2 dividing l;*
(2) the twisted ideals *$\langle x_n, \mathrm{Sq}^1(x_n)\rangle$ for $n \geq 2$, where x_n is recursively defined as $x_1 = u$ and $x_{n+1} = ux_n^2 + \mathrm{Sq}^1(x_n)^2$;*
(3) and the mixed ideals *$\langle v^i x_n^{2^m}, x_{n+1}^{2^{m-1}}\rangle$, where $m \geq 1$, and either $n = 1$ and $1 \leq i < 2^{m-1}$, or $n \geq 2$ and $0 \leq i < 2^{m-1}$.*

Proof. We have classified the twisted ideals in Theorem 3.15 and the fibered ideals in Theorem 5.8. The ideals in (3) are indeed Steenrod closed parameter ideals by Lemma 6.4 and Remark 6.2.

We will show that any Steenrod closed parameter ideal that is neither twisted nor fibered is of the form (3). We proceed by induction on the sum of the degrees of the parameters. Let I be a Steenrod closed parameter ideal that is neither fibered nor twisted.

Let us first consider the case where both parameters have even degrees. By Proposition 4.5, $I = J^{[2]}$ for some Steenrod closed parameter ideal J. By induction assumption J is either fibered, twisted, or mixed. If J is fibered, so is I. If J is mixed, so is I. If J is twisted, say $J = \langle x_n, \mathrm{Sq}^1(x_n)\rangle$ for $n \geq 2$, then $I = \langle x_n^2, ux_n^2 + \mathrm{Sq}^1(x_n)^2\rangle = \langle x_n^2, x_{n+1}\rangle$ and thus it is a mixed ideal with $i = 0$ and $m = 1$.

It remains to consider the case where at least one of the parameters has odd degree. By Proposition 4.8, they cannot both be odd. If one of the parameters has degree 3, then $I = \langle v, y^2\rangle$ by Proposition 4.8 and thus it is fibered by Definition 5.1. Otherwise we can find parameters $\{vx^2, y^2\}$ of I such that $J = \langle x, y\rangle$ is also a Steenrod closed parameter ideal by Proposition 4.8. By induction the ideal J is either fibered, twisted, or mixed.

First, consider the case that J is fibered and thus generated by a pair of the form $(X, Y) = (v^k, u^{2^t c})$ with $1 \leq k \leq 2^t$ and c odd. The ideal I has to be one of the ideals from Lemma 5.4:

i) $\langle vX^2, Y^2\rangle = \langle v^{2k+1}, u^{2^{t+1}c}\rangle$ is a Steenrod closed parameter ideal if and only if $k < 2^t$ by Lemma 5.7(1). In this case, it is again fibered.
ii) If $|X| < |Y|$, then the ideal I can be $\langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle$, but such ideals are not a Steenrod closed parameter ideals by Lemma 5.7(2).
iii) $\langle X^2, vY^2\rangle$ is not a Steenrod closed parameter ideal, since both generators are divisible by v.
iv) If $|X| > |Y|$, we can have $I = \langle X^2 + u^{|X|-|Y|}Y^2, vY^2\rangle$. By Lemma 5.7(3), this ideal is a Steenrod closed parameter ideal if and only if $k = 2^t$ and $c = 1$. This gives $I = \langle v^{2^{t+1}} + u^{3\cdot 2^t}, vu^{2^{t+1}}\rangle$ which is of the form (3) with $i = 1$, $n = 1$, and $m = t + 1$.

Secondly, consider the case that J is twisted and nonfibered. This means that $J = \langle x_n, \mathrm{Sq}^1(x_n)\rangle$ for some $n \geq 2$. By Remark 3.4, $\{x_n, \mathrm{Sq}^1(x_n)\}$ is the only choice of parameters for J. Neither of the ideals $\langle vx_n^2, \mathrm{Sq}^1(x_n)^2\rangle$ and $\langle x_n^2, v\,\mathrm{Sq}^1(x_n)^2\rangle$ is a Steenrod closed parameter ideal by Lemma 6.9.

Thirdly, consider the case that J is mixed, i.e., generated by a pair of the form

$$(X, Y) = (v^i x_n^{2^m}, u^{2^{m-1}} x_n^{2^m} + \mathrm{Sq}^1(x_n)^{2^m})$$

with $m \geq 1$, $n \geq 2$ and $0 \leq i < 2^{m-1}$ or $m \geq 1$, $n = 1$ and $1 \leq i < 2^{m-1}$. The ideal I has to be one of the ideals from Lemma 5.4:

i) $\langle vX^2, Y^2\rangle$ is again generated by a pair of the form (3) by Lemma 6.7.
ii) $\langle vX^2, Y^2 + u^{|Y|-|X|}X^2\rangle$ if $|Y| > |X|$, which is not a Steenrod closed parameter ideal by Lemma 6.7.
iii) $\langle X^2, vY^2\rangle$ is not a Steenrod closed parameter ideal by Lemma 6.8.
iv) $\langle X^2 + u^{|X|-|Y|}Y^2, vY^2\rangle$ if $|X| > |Y|$, which is not a Steenrod closed parameter ideal by Lemma 6.8.

This completes the proof of the theorem. □

To establish Theorem 1.2, we also need the following result.

Proposition 6.11. *For a given pair of natural numbers there is at most one Steenrod closed parameter ideal with parameters of these degrees. Additionally, the union of families listed in* (1)-(3) *of Theorem 6.10 is disjoint.*

Proof. For a Steenrod closed parameter ideal $I \subset H^*(BA_4)$, we will denote by $|I|$ the set of degrees $\{|X|, |Y|\}$, where $\{X, Y\}$ is a homogeneous system of parameters for I. We will show that the degrees of the Steenrod closed parameter ideals listed in Theorem 6.10 are all distinct.

There is no Steenrod closed parameter ideal $\langle X, Y\rangle$ such that $|X|$ and $|Y|$ are odd. Indeed, for twisted ideals this holds since the degrees of the parameters differ by 1. For nontwisted ideals this follows from Proposition 4.8.

Note that if I is a fibered ideal, then $|I| = \{3k, 2^{t+1}c\}$ with c odd and $1 \leq k \leq 2^t$. If I is twisted and not $\langle v, u\rangle$, then $|I| = \{2^{n+1}-2, 2^{n+1}-1\}$ for some $n \geq 2$. Finally if I is mixed then $|I| = \{3i + (2^r - 1)2^{m+1}, (2^{r+1} - 1)2^m\}$ for some $m \geq 1$ and $r \geq 1$, where $0 \leq i < 2^{m-1}$ if $r \geq 2$ and $1 \leq i < 2^{m-1}$ if $r = 1$.

Suppose that there is a fibered and a twisted ideal different from $\langle v, u\rangle$ with same degree. Then we must have $3k = 2^{n+1} - 1$ and $2^{t+1}c = 2(2^n - 1)$. The second equality implies that $t = 0$. Since $k \leq 2^t$, we have $k = 1$. Then the first equality gives $n = 1$ which is not possible since $n \geq 2$ by our assumption.

If there is a twisted and a mixed ideal with the same degree, then $2(2^n - 1) = (2^{r+1} - 1)2^m$ and $2^{n+1} - 1 = 3i + (2^r - 1)2^{m+1}$. The first equality gives $m = 1$. So we must have $i = 0$ by the given inequalities. But then the second equality does not hold since the right-hand side is even and the left-hand side is odd.

For the case of a fibered and a mixed ideal with the same degree, we need the following observations. If $I = \langle X, Y \rangle$ is a Steenrod closed parameter ideal with both parameters of even degrees, then there exists a unique Steenrod closed parameter ideal J such that $I = J^{[2]}$ by Proposition 4.5. Note that $|J| = \{|X|/2, |Y|/2\}$. For I fibered, the degree of J is again the degree of a fibered ideal. For I mixed, the degree of J is either the degree of a mixed ideal or the degree of a twisted ideal. Hence by induction it suffices to prove the statement when one parameter is of odd degree and the other parameter has even degree.

Suppose that there is a fibered and a mixed ideal with the same degree. By the discussion above we can assume that one of the parameters is of odd degree. Then k is odd, and we have $3k = 3i + (2^r - 1)2^{m+1}$ and $2^{t+1}c = (2^{r+1} - 1)2^m$. This implies that $t + 1 = m$. The inequality $k \leq 2^t$ gives $k \leq 2^{m-1}$. From the first equality we obtain

$$3i + (2^r - 1)2^{m+1} \leq 3 \cdot 2^{m-1},$$

which gives $0 \leq 3i \leq (3 - 4(2^r - 1))2^{m-1}$. Since for $r \geq 1$, we have $3 - 4(2^r - 1) < 0$, this gives a contradiction.

Hence we can conclude that no two Steenrod closed parameter ideals of different type have parameters of the same degrees. In particular, the union in Theorem 6.10 is disjoint.

To complete the proof, we also need to eliminate the possibility that two different Steenrod closed parameter ideals of the same type have parameters with equal degrees. This is clear in the twisted case since they are defined recursively. For the fibered case, assume that there are two pairs of numbers k, k', t, t', c, c' satisfying the conditions for the fibered ideal such that $\{3k, 2^t c\} = \{3k', 2^{t'} c'\}$. Since by induction we can assume that one of the parameters is of odd degree, then we must have $3k = 3k'$ since these two are the only possible odd numbers in the pair. This gives that $t = t'$ and $c = c'$, hence the two fibered ideals are equal. For the mixed case, suppose that for some i, i', r, r', m, m' satisfying the conditions for a mixed ideal, we have

$$\{3i + (2^r - 1)2^{m+1}, (2^{r+1} - 1)2^m\} = \{3i' + (2r' - 1)2^{m'+1}, (2^{r'+1} - 1)2^{m'}\}$$

Again by induction we can assume that one of the degrees is odd. The only odd degree terms are the ones that involve i and i', so we must have $(2^{r+1} - 1)2^m = (2^{r'+1} - 1)2^{m'}$. This gives $r = r'$ and $m = m'$. From this we obtain $i = i'$, and hence the two mixed ideals are equal. □

Theorem 6.10 and Proposition 6.11 together complete the proof of Theorem 1.2, the main theorem of the paper.

Remark 6.12. From the statement of Theorem 1.2, we can observe that parameters of a Steenrod closed parameter ideal can be chosen so that one of the parameters is divisible by v. In the fibered case this is clear from the definition. In the twisted case it follows from the recursion formula for x_n. In the mixed case, this is obvious when $i > 0$. If $i = 0$, then the mixed ideal is $\langle x_n^{2^m}, x_{n+1}^{2^m - 1} \rangle = \langle x_n^{2^m}, \mathrm{Sq}^1(x_n)^{2^m} \rangle$. So v divides one of these parameters since v divides one of the parameters in the twisted case.

Calculating the degrees of the parameters and using uniqueness of Proposition 6.11 yields the following result.

Corollary 6.13. *Let $I \subset H^*(BA_4)$ be a Steenrod closed parameter ideal. Then the (unordered) degrees of the parameters of I are of the form*

(1) $(3k, 2l)$ with $l \geq 1$ and $1 \leq k \leq 2^t$, where 2^t is the largest power of 2 dividing l, or

(2) $(3i + 2^{m+n+1} - 2^{m+1}, 2^{m+n+1} - 2^m)$ for $m \geq 0$, $n \geq 1$ and $0 \leq i < 2^{m-1}$.

For each such pair there is a unique Steenrod closed parameter ideal with parameters of these degrees.

Remark 6.14. Note that at least one of the degrees of the parameters of a Steenrod closed parameter ideal is divisible by three by Corollary 6.13.

We can also conclude the following:

Corollary 6.15. *Every Steenrod closed parameter ideal of $H^*(BA_4)$ is of the form $\langle X, Y\rangle$ with $X, Y \in \mathbb{F}_2[u, v] \subset H^*(BA_4)$.*

To obtain a classification of Steenrod closed parameter ideals in $H^*(B\,\mathrm{SO}(3))$, we need the following lemma.

Lemma 6.16. *The map sending a Steenrod closed parameter ideal in $H^*(B\,\mathrm{SO}(3))$ to its extension in $H^*(BA_4)$ is injective.*

Proof. Let I, J be parameter ideals of $H^*(B\,\mathrm{SO}(3))$ with parameters $\{X, Y\}$ and $\{X', Y'\}$, respectively. Suppose that $RI = RJ$ for $R = H^*(BA_4)$.

After renaming the parameters we may assume that $|X| < |Y|$ and $|X'| < |Y'|$. For degree reasons and since we work over $\mathbb{F}_2$, we have $X = X'$ and $Y' = Y + \lambda X$ for some homogeneous element $\lambda \in H^*(BA_4)$. We conclude from Lemma 2.11 that $\lambda \in \mathbb{F}_2[a, b]$ is $\mathrm{GL}_2(2)$-invariant. It follows that $I = J$. □

The following result is a consequence of Theorem 1.2 and Lemma 6.16.

Corollary 6.17. *Restriction of ideals yields a bijection between Steenrod closed parameter ideals in $H^*(BA_4)$ and Steenrod closed parameter ideals in the Dickson algebra $H^*(B\,\mathrm{SO}(3))$. Thus the Steenrod closed parameter ideals in $H^*(B\,\mathrm{SO}(3))$ are the restrictions of the ideals listed in Theorem 1.2.*

Proof. The extension from $H^*(B\,\mathrm{SO}(3))$ to $H^*(BA_4)$ defines an injection on Steenrod closed parameter ideals by Lemma 6.16 and Lemma 2.8. It is surjective by Corollary 6.15. The inverse of this bijection is restriction. □

7. The k-invariants of a free G-action

In [Oli79, Theorem 2], Oliver showed that A_4 can not act freely on any finite CW-complex X with $H^*(X; \mathbb{Z}) \cong H^*(S^n \times S^n; \mathbb{Z})$. The proof proceeds in two steps. First, A_4 can not act freely on any finite-dimensional X with mod-2 cohomology of a finite product of n-spheres

$$H^*(X; \mathbb{F}_2) \cong H^*(\prod_k S^n; \mathbb{F}_2)$$

and A_4 acting trivially on cohomology (see [Oli79, Theorem 1]). Secondly, a Lefschetz fixed point argument ensures that A_4 acts trivially on cohomology provided that $H^*(X; \mathbb{Z}) \cong H^*(S^n \times S^n; \mathbb{Z})$ and X is finite. We observe that this assumption can be weakened to mod-2 coefficients.

Theorem 7.1. *There is no finite, free A_4-CW-complex X with $H^*(X;\mathbb{F}_2) \cong H^*(S^n \times S^n;\mathbb{F}_2)$.*

Proof. Suppose that X is a finite, A_4-CW-complex with $H^*(X;\mathbb{F}_2) \cong H^*(S^n \times S^n;\mathbb{F}_2)$. By [Oli79, Theorem 1], it suffices to show that if A_4 acts nontrivially on $H^n(X;\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$, then the A_4-action on X is not free. Let $A_4 \to \mathrm{GL}_2(2) \cong S_3$ be the representation of A_4 on $H^n(X;\mathbb{F}_2)$ and assume it is nontrivial. Since its kernel is a proper, normal subgroup of A_4, it follows that the kernel is $\mathbb{Z}/2 \times \mathbb{Z}/2$ and that $C_3 \subset A_4$ acts nontrivially on $H^n(X;\mathbb{F}_2)$. Therefore, the trace of any generator of C_3 acting on $H^n(X;\mathbb{F}_2)$ is one. It follows from the Lefschetz fixed point theorem with mod-2 coefficients (see e.g. [Bro71, III.C Theorem 2]) that C_3 fixes a point of X. Hence A_4 does not act freely on X. □

In this section we consider free A_4-actions on a finite CW-complex X which has mod-2 cohomology of a product of two spheres $S^n \times S^m$ with $n \neq m$. We first recall the general definitions and methods for studying free group actions on a finite-dimensional CW-complex. Let G be a finite group and X be a finite-dimensional G-CW-complex. The Borel construction for the G-CW-complex X is a fibration

$$X \to EG \times_G X \xrightarrow{\pi} BG$$

where EG is the universal space for G, and BG is the classifying space for G. Let R be a commutative ring with unity. The Serre spectral sequence associated to this fibration has E_2-page

$$E_2^{s,t} = H^s(BG; H^t(X;R))$$

and it converges to $H^*(EG \times_G X;R)$. The cohomology ring $H^*(EG \times_G X;R)$ is called the equivariant Borel cohomology of X, denoted by $H^*_G(X;R)$.

If G acts freely on X, then $EG \times_G X$ is homotopy equivalent to the orbit space X/G which is a finite-dimensional CW-complex. This puts restrictions on the E_∞-page of the Serre spectral sequence. Since $H^n_G(X;R) \cong H^n(X/G;R) = 0$ for $n > \dim X$, we have $E_\infty^{i,j} = 0$ for $i + j > \dim X$.

The Serre spectral sequence is multiplicative and thus has a pairing

$$E_r^{s,t} \otimes E_r^{s',t'} \to E_r^{s+s',t+t'}$$

so that the differential d_r is a derivation with respect to this product. If for all $s, t \geq 0$, $H^s(BG;R)$ and $H^t(X;R)$ are free R-modules of finite type, and the G-action on the cohomology $H^*(X;R)$ is trivial, then

$$E_2^{*,*} = H^*(BG; H^*(X;R)) \cong H^*(X;R) \otimes H^*(BG;R)$$

as a bigraded algebra (see [McC01, Proposition 5.6]). In fact, by the Universal coefficient theorem [Spa95, Theorem 5.5.10] the isomorphism above holds under the weaker assumption that G acts trivially on $H^*(X;R)$ and either $H^t(X;R)$ is a finitely generated, free R-module for each t, or R is a field. Note that here the isomorphisms as bigraded algebras means that the product structure on $E_2^{*,*}$ coincides with the product induced by cup products on $H^*(X;R)$ and $H^*(BG;R)$.

The product structure on the Serre spectral sequence gives an $E_r^{*,0}$-module structure on $E_r^{*,*}$. There is also a $H^*(BG;R)$-module structure on $E_r^{*,*}$ induced by the constant map $X \to pt$. When X is a connected space, there is a canonical isomorphism

$$E_2^{*,0} \cong H^0(X;R) \otimes H^*(BG;R) \cong H^*(BG;R).$$

In this case, the $H^*(BG;R)$-module structure on $E_2^{*,*}$ coincides with the $E_2^{*,0}$-module structure described above. The $H^*(BG,R)$-module structure is used for calculating differentials of arbitrary elements given the differentials of the generators of the algebra $H^*(X;R)$; see for instance [Car80].

We will also use compatibility of two different Serre spectral sequences when there is a diagram of fibrations; in particular for the diagram

$$\begin{array}{ccccc} X & \longrightarrow & EG \times_G X & \longrightarrow & BG \\ \| & & \uparrow & & \uparrow \\ X & \longrightarrow & EH \times_H X & \longrightarrow & BH \end{array}$$

where H is a subgroup of G.

Assume that X is a finite-dimensional G-CW-complex with cohomology ring

$$H^*(X;\mathbb{F}_2) \cong H^*(S^n \times S^m;\mathbb{F}_2)$$

for some n, m satisfying $1 \leq n < m$. In fact, with the exception of Proposition 7.9, for all the results we obtain in this section we only use the fact that the mod-2 cohomology is isomorphic to $\mathbb{F}_2$ in degrees 0, n, m, $n+m$ such that a generator in degree $m+n$ is the product of two generators in degrees m and n. So we can also take this weaker cohomology condition as our assumption on X.

Since $\mathbb{F}_2$ has no nontrivial automorphisms, the G-action on $H^*(X;\mathbb{F}_2)$ is trivial. Hence we have

$$E_2^{s,t} \cong H^t(X;\mathbb{F}_2) \otimes H^s(BG;\mathbb{F}_2)$$

with $E_2^{s,t}$ nonzero only at dimensions $t = 0, n, m, n+m$. As it is often done, we identify $E_2^{0,*} \cong H^*(X;\mathbb{F}_2) \otimes H^0(BG;\mathbb{F}_2)$ with $H^*(X;\mathbb{F}_2)$ and write elements of $E_2^{*,*}$ as a product tx instead of a tensor product $t \otimes x$. If t_1, t_2 are generators for $H^*(X;\mathbb{F}_2)$ in dimensions n and m, then $E_2^{*,*}$ is a free $H^*(BG;\mathbb{F}_2)$-module with generators 1, t_1, t_2 and t_1t_2.

Since $n < m$, we have $d_i(t_1) = 0$ for all $i \leq n$, hence t_1 is transgressive. Let $d_{n+1}(t_1) = \mu_1 \in H^{n+1}(BG;\mathbb{F}_2)$. We call the cohomology class $\mu_1 \in H^{n+1}(BG;\mathbb{F}_2)$ the *k-invariant for the sphere S^n*. If we also have $d_i(t_2) = 0$ for $i \leq m$, then t_2 is also transgressive and there is a $\mu_2 \in H^{m+1}(BG;\mathbb{F}_2)$ such that $d_{m+1}(t_2) = \mu_2 \in H^{m+1}(BG;\mathbb{F}_2)$ modulo the ideal $\langle \mu_1 \rangle$. We call a choice of μ_2 in $H^*(BG;\mathbb{F}_2)$ a *k-invariant for the sphere S^m*.

A graded $\mathbb{F}_2$-vector space $A^* = \bigoplus_{i \geq 0} A^i$ is called *finite-dimensional* if there is a d such that $A^i = 0$ for all $i > d$. If P is a subgroup of G which acts freely on X, then $H_P^*(X;\mathbb{F}_2) \cong H^*(X/P;\mathbb{F}_2)$. Hence $H_P^i(X;\mathbb{F}_2) = 0$ for $i > \dim X$ and $H_P^*(X;\mathbb{F}_2)$ is finite-dimensional as a graded $\mathbb{F}_2$-vector space.

Lemma 7.2. *Let G be a finite group, and X be a finite-dimensional G-CW-complex such that (as $\mathbb{F}_2$-algebras)*

$$H^*(X;\mathbb{F}_2) \cong H^*(S^n \times S^m;\mathbb{F}_2)$$

for some $n, m \geq 1$ satisfying $n < m$. Assume that G has a subgroup $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ that acts freely on X. Then the k-invariant $\mu_1 \in H^{n+1}(BG;\mathbb{F}_2)$ for the sphere S^n is not equal to zero.

Proof. Let $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ be a subgroup of G that acts freely on X. Consider the spectral sequence

$$E_2^{i,j} = H^i(BP; H^j(X;\mathbb{F}_2)) \Rightarrow H_P^*(X;\mathbb{F}_2)$$

for the P-action on X. Since the P-action on X is free, $H_P^*(X;\mathbb{F}_2)$ is finite-dimensional as a graded $\mathbb{F}_2$-vector space. We denote the differentials of this spectral sequence by d_i'. Assume that $\mu_1 = 0$. Then by the compatibility of the Serre spectral sequences, we have

$$d'_{n+1}(t_1) = \mathrm{Res}^G_P(d_{n+1}(t_1)) = \mathrm{Res}^G_P \mu_1 = 0.$$

Note that $H^*(BP;\mathbb{F}_2) \cong \mathbb{F}_2[a,b]$ with $|a| = |b| = 1$. In particular, the cohomology ring $H^*(BP;\mathbb{F}_2)$ has no zero divisors. Assume that $d'_{m-n+1}(t_2) \neq 0$. Then for every nonzero cohomology class $\alpha \in H^*(BP;\mathbb{F}_2)$, we have

$$d'_{m-n+1}(t_2\alpha) = d'_{m-n+1}(t_2)\alpha \neq 0.$$

Hence the horizontal line $E_2^{*,m}$ does not survive to the E_{m+1}-page. In this case the only differentials that can hit the bottom line have to come from the top line. But $E_2^{*,0}$ has Krull dimension 2, which gives that $E_\infty^{*,0} = E_2^{*,0}/\langle d_{n+m+1}(t_1t_2)\rangle$ has Krull dimension at least 1. This is in contradiction with the fact that $E_\infty^{*,0}$ is finite-dimensional. So we must have $d'_{m-n+1}(t_2) = 0$.

Since $m > n$ and $d'_{m-n+1}(t_2) = 0$, both t_1 and t_2 survive to E_{n+1}. By the product structure on the spectral sequence, this gives that

$$d'_{n+1}(t_1t_2) = d'_{n+1}(t_1)t_2 + t_1 d'_{n+1}(t_2) = 0.$$

Let $\mu_2' \coloneqq d'_{m+1}(t_2)$ denote the k-invariant for the sphere S^m. If $\mu_2' = 0$, then the horizontal line at $j = m$ will survive to the E_∞-page which gives a contradiction because $H_P^*(X;\mathbb{F}_2)$ is finite-dimensional. Hence $\mu_2' = d'_{m+1}(t_2) \neq 0$. This gives that

$$d'_{m+1}(t_1t_2) = t_1\mu_2' \neq 0.$$

Hence the top horizontal line does not survive to the E_∞-page, therefore there is no other differential hitting the bottom line. We conclude that

$$E_\infty^{*,0} \cong H^*(BP;\mathbb{F}_2)/\langle \mu_2' \rangle.$$

This again gives a contradiction because the Krull dimension of $H^*(BP;\mathbb{F}_2)$ is 2, hence $E_\infty^{*,0}$ is not finite-dimensional. □

Lemma 7.3. *Let G and X be as in Lemma 7.2. Suppose that μ_1 is a non-zero divisor in $H^*(BG;\mathbb{F}_2)$. Let $S = H^*(BG,\mathbb{F}_2)/\langle\mu_1\rangle$ denote the quotient ring. Then $t_2 \in H^m(X;\mathbb{F}_2)$ is transgressive and there is a short exact sequence*

$$0 \to (S/\mu_2 S)_i \to H^i_G(X;\mathbb{F}_2) \to (\mathrm{Ann}_S(\mu_2))_{i-m} \to 0$$

where the first map is induced by $\pi^*\colon H^*(BG;\mathbb{F}_2) \to H^*_G(X;\mathbb{F}_2)$.

Proof. Consider the case where $m \leq 2n$. Write $d_{m-n+1}(t_2)$ in the form $t_1\gamma$ for some $\gamma \in H^*(BG;\mathbb{F}_2)$. Then $d_{n+1}(t_1\gamma) = 0$ because $t_1\gamma$ is in the image of d_{m-n+1}. This gives that $0 = d_{n+1}(t_1\gamma) = \mu_1\gamma$. Since μ_1 is a non-zero divisor we get $\gamma = 0$, hence $d_{m-n+1}(t_2) = 0$. Now consider the case where $m > 2n$. In this case $d_{m-n+1}(t_2)$ lies in the kernel of the map $d_{n+1}\colon E_{n+1}^{*,n} \to E_{n+1}^{*,0}$, which is defined by multiplication with μ_1. Since μ_1 is a non-zero divisor, we have $\ker d_{n+1} = 0$, hence $d_{m-n+1}(t_2) = 0$. Therefore t_2 survives to E_{m+1}, i.e., it is transgressive in both cases.

By the multiplicative structure of the spectral sequence, we have $d_{n+1}(t_1t_2) = t_2\mu_1 \neq 0$. This gives that E_{m+1} only has two nonzero lines which are at $j = 0$ and $j = m$ and both of them are isomorphic to S. The differential between these two nonzero lines is defined by the multiplication with $d_{m+1}(t_2) = \mu_2$. Hence we have

$E_\infty^{*,0} \cong S/\mu_2 S$ and $E_\infty^{*,m} \cong \mathrm{Ann}_S(\mu_2)$. Since the E_∞-page has only two nonzero horizontal lines at $j = 0$ and $j = m$, for every $i \geq 0$, there is a short exact sequence

$$0 \to E_\infty^{i,0} \to H_G^i(X; \mathbb{F}_2) \to E_\infty^{i-m,m} \to 0.$$

Combining with the above observations, this gives the short exact sequence given in the lemma. □

In the rest of the section we assume $G = A_4$. By Theorem 2.1, the mod-2 cohomology algebra of A_4 is

$$H^*(BA_4; \mathbb{F}_2) \cong H^*(B(\mathbb{Z}/2)^2; \mathbb{F}_2)^{C_3} \cong \mathbb{F}_2[u, v, w]/\langle u^3 + v^2 + vw + w^2 \rangle,$$

where $\deg(u) = 2$ and $\deg(v) = \deg(w) = 3$. From the first isomorphism it is clear that $H^*(BA_4, \mathbb{F}_2)$ has no zero divisors.

Proposition 7.4. *Let $G = A_4$ and X be a finite, free G-CW-complex such that*

$$H^*(X; \mathbb{F}_2) \cong H^*(S^n \times S^m; \mathbb{F}_2)$$

for some $n, m \geq 1$ with $n < m$. Then the following holds:

(1) The homomorphism

$$\pi^* \colon H^*(BG; \mathbb{F}_2) \to H_G^*(X; \mathbb{F}_2)$$

is surjective and its kernel is the ideal $J := \langle \mu_1, \mu_2 \rangle$ generated by the k-invariants μ_1 and μ_2.

(2) $H^*(BG, \mathbb{F}_2)/J$ is finite over $\mathbb{F}_2$.

(3) The ideal J is closed under Steenrod operations.

Proof. Consider the Serre spectral sequence in mod-2 coefficients. By Lemma 7.2, the k-invariant μ_1 is nonzero. The ring $H^*(BG; \mathbb{F}_2)$ has no zero-divisors, hence we can apply Lemma 7.3 to conclude that the generator t_2 is transgressive. So the k-invariant μ_2 is defined and as in the proof of Lemma 7.3, the E_{m+1}-page has only two nonzero lines at $j = 0$ and $j = m$. Since $S/\mu_2 S = H^*(BG; R)/J$ where $J = \langle \mu_1, \mu_2 \rangle$, we conclude that the kernel of π^* is the ideal J. The cohomology ring $H_G^*(X; \mathbb{F}_2)$ is finite-dimensional. This gives that $H^*(BG; \mathbb{F}_2)/J$ is finite-dimensional, hence finite. The homomorphism π^* is induced by a continuous map between topological spaces, thus its kernel J is closed under Steenrod operations.

Since $H^*(BG; \mathbb{F}_2)/J$ is finite, we conclude that μ_1, μ_2 is a regular sequence by Lemma 2.6. Thus μ_2 is a non-zero divisor in S. This gives that $(\mathrm{Ann}_S(\mu_2))_{i-m} = 0$ for $S = H^*(BG; \mathbb{F}_2)/\langle \mu_1 \rangle$. Thus $E_\infty^{i,j} = 0$ for all $j > 0$. Hence π^* is surjective. □

We obtain the following:

Theorem 7.5. *Let $G = A_4$ and X be a finite, free G-CW-complex such that*

$$H^*(X; \mathbb{F}_2) \cong H^*(S^n \times S^m; \mathbb{F}_2)$$

for some $n, m \geq 1$ with $n < m$. The ideal J generated by the k-invariants is equal to one of the ideals listed in Theorem 1.2.

Proof. By Proposition 7.4, the ideal J is a parameter ideal closed under Steenrod operations. Hence the result follows from Theorem 1.2. □

When the G-CW-complex X has the integral cohomology of a product of two spheres, the possibilities for the ideal J are more restricted. To see this we use the commutativity of Steenrod operations with transgressions.

Proposition 7.6. *Let $G = A_4$ and X be a finite, free G-CW-complex such that*

$$H^*(X; \mathbb{F}_2) \cong H^*(S^n \times S^m; \mathbb{F}_2)$$

for some $n, m \geq 1$ with $n < m$. Let t_1 and t_2 denote the generators of $H^*(X; \mathbb{F}_2)$ in dimensions n and m, and let μ_1 and μ_2 in $H^*(BA_4; \mathbb{F}_2)$ be the k-invariants for the A_4-action on X. If $\mathrm{Sq}^{m-n}(t_1) = t_2\lambda$, then $\mathrm{Sq}^{m-n}(\mu_1) = \mu_2\lambda$ modulo $\langle \mu_1 \rangle$.

Proof. By [McC01, Corollary 6.9], the transgressions in the Serre spectral sequence commute with the Steenrod operations. This means that if $x \in H^n(X; \mathbb{F}_2)$ survives to the E_{n+1}-page, then for every $i \geq 0$, $\mathrm{Sq}^i(x)$ survives to the E_{n+i+1}-page and the equality

$$d_{n+i+1}(\mathrm{Sq}^i(x)) = \mathrm{Sq}^i(d_{n+1}(x))$$

holds on the E_{n+i+1}-page. Using this, we can conclude that if $\mathrm{Sq}^{m-n}(t_1) = t_2\lambda$, then

$$\mathrm{Sq}^{m-n}(\mu_1) = \mathrm{Sq}^{m-n}(d_{n+1}(t_1)) = d_{m+1}(\mathrm{Sq}^{m-n}(t_1)) = d_{m+1}(t_2\lambda) = \mu_2\lambda$$

modulo $\langle \mu_1 \rangle$. This completes the proof. □

As a consequence we obtain the following theorem.

Theorem 7.7. *Let $G = A_4$ and X be a finite, free G-CW-complex such that*

$$H^*(X; \mathbb{Z}) \cong H^*(S^n \times S^m; \mathbb{Z})$$

for some $n, m \geq 1$ with $n < m$. Let J be the ideal generated by the k-invariants such that $H^*(X/G; \mathbb{F}_2) \cong H^*(BA_4; \mathbb{F}_2)/J$. Then the ideal J is equal to one of the ideals listed in Theorem 1.2 except the twisted ones listed in (2).

Proof. By the universal coefficient theorem for cohomology, the cohomology ring $H^*(X; \mathbb{F}_2)$ satisfies the conditions of Theorem 7.5, so we have $H^*(X/G; \mathbb{F}_2) \cong H^*(BG; \mathbb{F}_2)/J$, where the ideal J must be equal to one of the ideals listed in Theorem 1.2. Consider the homomorphism $m_2^*\colon H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{F}_2)$ induced by the mod-2 reduction map $m_2\colon \mathbb{Z} \to \mathbb{F}_2$. Since the generator t_1 is in the image of the mod-2 reduction map, $\beta_0(t_1) = 0$ where β_0 is the connecting homomorphism for $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{F}_2 \to 0$. The Bockstein homomorphism is $\beta = m_2^* \circ \beta_0$, hence we have $\beta(t_1) = 0$. Since $\beta(t_1) = \mathrm{Sq}^1(t_1)$, we can apply Proposition 7.6 and conclude that $\mathrm{Sq}^1(\mu_1) = 0 \bmod \langle \mu_1 \rangle$. This shows that the ideal $J = \langle \mu_1, \mu_2 \rangle$ can not be twisted. □

Removing the degrees of the parameters for the twisted ideals from Corollary 6.13 yields the list in the following result.

Corollary 7.8. *Let $G = A_4$ and X be a finite, free G-CW-complex such that*

$$H^*(X; \mathbb{Z}) \cong H^*(S^n \times S^m; \mathbb{Z})$$

for some $n, m \geq 1$.

Then the unordered pair $(n + 1, m + 1)$ must be one of the following:

(1) $(3k, 2l)$ where k is not larger than the highest power of 2 dividing l, or
(2) $(3i + 2^{s+r+1} - 2^{s+1}, 2^{s+r+1} - 2^s)$ for $s \geq 1$, $r \geq 1$ and $0 \leq i < 2^{s-1}$.

Another consequence of Proposition 7.6 is a slight generalization of a result by Blaszczyk [Bla13, Proposition 4.3].

Proposition 7.9. *The group $G = A_4$ can not act freely on a finite CW-complex X with $H^*(X; \mathbb{F}_2) \cong H^*(S^1 \times S^m; \mathbb{F}_2)$ for any $m \geq 1$.*

Proof. The case $m = 1$ holds by Theorem 7.1. If $m \geq 2$, then the k-invariant μ_1 must be equal to $u \in H^2(BA_4; \mathbb{F}_2)$. Since $\mathrm{Sq}^1(t_1) = t_1^2 = 0$, we must have $\mathrm{Sq}^1(u) = 0$ modulo the ideal $\langle u \rangle$. However $\mathrm{Sq}^1(u) = v$ is not in the ideal $\langle u \rangle$. □

Extending the above arguments we can say that a parameter ideal $\langle \mu_1, \mu_2 \rangle$ with $\mathrm{Sq}^{m-n}(\mu_1) = \mu_2$ can not be realized by a finite, free A_4-CW-complex X homotopy equivalent to $S^n \times S^m$.

Corollary 7.10. *Let $G = A_4$ and $\langle x, Sq^1(x) \rangle$ be a twisted ideal. For each $r \geq 0$, the ideals of the form $\langle x^{2^r}, \mathrm{Sq}^1(x)^{2^r} \rangle$ are not realizable by a finite, free G-CW-complex X homotopy equivalent to $S^n \times S^m$.*

Proof. The statement follows from Proposition 7.6 since $\mathrm{Sq}^{2^r}(x^{2^r}) = \mathrm{Sq}^1(x)^{2^r}$. □

In particular by Corollary 7.10, the fibered pairs (v^{2^r}, u^{2^r}) are not realizable by a finite, free A_4-CW-complex X homotopy equivalent to $S^n \times S^m$.

8. Constructions using fixity methods

Fixity methods for constructing free actions on products of spheres were first introduced in [ADU04] and they were also used in [UY10].

Let G be a nontrivial, finite group and let F denote the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$. For a real number x the conjugation is defined by $\overline{x} = x$, for a complex number $x = a + bi$ by $\overline{x} = a - bi$, and for a quaternion $x = a + bi + cj + dk$ by $\overline{x} = a - bi - cj - dk$. On the vector space F^n, we define an inner product by

$$(v, w) = v_1\overline{w}_1 + v_2\overline{w}_2 + \cdots + v_n\overline{w}_n.$$

The classical group $U_F(n)$ is defined as the subgroup of $\mathrm{GL}_n(F)$ formed by linear transformations that preserve the inner product (see [UY10, Section 2]).

Definition 8.1. Let V be a representation of G over F. We say V has *fixity* f if

$$f = \max_{1 \neq g \in G} \dim_F V^g.$$

By Lemma 2.2 and Corollary 2.3 in [ADU04], fixity of a unitary representation V is the smallest integer f such that G-action on the coset space

$$U_F(n)/U_F(n - f - 1)$$

is free. Here the G-action on $U_F(n)$ is given by the representation $\rho_V : G \to U_F(n)$. For each $1 \leq k \leq n$, the Stiefel manifold $V_k(F^n)$ is defined as the subspace of F^{nk} formed by k-tuples $(v_1, v_2, \ldots, v_k)$ of vectors v_i in F^n such that for every pair (i, j), we have $(v_i, v_j) = 1$ if $i = j$, and $(v_i, v_j) = 0$ if $i \neq j$. There is a homeomorphism between $V_k(F^n)$ and the coset space $U_F(n)/U_F(n-1)$. For each k there is a map $q_k : V_{k+1}(F^n) \to V_k(F^n)$ which takes $(v_1, \cdots, v_{k+1})$ to $(v_1, \cdots, v_k)$. The map q_k defines a sphere bundle with fiber $S^{d(n-k)-1}$ where $d = \dim_{\mathbb{R}} F$. As in [UY10, Section 2], we denote the corresponding vector bundle

$$\overline{q}_k : \overline{V}_{k+1}(F^n) \to V_k(F^n)$$

by ξ_k. The map ξ_k is stably trivial, see for example [UY10, Lemma 2.1]. If G acts on F^n unitarily, we get an induced G-action on $V_k(F^n)$ and $\overline{V}_{k+1}(F^n)$ such that $\overline{q}_k$ is an equivariant bundle.

For the octonions, there is no notion of Stiefel manifolds $V_k(\mathbb{O}^n)$ for arbitrary k. Nevertheless, there exists a bundle $\overline{V}_2(\mathbb{O}^n) \to V_1(\mathbb{O}^n)$, see [Jam58, Section 8]. If

V is an orthogonal G-representation of dimension n, we get an induced action on $\mathbb{O}^n \cong V \otimes_{\mathbb{R}} \mathbb{O}$. In this case we give an alternative description of $\overline{q_1}$ that provides equivariant bundles as well.

Recall that the tensor product of two finite-dimensional inner product spaces $(V, s_V), (W, s_W)$ is again an inner product space with inner product

$$s_{V\otimes W}(v_1 \otimes w_1, v_2 \otimes w_2) = s_V(v_1, v_2)s_W(w_1, w_2).$$

Moreover, if V, W are orthogonal G-representations of a group G, so is $V \otimes W$.

Lemma 8.2. *Let G be finite group and let V be a real G-representation of dimension ≥ 2 equipped with a G-invariant inner product s. Let W be a finite-dimensional, real vector space with a bilinear map $m\colon W \otimes W \to W$ with no zero divisors. Let G act trivially on W and pick an inner product on W. Let $M\colon (V \otimes W) \otimes (V \otimes W) \to W$ be the bilinear map*

$$M(v \otimes w, v' \otimes w') = s(v, v')m(w, w').$$

Then

$$E = \{(x, y) \mid x, y \in V \otimes W, |x| = 1, M(x, y) = 0\}$$

is a stably trivial vector bundle over the unit sphere $S(V \otimes W)$ in $V \otimes W$. The group G acts on E, and if V has fixity one, then the induced action on the sphere bundle $S(E)$ of E is free.

Proof. Note that E is the kernel of the map

$$\varphi\colon S(V \otimes W) \times (V \otimes W) \to S(V \otimes W) \times W, \quad (x, y) \mapsto (x, M(x, y)),$$

of vector bundles over $S(V \otimes W)$. We show that φ_x is surjective for any $x \in S(V \otimes W)$. Let $w \in W$. Write $x = \sum_i e_i \otimes x_i$, where $\{e_i\}_i$ is an orthonormal basis for V. Since $x \neq 0$, there exists an index j such that $x_j \neq 0$. Since $m(x_j, \_)$ is injective and thus an isomorphism of the finite-dimensional vector space W, there exists an element $w' \in W$ such that $m(x_j, w') = w$. It follows that $\varphi_x(e_j \otimes w') = w$. Hence φ_x is surjective. Since φ has constant rank, its kernel E is again a vector bundle. Since short exact sequences of vector bundles split, we obtain $E \oplus (S(V \otimes W) \times W) \cong S(V \otimes W) \times (V \otimes W)$. Hence E is stably trivial.

The tensor product $V \otimes W$ is an orthogonal G-representation with $g(v \otimes w) = gv \otimes w$. This induces actions on E, $S(E)$, and $S(V \otimes W)$. Moreover, the projection map of the bundle is equivariant.

We show that the G-action on $S(E)$ is free if V has fixity one. Let $g \in G$, $1 \neq g$. If $V^g = 0$, then g does not even fix a point on the base space of the bundle. If V^g is one-dimensional, let $u \in V^g$ be a generating unit vector. Then the fixed point space E^g is

$$E^g = \{(u \otimes w, u \otimes w') \mid |u \otimes w| = 1,\ M(u \otimes w, u \otimes w') = m(w, w') = 0\}.$$

Since $w \neq 0$, it follows that $w' = 0$ and hence $S(E)^g = \emptyset$. □

The assumption on W means that W is a finite-dimensional real division algebra. In particular, the lemma applies to the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The finite-dimensional division algebras are not classified, but they must have dimension 1, 2, 4 or 8; see [Dar10].

Lemma 8.3. *Let G be a finite group and $n \geq 2$. Suppose that G has an n-dimensional real representation with fixity one and let $d = 1, 2, 4, 8$. Then there is a G-action on a stably trivial $d(n-1)$-dimensional bundle E over S^{dn-1} such that the restriction of the action to its sphere bundle $S(E)$ is free.*

Proof. This follows from Lemma 8.2, where W is either $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ with their respective multiplications. □

Remark 8.4. The cases of $d = 1, 2, 4$ also follow from the discussion after Definition 8.1 by possibly first inducing the real representation up to a complex or quaternionic representation.

To get an action on a product of spheres, we need a trivial vector bundle, but we only have a stably trivial bundle. We thus use [UY10, Lemma 2.2]:

Lemma 8.5. *Let $F = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and $c = \dim_{\mathbb{R}} F$. Suppose that ξ is an m-dimensional, stably trivial F-vector bundle over an n-dimensional CW-complex B. If $n \leq c(m+1) - 2$, then ξ is trivial.*

We apply this lemma to construct free actions on products of spheres. In fact we will be using this lemma only in the case $F = \mathbb{R}$.

Proposition 8.6. *Let G be a finite group and $n \geq 2$. Suppose that G has an n-dimensional real representation with fixity one. Then for every $q \geq 2$ and $d = 1, 2, 4, 8$, there is a free G-action on $S^{dn-1} \times S^{dq(n-1)-1}$.*

Proof. The space is given by the sphere bundle $S(E^{\oplus q})$, where $E^{\oplus q}$ denotes the q-fold Whitney sum of the vector bundle E from Lemma 8.3 with itself. The bundle $E^{\oplus q} \to S^{dn-1}$ is a $qd(n-1)$-dimensional, stably trivial real vector bundle. By Lemma 8.5 applied to the case $F = \mathbb{R}$ and $c = 1$, this bundle is trivial if $dn - 1 \leq qd(n-1) - 1$. This inequality holds for $q \geq 2$ since $n \geq 2$. Hence $E^{\oplus q}$ is a trivial vector bundle and therefore its sphere bundle is a product. Since the base space is a sphere as well, we get a product of spheres. The dimensions are as stated in this proposition. □

Next we want to reduce the dimension of the base sphere.

Proposition 8.7. *Let G be a finite group and $n \geq 2$. Suppose that G has an n-dimensional real representation with fixity one. Then for every $q \geq 1$, $d = 1, 2, 4, 8$ and $k < d$, there is a free G-action on $S^{kn-1} \times S^{dq(n-1)-1}$.*

Proof. Let V be an n-dimensional real representation of G with fixity one. In Lemma 8.3, the stably trivial bundle E is over the unit sphere $S(U)$ where U is a direct sum of d copies of V. For $q \geq 1$, let $E^{\oplus q}$ denote the q-fold Whitney sum of E with itself. For every subrepresentation $U' \subset U$, we can restrict the bundle $E^{\oplus q}$ to $S(U') \subset S(U)$. For $k < d$, let U' be the subrepresentation given by the first k copies. Since the inclusion $S(U') \subset S(U)$ is nullhomotopic, the restriction of the bundle is trivial even before stabilization. The total space of the sphere bundle $S(E^{\oplus q}|_{S(U')})$ has a free G-action and it is homeomorphic to a product $S(U') \times S^{dq(n-1)-1}$. □

We are mostly interested in the case of A_4, which has a three-dimensional real representation of fixity one. In this case we can realize all fibered ideals $\langle v^k, u^l \rangle$ for $k \leq 8$. Recall that k and l must satisfy the condition that $1 \leq k \leq 2^t$, where 2^t is the largest power of 2 dividing l.

Theorem 8.8. *Any fibered ideal $I = \langle v^k, u^l \rangle$ with $k \le 8$ can be realized by a free A_4-action on the total space of an S^{2l-1}-bundle over S^{3k-1}. If $I \neq \langle v^c, u^c \rangle$ for all $c = 1, 2, 4, 8$, then I can be realized by a trivial bundle, and thus by a free A_4-action on a product of spheres.*

Proof. Suppose that $I = \langle v^k, u^l \rangle$ is a fibered ideal with $k \le 8$. Writing $l = 2^t c$ with c odd, we have $k \le 2^t$. Let $d = \min(2^t, 8)$ and $q = l/d$ We construct a free A_4-action on a S^{2dq-1}-bundle over S^{3k-1}. This will realize I since there is at most one Steenrod closed parameter ideal with parameters in given degrees by Proposition 6.11. If $k < d$, then Proposition 8.7 provides such a trivial bundle. If $k = d$ and $q \ge 2$, then Proposition 8.6 provides such a trivial bundle. If $k = d$ and $q = 1$, then $I = \langle v^k, u^k \rangle$, and in this case Lemma 8.3 yields the desired bundle. □

It is already observed by Oliver [Oli79] that A_4 acts freely on $S^2 \times S^3$. This is also stated in [Bla13] where the construction of a free A_4-action on $S^2 \times S^3$ is done by constructing a trivial sphere bundle over S^2 using the group structure of S^3.

Remark 8.9. One can take Whitney sums of the trivial bundles constructed in [Oli79] and [Bla13] to get a free action on a product of higher dimensional spheres, however using this method one can only get free A_4 actions on $S^2 \times S^{4l-1}$ realizing $\langle v, u^{2l} \rangle$ for $l \ge 2$. So the argument we give above gives more examples of free actions on a product of spheres.

Remark 8.10. In [Bla13], the author considers the problem of finding all pairs (n, m) such that A_4 acts freely on $S^n \times S^m$. One of the results listed there says A_4 acts freely on $S^{2n-1} \times S^{4n-5}$ for all $n \ge 3$. Unfortunately this result is not true for all $n \ge 3$. For example, for $n = 5$ such an action would realize a Steenrod closed parameter ideal with parameters in degrees $(10, 16)$ by Theorem 7.5, but there doesn't exist a Steenrod closed parameter ideals with these degrees by Remark 6.14.

The question of the topological realizability of unstable modules over the Steenrod algebra of the form

$$\mathbb{F}_2[a, b]/\langle v^k, u^l \rangle$$

has been studied in [MS03], and they prove that the quotient modules of the form

$$\mathbb{F}_2[a, b]/\langle v^{2^r}, u^{2^r} \rangle$$

are only realizable for $r = 0, 1, 2, 3$ (see [MS03, Theorem 1.2]). As a consequence we can conclude the following:

Proposition 8.11. *For $r \ge 4$, the ideal $\langle v^{2^r}, u^{2^r} \rangle$ in $H^*(BA_4; \mathbb{F}_2)$ is not realizable by a free A_4-action on a finite CW-complex X with four-dimensional mod-2 cohomology such that the top class is a product of two lower-dimensional classes.*

Proof. If such a free A_4-action existed then we can restrict the action to the elementary abelian subgroup $P \cong C_2 \times C_2$ in A_4 and obtain a free P-action on the product of spheres X with k-invariants u^{2^r} and v^{2^r} as cohomology classes in $H^*(P; \mathbb{F}_2)$. The orbit space of this action will have the cohomology ring

$$H^*(X/P; \mathbb{F}_2) \cong \mathbb{F}_2[a, b]/\langle v^{2^r}, u^{2^r} \rangle$$

as a module over the Steenrod algebra. This can be shown by studying the Serre spectral sequence as we did for A_4-actions in Proposition 7.4. However, by [MS03, Theorem 1.2], for $r \ge 4$ such Steenrod modules are not realizable. So such free actions are not possible. □

The following question is completely open to us.

Question 8.12. Which nonfibered ideals can be realized?

While we have realized all fibered ideals $\langle v^k, u^l \rangle$ with $k \leq 8$, we will provide partial results to the following question in Section 9.

Question 8.13. Which fibered ideals $\langle v^k, u^l \rangle$ with $k > 8$ can be realized?

9. Constructions using Adem-Smith method

In this section we prove the following theorem:

Theorem 9.1. *For every $k \geq 1$, there exists an integer $l_0 \geq 1$, depending on k, such that for every $s \geq 1$ the ideal $\langle v^k, u^{l_0 s} \rangle$ in $H^*(BA_4; \mathbb{F}_2)$ is realized by a free A_4-action on a finite CW-complex homotopy equivalent to a product of two spheres.*

To prove this theorem we will use a construction method introduced by Adem and Smith [AS01]. In fact the proof essentially follows from [AS01, Theorem 3.6], but here rather than directly applying this theorem, we will explain also the steps of this construction to observe the special refinements coming from the fact that the group is A_4. We start with recalling a definition.

Definition 9.2. A CW-complex X has *periodic cohomology* if there is a cohomology class $\alpha \in H^*(X; \mathbb{Z})$ with $|\alpha| > 0$ and an integer $d \geq 0$ such that

$$\alpha \cup - \colon H^i_{loc}(X; \mathfrak{B}) \to H^{i+|\alpha|}_{loc}(X; \mathfrak{B})$$

is an isomorphism for every local coefficient system $\mathfrak{B}$ and every integer $i \geq d$.

The main theorem of the Adem-Smith paper is the following:

Theorem 9.3 (Theorem 1.2, [AS01])**.** *Let X be a connected CW-complex. The cohomology of X is periodic if and only if there is a spherical fibration $E \to X$ such that the total space E is homotopy equivalent to a finite-dimensional CW-complex.*

To apply this theorem we need to find a suitable base space X whose cohomology is periodic. Since we want one of the k-invariants to be v^k, we take

$$X := EA_4 \times_{A_4} S(W)$$

where $W = \oplus_k V$ is the k-fold direct sum of the 3-dimensional irreducible real A_4-representation V. Since the mod-2 Euler class of the representation V is equal to $v \in H^3(BA_4; \mathbb{F}_2)$, the mod-2 Euler class of the sphere bundle $\xi \colon X \to BA_4$ is equal to $v^k \in H^{3k}(BA_4; \mathbb{F}_2)$.

To show that X has periodic cohomology, we need to consider the integral cohomology of A_4. The integral cohomology of a finite group in positive dimensions is a finite abelian group, hence for $i > 0$, we have

$$\begin{aligned} H^i(BA_4; \mathbb{Z}) &\cong H^i(BA_4; \mathbb{Z}_{(2)}) \oplus H^i(BA_4; \mathbb{Z}_{(3)}) \\ &\cong H^i(B(\mathbb{Z}/2 \times \mathbb{Z}/2); \mathbb{Z}_{(2)})^{C_3} \oplus H^i(BC_3; \mathbb{Z}_{(3)}). \end{aligned}$$

We have $H^*(BC_3; \mathbb{Z}_{(3)}) \cong \mathbb{Z}[z]/\langle 3z \rangle$, where $z \in H^2(BC_3; \mathbb{Z}_{(3)})$ is the Euler class of a nontrivial one-dimensional complex representation of C_3. For the 2-local cohomology we have the following result (see [Ber06, Lemmas 3.4 and 3.5]) using the notation $I^{[2]}$ from Definition 4.2:

Proposition 9.4. *The group cohomology $H^*(BA_4;\mathbb{Z}_{(2)})$ is isomorphic to the kernel of the Bockstein homomorphism*

$$\beta\colon H^*(BA_4;\mathbb{F}_2)\to H^{*+1}(BA_4;\mathbb{F}_2).$$

Hence there is an isomorphism

$$H^*(BA_4;\mathbb{Z}_{(2)})\cong H^*(BA_4;\mathbb{F}_2)^{[2]}\oplus vH^*(BA_4;\mathbb{F}_2)^{[2]}.$$

Proof. Consider the long exact sequence induced by $0\to\mathbb{Z}_{(2)}\xrightarrow{\times 2}\mathbb{Z}_{(2)}\xrightarrow{m_2}\mathbb{F}_2\to 0$:

$$\cdots\to H^i(BA_4;\mathbb{Z}_{(2)})\xrightarrow{\times 2}H^i(BA_4;\mathbb{Z}_{(2)})\xrightarrow{m_2^*}H^i(BA_4;\mathbb{F}_2)\xrightarrow{\beta_0}H^{i+1}(BA_4;\mathbb{Z}_{(2)})\xrightarrow{\times 2}\cdots$$

Since $H^*(B(C_2\times C_2);\mathbb{Z}_{(2)})$ is an $\mathbb{F}_2$-vector space by the Künneth theorem, $H^*(BA_4;\mathbb{Z}_{(2)})\cong H^*(B(C_2\times C_2);\mathbb{Z}_{(2)})^{C_3}$ is an $\mathbb{F}_2$-vector space. Hence the homomorphism $\times 2$ in the above long exact sequence is the zero map. This gives that m_2^* is injective and

$$H^*(BA_4;\mathbb{Z}_{(2)})=\ker(\beta_0).$$

Since the Bockstein operator β is equal to $m_2^*\circ\beta_0$, and since m_2^* is injective, we have $\ker(\beta_0)=\ker(\beta)$. The second statement follows from Lemma 3.5. □

Since A_4 cannot act nontrivially on $\mathbb{Z}$, the bundle $\xi\colon X=EA_4\times_{A_4}S(W)\to BA_4$ is orientable. The integral Euler class of ξ is equal to

$$\gamma=(v^k,0)\in H^{3k}(BA_4;\mathbb{Z})\cong H^{3k}(BA_4;\mathbb{Z}_{(2)})\oplus H^{3k}(BA_4;\mathbb{Z}_{(3)}).$$

The second coordinate is zero since C_3 fixes a point on $S(W)$. Furthermore we can say the following:

Lemma 9.5. *The space $X=EA_4\times_{A_4}S(W)$ has periodic cohomology in the sense defined in Definition 9.2.*

Proof. Note that the A_4-action on $S(V)$ comes from identifying A_4 as the group of rotational symmetries of a tetrahedron. In particular, all the isotropy subgroups of the A_4-action on $S(V)$ are isomorphic to C_3 or C_2. The A_4-action on $S(W)$ also has the same isotropy subgroups, hence all the isotropy subgroups of the A_4-action on $S(W)$ have periodic group cohomology. By [AS01, Proposition 3.1], this implies that the CW-complex X has periodic cohomology in the sense defined in Definition 9.2. □

To find a periodicity generator in the sense of Definition 9.2, we need a cohomology class $\alpha'\in H^*(BA_4;\mathbb{Z})$ which restricts nontrivially to cyclic subgroups C_2 and C_3. We can find a cohomology class satisfying this condition explicitly using the cohomology of A_4. Let $\alpha'=u^2\oplus z^2$ denote the cohomology class in

$$H^4(BA_4;\mathbb{Z})\cong H^4(BA_4;\mathbb{Z}_{(2)})\oplus H^4(BA_4;\mathbb{Z}_{(3)}),$$

where $z\in H^2(BA_4;\mathbb{Z}_{(3)})\cong H^2(BC_3;\mathbb{Z}_{(3)})$ is the class defined above. Note that α' is an integral class because $\beta(u^2)=0$. Observe that α' restricts nontrivially to C_2 and C_3. This follows from the fact that the cohomology class $u=a^2+ab+b^2\in H^2(B(\mathbb{Z}/2\times\mathbb{Z}/2);\mathbb{F}_2)$ restricts nontrivially to every cyclic subgroup isomorphic to C_2.

Lemma 9.6. *Let $\alpha\in H^*(X;\mathbb{Z})$ be the cohomology class $\pi^*(\alpha')$ where $\pi\colon EA_4\times_{A_4}S(W)\to BA_4$ is the canonical projection map. Then $\alpha\in H^4(X;\mathbb{Z})$ is a periodicity generator for X and the integer d in Definition 9.2 can be taken as*

$$d=\dim S(W)+1=3k.$$

Proof. This is clear from the definition of the cohomology class $\alpha' \in H^4(BG;\mathbb{Z})$ and from the proof of [AS01, Proposition 3.1]. □

To complete the proof of Theorem 9.1, we follow the construction given in the proof of [AS01, Theorem 3.6].

Proof of Theorem 9.1. Let $G = A_4$ and $X := EG \times_G S(W)$ denote the Borel construction for the representation sphere $S(W)$ as defined above. The Euler class of the oriented spherical fibration $X \to BG$ is equal to $\gamma = (v^k, 0) \in H^{3r}(BG;\mathbb{Z})$. By Lemmas 9.5 and 9.6, the space X has periodic cohomology with periodicity generator $\alpha = \pi^*(\alpha')$ where $\alpha' = (u^2, z^2) \in H^4(BG;\mathbb{Z})$ and the integer d in the definition of periodicity of X is equal to $3k$.

By Theorem 9.3, there is a spherical fibration $E \to X$ such that E is homotopy equivalent to a finite-dimensional CW-complex and the fundamental class of the fiber transgresses to α^m for some m. The fibers of $E \to X$ are homotopy equivalent to the sphere S^{4m-1}. The power m depends on $d \geq 0$ in the definition of the periodicity of the cohomology of X.

Since the CW-complex X is of finite type, it follows that the total space of the fibration $E \to X$ is finitely dominated, hence the finiteness obstruction $\mathcal{O}_E \in \widetilde{K}_0(\mathbb{Z}A_4)$ for E is defined (see the proof of [AS01, Theorem 2.13]). By a theorem of Swan the reduced projective class group $\widetilde{K}_0(\mathbb{Z}A_4)$ is isomorphic to the locally free class group $Cl(\mathbb{Z}A_4)$ (see [RU74, Theorem 1]), hence by [RU74, Theorem 3.2], we have $\widetilde{K}_0(\mathbb{Z}A_4) \cong Cl(\mathbb{Z}A_4) = 0$. So the finiteness obstruction $\mathcal{O}_E$ is already zero without taking any further fiber joins as in the proof of [AS01, Theorem 2.13]. We can conclude that E is homotopy equivalent to a finite CW-complex.

Let $\widetilde{X}$ denote the universal cover of X. For each $q \geq 1$, let $\zeta_q \colon P_q \to \widetilde{X}$ be the fibration defined by the pullback diagram

$$\begin{array}{ccc} P_q & \longrightarrow & E_q := *^q_X E\ . \\ \downarrow{\scriptstyle \zeta_q} & & \downarrow \\ \widetilde{X} & \longrightarrow & X \end{array}$$

where $*^q_X E$ is the q-fold fiber join of the fibration $E \to X$ (see [AS01, Definition 2.2]). The fibers of ζ_q are homotopy equivalent to the sphere S^{4mq-1}. The universal cover $\widetilde{X} = EA_4 \times S(W)$ is homotopy equivalent to $S(W)$, so the covering map $\widetilde{X} \to X$ factors through some n-skeleton $\mathrm{sk}_n X$ of X. By [AS01, Lemma 2.8], there is an integer q such that the fibration $\zeta_q \colon P_q \to \widetilde{X}$ is a product fibration. Since $\widetilde{X}$ is homotopy equivalent to $S(W)$, the space P_q is homotopy equivalent to $S(W) \times S^{4mq-1}$ for some large q. There is a free A_4-action on P_q and we can assume P_q is a finite A_4-CW-complex since E is homotopy equivalent to a finite CW-complex. This is because if a space Z is homotopy equivalent to a finite connected CW-complex T with $G = \pi_1(T)$, then their universal covers are G-homotopy equivalent. Since the universal cover of the finite CW-complex T is a finite G-CW-complex, we can conclude that $\widetilde{Z}$ is G-homotopy equivalent to the finite G-CW-complex $\widetilde{T}$.

Taking further fiber joins of the spherical fibration $E_q \to X$, i.e., by replacing q with qs gives a free, finite A_4-CW-complex $P_{qs} \simeq S(W) \times S^{4mqs-1}$ for every $s \geq 1$. We claim that the mod-2 k-invariants of the A_4-action on P_{qs} generate the ideal

$\langle v^k, u^{2mqs}\rangle$ in $H^*(BA_4)$. To see this, note that for $G = A_4$ we have

$$EG \times_G \widetilde{X} \simeq EG \times_G (EG \times S(W)) \simeq EG \times_G S(W) = X.$$

This gives a commuting diagram

$$\begin{array}{ccccc}
S^{4mqs-1} & \longrightarrow & P_{qs} \simeq \widetilde{X} \times S^{4mqs-1} & \longrightarrow & \widetilde{X} \simeq S(W) \\
\downarrow & & \downarrow & & \downarrow \\
S^{4mqs-1} & \longrightarrow & EG \times_G P_{qs} & \xrightarrow{\pi_{qs}} & EG \times_G \widetilde{X} \simeq EG \times_G S(W) \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
 & & BG & \xrightarrow{=} & BG
\end{array}$$

where π_{qs} is induced by the projection map of the product bundle $\zeta_{qs} : P_{qs} \to \widetilde{X}$. Applying the comparison theorem for the mod-2 Serre spectral sequences for the bundles with projection maps π_1 and π_2, we see that the first k-invariant μ_1 of the G-action on P_{qs} is equal to the k-invariant of the bundle

$$S(W) \to EG \times S(W) \xrightarrow{\pi_2} BG$$

which is v^k since $W = \oplus_k V$.

Since $\mu_1 = v^k$ is a nonzero divisor in $H^*(BG; \mathbb{F}_2)$, the spectral sequence on the E_{4mqs}-page has only two vertical lines with the differential given by product with the second k-invariant $\mu_2 \in H^*(BG; \mathbb{F}_2)$, defined only modulo $\langle v^k\rangle$. Since the action is free, the k-invariant μ_2 is nonzero modulo $\langle v^k\rangle$, and it lies in the kernel of

$$\pi_1^* : H^*(BG; \mathbb{F}_2) \to H^*(EG \times_G P_{qs}; \mathbb{F}_2).$$

Since $\pi_1 = \pi_2 \circ \pi_{qs}$, the cohomology class $\pi_2^*(\mu_2)$ lies in the kernel of

$$\pi_{qs}^* : H^*(EG \times_G \widetilde{X}; \mathbb{F}_2) \to H^*(EG \times_G P_{qs}; \mathbb{F}_2).$$

The integral Euler class of the spherical fibration with the projection map π_{qs} is equal to $\alpha^{mqs} \in H^*(X; \mathbb{Z}) \cong H^*(EG \times_G S(W); \mathbb{F}_2)$ by construction, so the mod-2 Euler class for this spherical fibration is equal to $\pi_2^*(u^{2mqs})$ by our choice of α. This gives that the kernel of π_{qs}^* is equal to the ideal generated by $\pi_2^*(u^{2mqs})$. Hence we can conclude that $\mu_2 \equiv u^{2mqs}$ modulo $\langle v^k\rangle$, and therefore the ideal generated by the mod-2 k-invariants of the free A_4-action on $P_{qs} \cong S(W) \times S^{4mqs-1}$ is equal to $\langle v^k, u^{2mqs}\rangle$ for $s \geq 1$. So the integer l_0 in the statement of the theorem can be taken as $2mq$. □

The construction above gives a finite, free A_4-CW-complex homotopy equivalent to a product of two spheres. It is not clear if the construction can be improved to get a smooth, free A_4-action on a product of two spheres. However, applying [Dav16, Theorem 1], it is always possible to get a smooth, free A_4-action on a product of three spheres.

Remark 9.7. An ideal $\langle v^k, u^l\rangle$ is a Steenrod closed parameter ideal if $k \leq 2^t$ where 2^t is the largest power of 2 dividing l. In the geometric construction above the condition on l will be much more severe than this algebraic condition. Even if we ignore the number of joins required in the last step, i.e., if we try to construct a free action on the total space of a sphere bundle over a sphere, the number of joins required will be much bigger. Note that in the first step the number of joins depend on the exponent of the obstructions and we need to take joins until we reach to the dimension $d = 3k$ (see [AS01, Lemma 2.6]). So even in the best possible case where these exponents are just equal to 2, we will have $3k \leq 2^t$ when $l = 2^t c$ with c odd. So unless one finds specific reasons for some of the obstructions to be zero, it seems that using the construction we give here, it will not be possible to realize all fibered Steenrod closed parameter ideals.

Appendix A. Table of Steenrod closed parameter ideals

The following table summarizes which Steenrod closed parameter ideals in the group cohomology of A_4 can be realized by a free A_4-action on a finite CW-complex X whose total cohomology is four-dimensional with top class the product of two lower-dimensional classes. The axes represent the degrees of the parameters with larger degree on the x-axis.

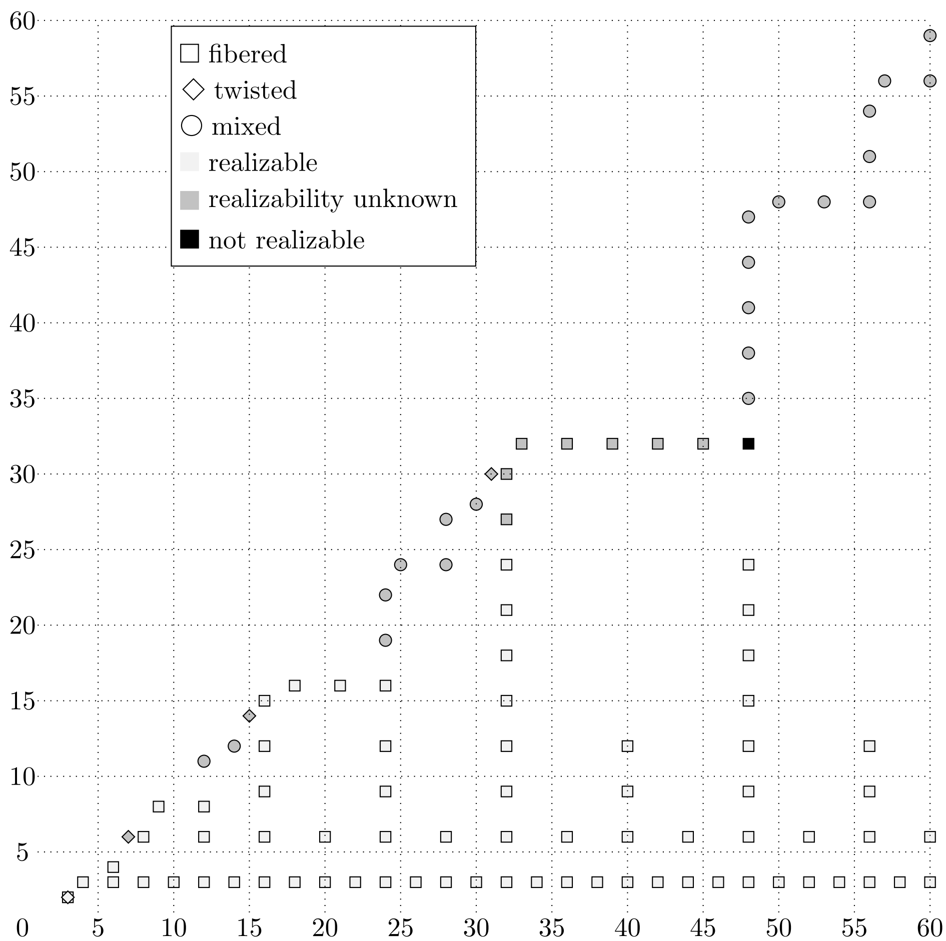


Figure 1. Steenrod closed parameter ideals with parameters of degrees at most 60.

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