MATH 220
MIDTERM II
Spring 2017

NAME : Solutions

STUDENT I.D. :

THIS EXAMINATION PAPER CONTAINS 7 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 110 minutes, starting at 15:40 and ending at 17:30.

GOOD LUCK !

1	2	3	4	Total
25	30	25	20	100

1. a) (10 points) Let V be the vector space of all $n \times n$ matrices (over $\mathbb{R}$), and B be a fixed matrix in V. Let W be the set of all matrices A in V such that $AB = BA$. Show that W is a subspace of V.

$W = \{A \mid AB = BA\} \subseteq M_{n\times n} = V$. Since $I \in W$, $W \neq \emptyset$.

We need to check two conditions:

(1) If $A_1, A_2 \in W$, then $A_1 B = BA_1$
$A_2 B = BA_2$.

Then $(A_1 + A_2)B = A_1 B + A_2 B = BA_1 + BA_2 = B(A_1 + A_2)$

So, $A_1 + A_2 \in W$.

(2) Let $A \in W$ and $\lambda \in \mathbb{R}$. This means $AB = BA$.

Then, $(\lambda A)B = \lambda \cdot AB = \lambda \cdot BA = B(\lambda A)$

So, $\lambda A \in W$.

Since these two conditions hold, W is a subspace of V.

b) Decide whether the following sentences are true or sometimes false. If it is true, then give a logical argument to justify it. If it is false, then give an example that illustrates why it is false.

i) (5 points) Let V be a vector space and S_1, S_2 be a set of vectors in V such that $S_1 \subseteq S_2$. If S_1 spans V, then S_2 also spans V.

TRUE. Assume $S_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $S_2 = \{\vec{v}_1, \ldots, \vec{v}_n, \vec{v}_{n+1}, \ldots, \vec{v}_m\}$.
Suppose S_1 spans V. Let $\vec{v} \in V$. Then there exists $\lambda_1, \ldots, \lambda_n$ s.t. $\vec{v} = \lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n$. Then,

$$\vec{v} = \lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n + 0 \cdot \vec{v}_{n+1} + \cdots + 0 \cdot \vec{v}_m.$$

So, S_2 spans V.

ii) (5 points) Let A be a 2×2-matrix and I_2 denote the 2×2 identity matrix. Then there exist real numbers $\lambda_0, \ldots, \lambda_4$ such that not all of the $\lambda_i' s$ are equal to zero, but $\lambda_0 I_2 + \lambda_1 A + \lambda_2 A^2 + \lambda_3 A^3 + \lambda_4 A^4 = 0$.

TRUE. $\{I, A, A^2, A^3, A^4\}$ is a subset of $M_{2\times2}$.
Since dimension of $M_{2\times2}$ is 4, this set with 5 elements can not be linearly independent. So, there must exist $\lambda_0, \ldots, \lambda_4$ (not all zero) such that $\lambda_0 I + \lambda_1 A + \cdots + \lambda_4 A^4 = 0$.

iii) (5 points) Let A be an $m \times n$ matrix which is already in row echelon form. The nullity of A is equal to the number of zero rows in A.

FALSE Let

$$A = \begin{bmatrix} \boxed{1} & 0 & 2 & 3 & 0 \\ 0 & \boxed{1} & -1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then Nullity of A is $5-2=3$ but there is only one zero row.

2. a) (15 points) Show that the following set of vectors is a basis for M_{22}.

$$\begin{bmatrix} 2 & 4 \\ 2 & -4 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -4 \\ -6 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

It is enough to check determinant of the matrix obtained by coordinate matrices of these vectors

$$\overset{+2}{}$$

$$\begin{vmatrix} 2 & 0 & 0 & -1 \\ 4 & -1 & -4 & 0 \\ 2 & -1 & -6 & 1 \\ -4 & 0 & -2 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 4 & -1 & -4 & 0 \\ 4 & -1 & -6 & 1 \\ -8 & 0 & -2 & -2 \end{vmatrix} = -(-1) \cdot \begin{vmatrix} 4 & -1 & -4 \\ 4 & -1 & -6 \\ -8 & 0 & -2 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 1 & -1 & -4 \\ 1 & -1 & -6 \\ -2 & 0 & -2 \end{vmatrix} = 4 \begin{vmatrix} 1 & -1 & -4 \\ 0 & 0 & -2 \\ -2 & 0 & -2 \end{vmatrix} = 4 \begin{vmatrix} 0 & -2 \\ -2 & -4 \end{vmatrix}$$

$$= (-8) \begin{vmatrix} 0 & 1 \\ -2 & 2 \end{vmatrix} = (-8)(2) = -16 \neq 0.$$

So, this set of vectors form a basis for M_{22}.

b) (15 pts) Let $\mathbb{R}_4$ denote the vector space of 4-tuples (a, b, c, d). Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$$

be a set of vectors in $\mathbb{R}_4$, where $\mathbf{v}_1\cancel{(x)} = (1, 0, 1, 1)$, $\mathbf{v}_2\cancel{(x)} = (3, 0, 3, 3)$, $\mathbf{v}_3\cancel{(x)} = (0, 2, 2, 2)$, $\mathbf{v}_4\cancel{(x)} = (1, 1, 2, 2)$, $\mathbf{v}_5\cancel{(x)} = (1, 2, 1, 1)$. Find a subset of S which is a basis for $\cancel{\mathbb{R}_4}$. Span(S).

$\vec{v}_2 = 3\vec{v}_1$. We can ignore $\vec{v}_2$.

$$\begin{array}{c} \\ -1 \\ \\ -1 \end{array}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1/2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ v_1 & v_3 & v_4 & v_5 \end{array}$$

$\{\vec{v}_1, \vec{v}_3, \vec{v}_5\}$ is a basis for span of S.

** The way the problem stated, the answer is these is no subset of S which is a basis for $\mathbb{R}_4$. I accepted both answers as correct.

3. a) (15 points) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be ordered basis for R^3, where $\mathbf{v}_1 = \mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3$, $\mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_3$, and $\mathbf{v}_3 = \mathbf{w}_2 + \mathbf{w}_3$. Calculate the change of basis matrix $P_{S \leftarrow T}$

$$v_1 = w_1 - w_2 + w_3$$
$$v_2 = w_1 - w_3$$
$$v_3 = w_2 + w_3$$

$$v_1 + v_2 = 2w_1 - w_2$$
$$v_2 + v_3 = w_1 + w_2$$

$$v_1 + 2v_2 + v_3 = 3w_1 \Rightarrow \boxed{w_1 = \tfrac{1}{3}v_1 + \tfrac{2}{3}v_2 + \tfrac{1}{3}v_3}$$

$$\boxed{w_3 = w_1 - v_2 = \tfrac{1}{3}v_1 - \tfrac{1}{3}v_2 + \tfrac{1}{3}v_3}$$

$$\boxed{w_2 = v_3 - w_3 = -\tfrac{1}{3}v_1 + \tfrac{1}{3}v_2 + \tfrac{2}{3}v_3}$$

$$P_{S \leftarrow T} = \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \end{bmatrix}.$$

b) (10 points) Find a basis for the subspace of 4-dimensional Euclidean space consisting of vectors (a, b, c, d) satisfying the equation $a - 2b + 3c - 4d = 0$.

$$a = 2b - 3c + 4d$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} b + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} d$$

$$\text{basis} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. (20 points) Consider the following matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ -1 & -2 & 0 & 2 & 2 \\ 0 & -3 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix}$$

a) Find a basis for the row space of A. What is the rank of A?
c) Find a basis for the null space of A. What is the nullity of A?

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ -1 & -2 & 0 & 2 & 2 \\ 0 & -3 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & -3 & 2 & 0 & 1 \\ 0 & -2 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 & 7 \\ 0 & 0 & -1 & 3 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 3 & -1 & -1 & 0 \\ 0 & \boxed{1} & -1 & 1 & 2 \\ 0 & 0 & \boxed{1} & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -4 & -7 \\ 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{ccccc} & & & t & s \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Basis for row space $\{[1\ 0\ 0\ 2\ 8], [0\ 1\ 0\ -2\ -5], [0\ 0\ 1\ -3\ -7]\}$

Rank$(A) = 3$

b) Solutions for $A\underline{x} = \underline{0}$ are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t-8s \\ 2t+5s \\ 3t+7s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -8 \\ 5 \\ 7 \\ 0 \\ 1 \end{bmatrix} s$$

Basis for null-space $= \left\{ \begin{bmatrix} -2 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \\ 0 \\ 1 \end{bmatrix} \right\}$. Nullity$(A) = 2$.