MATH 220
MIDTERM I
Spring 2017

NAME : Solutions

STUDENT I.D. :

THIS EXAMINATION PAPER CONTAINS 6 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 110 minutes, starting at 15:40 and ending at 17:30.

GOOD LUCK !

1	2	3	4	Total
25	25	25	25	100

1. (25 points) Consider the following system of linear equations. Find for which values of c the system has a solution. For these values write solutions using parameters. You are expected to solve this problem using matrix methods, i.e., Gauss-Jordan elimination, by reducing the augmented matrix to reduced row echelon form. Do not use elimination method.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 + x_3 - x_4 + x_5 &= -2 \\ 2x_1 \quad\quad - x_3 + 3x_4 + x_5 &= 3 \\ x_1 - x_2 + x_3 + x_4 \quad\quad &= c + 2 \end{aligned}$$

$$\begin{bmatrix} 2 & -3 & 1 & -1 & 2 & -2 \\ 1 & -2 & 1 & -1 & 1 & -2 \\ 2 & 0 & -1 & 3 & 1 & 3 \\ 1 & -1 & 1 & 1 & 0 & c+2 \end{bmatrix} \overset{r_1 \leftrightarrow r_2}{\sim} \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & -2 \\ 2 & -3 & 1 & -1 & 2 & -2 \\ 2 & 0 & -1 & 3 & 1 & 3 \\ 1 & -1 & 1 & 1 & 0 & c+2 \end{bmatrix}$$

$-2r_1+r_2 \to r_2$
$-2r_1+r_3 \to r_3$
$-r_1+r_4 \to r_4$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 4 & -3 & 5 & -1 & 7 \\ 0 & 1 & 0 & 2 & -1 & c+4 \end{bmatrix}$$

$-4r_2+r_3 \to r_3$
$-r_2+r_4 \to r_4$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & c+2 \end{bmatrix}$$

$-r_3+r_4 \to r_4$

$$\sim \begin{bmatrix} \boxed{1} & -2 & 1 & -1 & 1 & -2 \\ 0 & \boxed{1} & -1 & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & c+3 \end{bmatrix}$$

We need $c+3=0 \Rightarrow \boxed{c=-3}$ to have a solution.

set $x_4 = t$
$x_5 = s$

$x_3 + x_4 - x_5 = -1$
$\Rightarrow x_3 = -t + s - 1$

$x_2 - x_3 + x_4 = 2$
$\Rightarrow x_2 = 2 + (-t+s-1) - t$
$= 1 - 2t + s$

$x_1 - 2x_2 + x_3 - x_4 + x_5 = -2$
$x_1 = 2(1-2t+s) - (-t+s-1) + t - s - 2$
$= 2 - \cancel{4t} + 2s + \cancel{t} - s + 1 \cancel{+t} - s - 2$
$= 1 - 2t$

$$\boxed{\begin{aligned} x_1 &= 1-2t \\ x_2 &= 1-2t+s \\ x_3 &= -1-t+s \\ x_4 &= t \\ x_5 &= s \end{aligned}}$$

2. (25 points) Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & -2 \\ -1 & 0 & 1 & 1 \\ 2 & -1 & -1 & -1 \end{bmatrix}$$

using row operations. Solve the equation $A\mathbf{x} = \mathbf{b}$ for the matrix $\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 3 \end{bmatrix}$ using A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 & -2 & 1 & 0 & 0 \\ 0 & -3 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}b = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{2} + \frac{3}{2} \\ 1 - \frac{1}{2} - \frac{3}{2} \\ \frac{5}{2} + 1 \\ 1 - \frac{5}{2} + \frac{1}{2} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ \frac{7}{2} \\ \frac{1}{2} \end{bmatrix}$$

3. a) (12 points) Let

$$A = \begin{bmatrix} 2 & -1 & 2 & -1 \\ 0 & -1 & 3 & 0 \\ -2 & 0 & -1 & 2 \\ 3 & 1 & 0 & -1 \end{bmatrix}.$$

Calculate $\det(A)$.

$$\begin{vmatrix} 2 & -1 & 2 & -1 \\ 0 & -1 & 3 & 0 \\ -2 & 0 & -1 & 2 \\ 3 & 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & -1 & 2 \\ 3 & 1 & 3 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & -1 & -1 \\ -2 & -1 & 2 \\ 3 & 3 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 0 & -1 & 0 \\ -4 & -1 & 3 \\ 9 & 3 & -4 \end{vmatrix}$$

$$= (-1)(+1)\cdot\begin{vmatrix} -4 & 3 \\ 9 & -4 \end{vmatrix} = -(16-27) = \underline{\underline{+11}}.$$

b) (13 points) Let B be a 5×5-matrix such that $\det(B) = 2$. Let C be the matrix obtained from B by multiplying its second column by -7. Let D be the matrix obtained from B by replacing its first row with its fourth row (Caution: replacing not interchanging !). Find the determinants $\det(B^{-1})$, $\det(C)$, $\det(D)$, $\det(BC)$, and $\det(B+C)$.

$\det(B^{-1}) = \frac{1}{\det B} = \frac{1}{2}$ $\qquad$ $\det(BC) = \det B \cdot \det C = 2\cdot(-14) = -28$

5 $\quad \det(C) = -7\cdot\det B = -14$

2 $\quad \det(D) = 0$ because D has two rows which are identical. We can subtract one from other to get a zero row.

$$\det(B+C) = \det\left(\begin{bmatrix} b_{11} & \cdots & b_{15} \\ b_{21} & \cdots & b_{25} \\ \vdots & & \\ b_{51} & \cdots & b_{55} \end{bmatrix} + \begin{bmatrix} b_{11} & -7b_{12} & \cdots & b_{15} \\ b_{21} & -7b_{22} & \cdots & b_{25} \\ \vdots & \vdots & & \vdots \\ b_{51} & -7b_{52} & & b_{55} \end{bmatrix}\right)$$

6 $$= \det\left(\begin{bmatrix} 2b_{11} & -6\cdot b_{12} & 2b_{13} & \cdots 2b_{15} \\ 2b_{21} & -6\cdot b_{22} & 2b_{23} & 2b_{25} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{51} & -6\, b_{52} & 2b_{53} & 2b_{55} \end{bmatrix}\right) = 2\cdot(-6)\cdot 2\cdot 2\cdot 2\cdot \underbrace{\det B}_{2}$$

$$= 2^5\cdot(-6)$$

$$= -192$$

4. (a) For each of the statements below indicate whether the statement is always true or sometimes false. Justify your answer with a logical argument (for true statements) or give a counterexample (for false statements).

(i) (5 points) Let A and B be $n \times n$ matrices. If AB is invertible, then both A and B are invertible.

TRUE. If $A \cdot B$ is invertible then $\det(A \cdot B) \neq 0$

$\det(A \cdot B) = \det A \cdot \det B \neq 0$

$\Rightarrow$ $\det A \neq 0$ and $\det B \neq 0$

So, both A and B are invertible.

TRUE (ii) (5 points) If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then there is a nonzero matrix C such that $AC = 0$.

If $A\underline{x} = \underline{b}$ has infinitely many solutions then $A\underline{x} = \underline{0}$ has a nontrivial (nonzero) solution.

Take C as $n \times 1$-matrix which is a nonzero solution of $A\underline{x} = \underline{0}$.

(iii) (5 points) If A and B are invertible $n \times n$ matrices, then $(A+B)^{-1} = A^{-1}+B^{-1}$.

FALSE $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ both invertible

but $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

4. (b) (10 points) The trace $\mathrm{Tr}(A)$ of an $n \times n$ matrix $A = (a_{ij})$ is defined as the sum of its diagonal entries, that is, $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$. Show that for every elementary matrix E, the equality $\mathrm{Tr}(AE) = \mathrm{Tr}(EA)$ holds.

Let $A = (a_{ij})$ and $E = (e_{ij})$. Then

$$(AE)_{ii} = \sum_{j=1}^{n} a_{ij} e_{ji} = a_{i1} e_{1i} + a_{i2} e_{2i} + \cdots + a_{in} e_{ni}$$

$$(EA)_{ii} = \sum_{j=1}^{n} e_{ij} a_{ji} = e_{i1} a_{1i} + a_{i2} e_{2i} + \cdots + e_{in} a_{ni}$$

$$\mathrm{Tr}(AE) = \sum_{i=1}^{n} (AE)_{ii} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} e_{ji} \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} e_{ji} a_{ij} \right)$$

$$= \sum_{j=1}^{n} (EA)_{jj} = \mathrm{Tr}(EA)$$

You can also solve this by considering 3 types of elementary row operations. The multiplication EA corresponds to row operations and the product AE corresponds to column operations. One can check applying 3 types of row op. or corresponding column operations have the same effect on trace of a matrix.