MATH 220
FINAL EXAM
Spring 2017

NAME : Solutions

STUDENT I.D. :

THIS EXAMINATION PAPER CONTAINS 6 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 10:00 and ending at 12:00.

GOOD LUCK !

1	2	3	4	Total
25	25	25	25	100

1.a) Decide whether the following sentences are true or sometimes false. Justify your answer using mathematical arguments (examples, known theorems, or proofs).

i) (4 points) If A is a 3×5 matrix, then the equation $A\mathbf{x} = \mathbf{0}$ has a two-dimensional solution space.

FALSE. When $A=0$, the equation $Ax=0$ has 5-dimensional solution space.

ii) (4 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set with nonzero vectors, then each vector in the set is expressible as a linear combination of the other two.

FALSE. $\{(1,0), (2,0), (0,1)\}$ is a linearly dependent set but $(0,1)$ can not be written as a linear combination of vectors $(1,0), (2,0)$.

iii) (4 points) Consider the set V of vectors of the form (a, b, c, d) where $a = d - b$, $b = a - c$, and $d = c + 2b$. Then V is a two-dimensional vector space.

TRUE. $d = a+b$, $c = a-b$, $d-c = 2b$

$(a, b, a-b, a+b) = a(1,0,1,1) + b(0,1,-1,1)$

So, $\{(1,0,1,1), (0,1,-1,1)\}$ is a basis for V.

iv) (4 points) The set of 2×2 matrices M_{22} has a basis consisting of invertible matrices.

TRUE. $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis for M_{22} and all the matrices have $\det \neq 0$, so they are invertible.

1.b) (9 points) Prove that the formula

$$A \cdot \operatorname{Adj}(A) = \det(A) I_n$$

holds for an arbitrary $n \times n$ matrix A.

Let $A = (a_{ij})$ and

$\operatorname{Adj}(A) = (b_{ij})$ where $b_{ij} = (-1)^{i+j} \cdot \det(B_{ij})$

where B_{ij} is the matrix obtained from A by erasing jth row and ith column.

$$[A \cdot \operatorname{Adj}(A)]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

If $i = j$, then $[A \cdot \operatorname{Adj}(A)]_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$

$$= \underbrace{a_{i1} b_{1i}}_{b_{1i} = (-1)^{i+1} \det B_{1i}} + \cdots + \underbrace{a_{in} b_{ni}}_{b_{ni} = (-1)^{n+i} \det B_{ni}} = \det \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \boxed{a_{i1}} & \cdots & \cdots & \boxed{a_{in}} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow i\text{th row} \\ \\ \\ \end{array} \end{array}$$

$= \det(A)$ by cofactor expansion

If $i \neq j$, then $(A \cdot \operatorname{Adj}(A))_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$= a_{i1} b_{1j} + \cdots + a_{in} b_{nj} = \det \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{i1} & \cdots & \cdots & a_{in} \\ a_{i1} & \cdots & \cdots & a_{in} \\ \vdots & & & \end{bmatrix} & \begin{array}{l} \\ \leftarrow j\text{th row} \\ \leftarrow i\text{th row} \\ \\ \end{array} \end{array}$$

$= 0$ since this matrix has two rows with identical entries.

So, $A \cdot \operatorname{Adj}(A) = (\det A) \cdot I_n$.

2. Let $L : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by the matrix multiplication $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 & 3 \\ 3 & 6 & -3 & -3 & -3 \\ 2 & 4 & 2 & 0 & 8 \\ 5 & 10 & -2 & -3 & 2 \end{bmatrix}.$$

a) (10 points) Convert A into reduced row echelon form.
b) (8 points) Find a basis for the kernel of L. What is the nullity of A?
b) (7 points) Find a basis for the range of L. What is the rank of A?

a) $$\begin{bmatrix} 4 & 8 & -1 & -2 & 3 \\ 3 & 6 & -3 & -3 & -3 \\ 2 & 4 & 2 & 0 & 8 \\ 5 & 10 & -2 & -3 & 2 \end{bmatrix} \xrightarrow[r_3 \to \frac{1}{2} r_3]{r_2 \to \frac{1}{3} r_2} \begin{bmatrix} 4 & 8 & -1 & -2 & 3 \\ 1 & 2 & -1 & -1 & -1 \\ 1 & 2 & 1 & 0 & 4 \\ 5 & 10 & -2 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow{r_3 \leftrightarrow r_1} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 1 & 2 & -1 & -1 & -1 \\ 4 & 8 & -1 & -2 & 3 \\ 5 & 10 & -2 & -3 & 2 \end{bmatrix} \xrightarrow[\substack{-4r_1 + r_3 \to r_3 \\ -5r_1 + r_4 \to r_4}]{-r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & -2 & -1 & -5 \\ 0 & 0 & -5 & -2 & -13 \\ 0 & 0 & -7 & -3 & -18 \end{bmatrix}$$

$$\xrightarrow[\substack{-r_2 \to r_2 \\ -2r_2 + r_3 \to r_3}]{r_2 + r_3 - r_4 \to r_4} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[2r_2 + r_3 \to r_3]{\substack{r_2 \leftrightarrow r_3 \\ -r_2 \to r_2}} \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccccc} & t & & & s \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) $x_2 = t$, $x_5 = s$
$x_1 = -2t - s$, $x_3 = -3s$, $x_4 = s$

$$x = \begin{bmatrix} -2t-s \\ t \\ -3s \\ s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 1 \end{bmatrix} s$$

Basis vectors for the Null-space

$\boxed{\text{Nullity}(A) = 2}$

c) The Range of L is the same as column space of A. A basis for it is $\left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ -3 \end{bmatrix} \right\}$. $\boxed{\text{Rank}(A) = 3}$.

3. Let P_2 denote the space of polynomials of degree ≤ 2, and let

$$S = \{p_1(t), p_2(t), p_3(t)\}$$

be the basis given by $p_1(t) = t^2 - t + 1$, $p_2(t) = t + 2$, $p_3(t) = t^2 + t + 3$.
a) (13 points) Use Gram-Schmidt process to transform the basis S into an orthogonal basis T (using standard inner product with respect to standard basis).
b) (12 points) Find the change of basis matrix $P_{S \leftarrow T}$.

a) In standard basis $\{t^2, t, 1\}$, the coordinate vectors are $u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Applying Gram-Schmidt we find:

$$v_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 4/3 \\ 5/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \frac{\overbrace{-1+4+15}^{18}}{\underbrace{1+16+25}_{42}} \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} - \frac{1-1+3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \frac{3}{7} \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7+3-7 \\ 7-12+7 \\ 21-15-7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 2/7 \\ -1/7 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis.

b)

$$\begin{array}{c} \\ +1 \\ -1 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 3 \\ -1 & 1 & 1 & -1 & 4 & 2 \\ 1 & 2 & 3 & 1 & 5 & -1 \end{array}\right] \sim \begin{array}{c} \\ \\ -2 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 2 & 2 & 0 & 6 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & -2 & 0 & 0 & -14 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & -1 & -4 \\ 0 & 3 & -9 \\ 0 & 0 & 7 \end{matrix}}_{P_{S \leftarrow T}} \\ 0 & 0 & 1 & \end{array}\right]$$

4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$.

a) (13 points) Find the eigenvalues and associated eigenvectors of A. Find a nonsingular matrix P such that $P^{-1}AP = D$ is a diagonal matrix. (For the matrix P that you found, check whether this equation really holds).

b) (12 points) Calculate the inverse A^{-1} using row operations. Could we also calculate A^{-1} using the decomposition $A = PDP^{-1}$? Explain your answer.

a) $\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda+1 & 0 \\ 0 & -4 & \lambda-3 \end{bmatrix} = (\lambda-1)(\lambda+1)(\lambda-3)$

$\lambda = 1, -1, 3$ eigenvalues. 3

$\underline{\lambda=1}$ $\begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & -4 & -2 \end{bmatrix} \cdot v_1 = 0 \sim v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\underline{\lambda=-1}$ $\begin{bmatrix} -2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -4 & -4 \end{bmatrix} v_2 = 0 \sim v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ 6

$\underline{\lambda=3}$ $\begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \end{bmatrix} v_3 = 0 \sim v_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$AP = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{bmatrix}$ =

$PD = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 1 & 6 \end{bmatrix}$ 4

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right]$

$\underbrace{\qquad\qquad}_{P^{-1}}$

b) $\begin{array}{c} \\ -1\times \\ +4 \end{array}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \sim \begin{array}{c} -1 \\ \\ \frac{1}{3}\times \end{array}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & 4 & 1 \end{array}\right]$

$\sim \begin{array}{c} \\ \\ -1 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 4/3 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/3 & -1/3 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 4/3 & 1/3 \end{array}\right]$

7

$A^{-1} = \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & -1 & 0 \\ 0 & 4/3 & 1/3 \end{bmatrix}.$

We could calculate A^{-1} using $A = PDP^{-1}$ by taking $A^{-1} = PD^{-1}P^{-1}$. Then

5

$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & -1 & 0 \\ 0 & 1 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & -1 & 0 \\ 0 & 4/3 & 1/3 \end{bmatrix}$